CHAPTER : 8

SOME GODEL TYPE UNIVERSES WITH MATTER, A PURE RADIATION FIELD AND A SCALAR FIELD
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A PURE RADIATION FIELD AND A SCALAR FIELD

8.1: INTRODUCTION

The first rotating model of the universe is presented by Godel (1949). This universe is filled with perfect fluid and is stationary. Many investigators have discussed various aspects of Godel-type space-times. The detailed study of geodesics in Godel-type space-times has been done by Calvao, Damiao Soares and Tiomno (1990). It is well known fact that our universe is expanding. Therefore the nonstatic cosmological models are more realistic than the stationary ones. Attempts have been made by several relativists to construct non-static rotating cosmological models. This is an urgent task, especially if the indications of the rotation in our universe found by Birch (1982), prove to be correct.

Raval and Vaidya (1966) have discussed expanding rotating models filled with the imperfect fluid. The flow vector of the fluid is geodetic and has non-zero shear. Rosquist (1983) has obtained a radiation filled rotating and expanding universe.
Agakov (1984) has discussed a non-stationary generalization of Godel universe filled with perfect fluid. The stream lines of the perfect fluid filling the universe are not geodetic.

In recent years, the problem of finding cosmological solutions with heat flow of Einstein's equations has attracted wide attention. Such solutions are helpful in understanding the early universe. Novello and Rebaucas (1978) and Ray (1980) have discussed some remarkably simple exact solutions representing expanding Godel-type universes with matter and heat flow. A detailed discussion of some rotating Bianchi type VIII time depended cosmologies with non-zero heat flow has been given by Bradley and Svetins (1984). On the similar lines, Svetins (1985) has studied