CHAPTER 9

A RADIATING KERR PARTICLE EMBEDDED IN A HOMOGENEOUS UNIVERSE WITH AN ELECTROMAGNETIC FIELD
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9.1 : INTRODUCTION

It is well known that the exterior gravitational field of a rotating blackhole is described by the Kerr metric (Kerr, 1963). The Kerr solution is most interesting and intriguing. The cause of its complexity as well as beauty is the inherent rotation. There is a considerable interest in generalizing Kerr metric to describe the non-static field of a radiating rotating star. Vaidya (1974) has given a very general form of the Kerr-Schild (1965) metric satisfying the Einstein's field equations $R_{ik} = \alpha \xi_i \xi_k$, $\xi_i \xi^i = 0$. Vaidya, Patel and Bhatt (1976) have also discussed some radiating Kerr solutions which are not of the Kerr-Schild type.

Vaidya and Patel (1989) have obtained an exact solution of the Einstein's equations which describes the field of a radiating Kerr particle embedded in Einstein static universe. The main aim of the present chapter is to derive an exact solution of Einstein's equations which describes the field of a radiating Kerr source embedded in a rotating homogeneous universe with an electromagnetic field.
In the present chapter we shall use the following field equations:

\[ R_{ik} + \frac{1}{2} R g_{ik} = -8\pi T_{ik} + \Lambda g_{ik} \]  

...(9.1.1)

where

\[ T_{ik} = (p+\rho)\nu_i \nu_k - p g_{ik} + E_{ik} + \sigma \omega_k \omega_i \]  

...(9.1.2)

and \( F_{ik} = A_{i,k} - A_{k,i} \) is the electromagnetic field tensor satisfying the Maxwell's equations

\[ F^{ik}{};_k = 0 \]  

...(9.1.3)

A comma indicates ordinary derivative and semi-colon indicates covariant derivative. \( \sigma \omega_k \omega_i \) is the tensor arising out of the flowing null radiation. \( p, \rho \) and \( \Lambda \) are respectively the fluid pressure, the material density and the cosmological constant. The velocity vectors \( \nu_i \) and \( \omega_i \) satisfy

\[ \nu_i \nu^i = 1, \omega_i \omega^i = 0, \nu^i \omega_i = 1 \]  

...(9.1.4)

Vaidya and Patel (1984) have obtained an exact solution of the above field equations. The geometry of this solution is of Robertson-Walker type but the physical properties of this solution are different than those of the Robertson-Walker model. The metric of this solution can be expressed in the form
where \( R \) is an arbitrary function of time \( t \) and \( R_0 \) is a constant. The metric (9.1.5) is the transform of the Robertson-Walker metric. When \( R = \text{constant} = R_0 \), (9.1.5) reduces to the metric of the Einstein static universe. Vaidya and Patel (1984) have shown that the metric (9.1.5) describes a rotating homogeneous universe with an electromagnetic field.

The metric can be expressed in the Kerr-NUT form (Vaidya, et al 1976) as

\[
\begin{align*}
\text{ds}^2 &= 2\left[ du + R_o \sin^2(\alpha/2) d\beta \right] \text{dt} - \left[ \frac{R}{R_0} \right] \left[ du + R_o \sin^2(\alpha/2) d\beta \right]^2 \\
&\quad - \frac{1}{4} RR_o \left[ d\alpha^2 + \sin^2 \alpha \ d\beta^2 \right] \quad \cdots (9.1.5)
\end{align*}
\]

So we have at our disposal the algebra and geometry of this metric as developed by Vaidya et al (1976). Here for (9.1.5) we have

\[
\begin{align*}
g \sin \alpha &= R_0 \sin^2(\alpha/2) \quad 2L = \frac{R}{R_0} \quad M^2 = \frac{1}{4} RR_o \\
\end{align*}
\]

we now assume that

\[
\begin{align*}
g &= g(\alpha), \quad M = M(u,t,\alpha), \quad L = L(u,t,\alpha) \quad \cdots (9.1.7)
\end{align*}
\]
\[ \theta^1 = du + g \sin \alpha \, d\beta, \quad \theta^2 = M \, d\alpha \]
\[ \theta^3 = M \sin \alpha \, d\beta, \quad \theta^4 = dt - L \theta^1 \]

the metric (9.1.6) becomes

\[ ds^2 = 2 \theta^1 \theta^4 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)} \theta^a \theta^b \quad \ldots (9.1.9) \]

Here and in what follows bracketed indices indicate tetrad components with respect to the tetrad (9.1.9). Using the results of Vaidya et al (1976) one can determine the tetrad components \( R_{(ab)} \) of the Ricci tensor for the metric (9.1.6). They are recorded below for ready reference:

\[ R_{(23)} = 0 \]

\[ R_{(44)} = \frac{2}{M} \left[ H_{tt} - \frac{f^2}{M^3} \right] \]

\[ R_{(24)} = \frac{g}{M} \left[ \left( \frac{M_t}{M} \right)_y \left( \frac{f}{M^2} \right)_u \right] \]

\[ R_{(34)} = \left( - \frac{g}{M} \right) \left[ \left( \frac{M_t}{M} \right)_u + \left( \frac{f}{M^2} \right)_y \right] \]

\[ R_{(44)} = L_{tt} + \left( \frac{2}{M} \right) \left[ H_{tu} + (LM_t)_t + \frac{Lf^2}{M^3} \right] \]
Here $2f = g_\alpha + g\cot\alpha$. A suffix indicates partial derivative with respect to the corresponding variable, e.g., $g_\alpha = \frac{\partial g}{\partial \alpha}$, $H_u = \frac{\partial H}{\partial u}$, $L_u = \frac{\partial^2 L}{\partial t \partial u}$, etc. The variable $y$ is defined by the relation $gda = dy$.

9.2: A RADIATING KERR PARTICLE IN A ROTATING EXPANDING UNIVERSE

We propose to introduce inhomogeneity in the rotating homogeneous universe, described by the metric (9.1.5). If the metric tensor in (9.5.5) is designated as $g_{ik}$, we shall introduce this...
inhomogeneity by adding a term $H l_i \cdot l_k$ to $g_{ik}$ ie by taking the new metric tensor $\bar{g}_{ik}$ as representing what has been termed by Taub (1981) as generalised Kerr-Schild metric,

$$\bar{g}_{ik} = g_{ik} + H l_i \cdot l_k \quad \ldots \text{(9.2.1)}$$

where $l_i$ is a null shear-free congruence in (9.1.5) and so

$$l_i dx^i = du + R_o \sin^2(\alpha/2) \, d\beta \quad \ldots \text{(9.2.2)}$$

Here $H$ is a function of $\alpha$, $u$ and $t$ which is to be determined from the field equations. Consequently we consider the metric

$$ds^2 = \bar{g}_{ik} dx^i dx^k \quad \text{ie.}$$

$$ds^2 = 2 \left[ du + R_o \sin^2(\alpha/2) \, d\beta \right] dt - \frac{1}{4} R R_o \left[ d\alpha^2 + \sin^2 \alpha \, d\beta^2 \right]$$

$$- \left[ \frac{R}{R_o} + H(\alpha, u, t) \right] \left[ du + R_o \sin^2(\alpha/2) \, d\beta \right]^2 \quad \ldots \text{(9.2.3)}$$

The metric (9.2.3) is a particular case of the metric (9.1.6) and (9.1.7).

Vaidya and Patel (1984) have shown that the source free electromagnetic field for the metric (9.2.3) can be described by the 4-potential $A_i$ given by

$$A_i = \left[ A(t), 0, A(t) R_o \sin^2(\alpha/2), 0 \right] \quad \ldots \text{(9.2.4)}$$
where \( A(t) \) satisfies

\[
A_t^2 + \frac{4A^2}{R^2} = \frac{b^2 R_o^2}{R^2}, \quad b = \text{constant}
\]  \( \ldots (9.2.5) \)

Here \( x^1 = u, x^2 = \alpha, x^3 = \beta, x^4 = t \)

Using the 4-potential \( A_t \) given by (9.2.4) and (9.2.5), one can determine \( F_{ik} \) and \( E_{ik} \). For the sake of brevity we shall not give here the expressions for \( F_{ik} \) and \( E_{ik} \). With the help of the results

\[
E_{(ab)} = e^i_{(a)} e^k_{(b)} E_{ik}, \quad e^i_{(a)} \theta^a = dx^i
\]

and \( E_{ik} \) one can find the tetrad components \( E_{(ab)} \) of \( E_{ik} \). \( E_{(ab)} \) are given by

\[
E_{(23)} = E_{(24)} = E_{(34)} = E_{(12)} = E_{(13)} = E_{(11)} = E_{(44)} = 0
\]

\[
E_{(14)} = E_{(22)} = E_{(33)} = \frac{b^2 R_o}{2 R^2}
\]  \( \ldots (9.2.6) \)

It is painless to verify that the field equations (9.1.1) with (9.1.2) can be expressed in the tetrad basis as

\[
R_{(ab)} = A g_{(ab)} - 8\pi E_{(ab)} - 8\pi \sigma \omega_{(a)} \omega_{(b)} - 8\pi \left[ (p+\rho) \nu_{(a)} \nu_{(b)} - \frac{1}{2}(\rho-p) g_{(ab)} \right]
\]  \( \ldots (9.2.7) \)

Where \( \nu_{(a)}, \omega_{(a)} \) and \( g_{(ab)} \) are tetrad components of \( \nu_t, \omega_t \) and \( g_{ik} \) respectively. For the metric (9.2.3) we can take
\[ \nu_{(\alpha)} = \left( \frac{1}{2z}, 0, 0, z \right), \quad \omega_{(\alpha)} = \left( \frac{1}{2}, 0, 0, 0 \right) \quad \ldots(9.2.8) \]

where \( z \) is a function of the coordinates to be determined from the field equations. Using (9.2.6) and (9.2.8) in (9.2.7), we get

\[ R_{(24)} = R_{(34)} = 0 \quad \ldots(9.2.9) \]

\[ R_{(12)} = R_{(13)} = 0 \quad \ldots(9.2.10) \]

\[ R_{(14)} = \frac{-2\pi(p+\rho)}{z^2} - \frac{8\pi\rho}{z^2} \quad \ldots(9.2.11) \]

\[ R_{(22)} = -\Lambda - 4\pi(p - \rho) - 6\pi E_{(22)} \quad \ldots(9.2.12) \]

\[ R_{(44)} = \Lambda - 8\pi p - 8\pi E_{(44)} \quad \ldots(9.2.13) \]

\[ R_{(44)} = -8\pi(p + \rho)z^2 \quad \ldots(9.2.14) \]

Where \( R_{(ab)} \) are given by (9.2.11).

It can be easily seen that the equations (9.2.9) are identically satisfied for the metric (9.2.3). The equations (9.2.10) can be written in the explicit forms as

\[ H_{\text{ty}} - \frac{2H_y}{R} = 0, \quad H_{\text{tu}} - \frac{2H_y}{R} = 0 \quad \ldots(9.2.15) \]
By introducing the variable $T$ defined by $dT = \frac{dt}{R}$, the above two equations can be solved for the function $H$. The solution can be expressed in the form

$$H = A(u,y)\cos 2T + B(u,y)\sin 2T + G(T) \quad \ldots(9.2.16)$$

where the functions $A$ and $B$ satisfy the relations

$$A_y = B_u, \quad A_u = -B_y \quad \ldots(9.2.17)$$

As an example solution of (9.2.17) we can take

$$A = e^{ky}\cos ku, \quad B = e^{ky}\sin ku \quad \ldots(9.2.18)$$

where $k$ is a constant. The four physical parameters $\rho, \rho, \sigma$ and $z$ can be determined from (9.2.11), (9.2.12), (9.2.13) and (9.2.14). They are given by the following expressions

$$8\pi \rho = A - 8\pi b^2 \frac{R_o^2}{R^2} - \frac{R}{R_o} \left[ \frac{R_t^2}{4R^2} + \frac{R_{tt}}{R^2} + \frac{1}{R^2} \right]$$

$$- \frac{1}{2R^2} \left[ H \left( R_{tt} - \frac{1}{2} R_t^2 + 2 \right) + H_{TT} \right] \quad \ldots(9.2.19)$$

$$z^2 = \frac{R_o}{R} \left[ 1 + \frac{1}{4} R_t^2 - \frac{RR_{tt}}{2} \right] \left[ 1 + \frac{R_t^2}{4} - \frac{RR_{tt}}{2} - 8\pi b^2 \frac{R_o^3}{R} \right]$$

$$+ \frac{1}{2R^2} \left\{ 6H - \frac{H}{2} R_t^2 - H_{RR} R_t + H_{TT} \right\}^{-1} \quad \ldots(9.2.20)$$
\[ 8\pi\rho = -\Lambda - 12\pi b^2 \frac{R_o^2}{R^2} + \frac{R}{R_o} \left[ \frac{3}{R^2} + \frac{3R_t^2}{4R^2} \right] + \frac{R_t H_T}{R^2} - \frac{H_{TT}}{2R^2} \]

\[ + \frac{H}{2R^2} \left[ RR_{tt} + \frac{R_t^2}{2} - 10 \right] \quad \ldots (9.2.21) \]

\[ 8\pi\sigma = \frac{R^2}{4R_o^2} R_{(44)}^2 + 16\pi b^2 \frac{R_o}{R} R_{(44)} + 64\pi b^4 \frac{R_o^4}{R^4} - R_{(11)} R_{(44)} \]

\[ + \left[ 16\pi b^2 \frac{R^2}{R_o^2} + \frac{R}{R_o} R_{(44)} \right] \left[ \frac{1}{4R^2} \left\{ 6H + \frac{H_T}{2} \frac{R_t^2}{R^2} - \frac{H_T}{R} R_t \right\} \right] \quad \ldots (9.2.22) \]

where \( R_{(44)} = \frac{R_{tt}}{2} - \frac{R_t^2}{2R^2} - \frac{2}{R} \) and

\[ R_{(11)} = \frac{1}{4} R_{(44)} \left[ \frac{R^2}{R_o^2} + \frac{2HR_o}{R} + H^2 \right] \]

\[ + \frac{4}{RR_o} - \frac{g^2}{2} \left( H_{uu} + H_{yy} \right) + 2H_y + \frac{H_y}{g} \frac{R_o}{R} R_t - \frac{H_T}{g} \frac{R_t}{R} R_o \]

The form of the function \( H \) can be explicitly expressed as

\[ H = m e^{ky} \cos \left[ \frac{2r}{R_o} + \left[ k - \frac{2}{R_o} \right] u \right] \quad \ldots (9.2.23) \]

where \( r = u - R_o T \) and \( m \) is a constant.

The metric (9.2.3) with \( H \) given by (9.2.23) represents the external field of a radiating Kerr particle embedded in a rotating
expanding universe with an electromagnetic field. When \( k = \frac{2}{R_0} \), then the \( u \) - dependence of \( H \) disappears. In this case we recover the solution given by Patel and Pandya (1984) which represents the field of rotating mass particle in a rotating homogeneous universe with an electromagnetic field. Vaidya and Patel (1984) have shown that, in the case \( H = 0 \), the flow lines of the fluid have non-zero rotation, provided \( b = 0 \). If the electromagnetic field is switched off, the rotation of the flow vector becomes zero. Therefore when \( b = 0 \), the solution discussed in this section represents the field of a radiating Kerr Particle embedded in the usual Robertson Walker universe. When \( b = 0 \) and \( k = \frac{2}{R_0} \), our solution gives Kerr metric in the cosmological background of Robertson Walker universe (Vaidya, 1977). We have verified that the radiation density \( \sigma \) does not vanish in the case \( b = 0 \) and \( k = \frac{2}{R_0} \). \( \sigma \neq 0 \) means that the fluid pressure in the model is not isotropic. This fact has been noted by Vaidya (1977). In the next section we shall discuss briefly a particular solution.

9.3: A RADIATING KERR PARTICLE IN ROTATING EINSTEIN UNIVERSE

In the solution of section 9.2, let us put \( R = \text{constant} = R_0 \). The metric in this case reduces to

\[
ds^2 = 2\left[ du + R_0 \sin^2(\alpha/2) \, d\beta \right]dt - \frac{1}{4} R_0^2 \left[ d\alpha^2 + \sin^2\alpha \, d\beta^2 \right] \\
- (1 + H) \left[ du + R_0 \sin^2(\alpha/2) \, d\beta \right]^2 \tag{9.3.1}
\]
where $H$ is given by (9.2.16) and (9.2.17). In this case the physical parameters $\rho$, $\rho$, $\beta$ and $\sigma$ are given by

$$8\pi\rho = \Lambda - 4\pi b^2 - \frac{1}{R_o^2} - \frac{1}{2R_o^2} \left[ H_{TT} + 2H \right] \quad \ldots(9.3.2)$$

$$8\pi\rho = -\Lambda - 12\pi b^2 + \frac{3}{R_o^2} - \frac{1}{2R_o^2} \left[ H_{TT} + 10H \right] \quad \ldots(9.3.3)$$

$$z^2 = \left[ 1 - 8\pi b^2 R_o^2 + \frac{1}{2R_o^2} \left[ H_{TT} + 6H \right] \right]^{-1} \quad \ldots(9.3.4)$$

$$8\pi\sigma \left[ 1 - 8\pi b^2 R_o^2 + \frac{1}{2R_o^2} \left[ 6H + 4a^2 H \right] \right] = 8\pi b^2 \left[ 1 - 4b^2 R_o^2 \right]$$

$$+ \left[ 6 + 4a^2 \right] H \left[ 2nb^2 \left( 1 + \frac{1}{R_o^2} \right) - 1 - \frac{1}{4R_o^2} \left( 1 + \frac{H}{2} \right) \right]$$

$$+ H \left[ \frac{1}{2R_o^2} (H + 2) - \frac{K}{R_o} \right]$$

When $b = 0$, the metric (9.3.1) describes the solution given by Vaidya and Patel (1989) in connection with the field of a radiating Kerr particle in Einstein universe.

When $H = 0$, our solution reduces to the rotating Einstein universe discussed by Vaidya and Patel (1984). Also when $k = 2/R_o$, the above solution reduces to the solution given by Patel and Pandya (1984) which represents the field of Kerr particle embedded
in the Einstein rotating universe. If \( k = 2/R_0 \) and \( b = 0 \) we recover the Kerr metric in the background of Einstein static universe given by Vaidya (1977).

Thus the metric (9.3.1) represents the field of a radiating Kerr particle embedded in the rotating Einstein universe with an electromagnetic field.

We have also verified that the flow vector \( v_\perp \) for the solution of this section has non-zero rotation.

9.4 : CONCLUDING REMARKS

In the present chapter, an exact solution of Einstein-Maxwell equations corresponding to a mixture of perfect fluid, a pure radiation field and a source-free electromagnetic field is obtained. The metric of the solution represents the external field of a radiating Kerr particle embedded in a rotating homogeneous universe with an electromagnetic field.

In the next chapter, several exact non-static solutions of Einstein-Maxwell equations corresponding to a field of flowing null radiation plus an electromagnetic field will be presented. These solutions are generalisations of the well-known Kerr-Newman solution. Thus the next chapter is intended mainly to investigate some new radiating Kerr-Newman solutions.
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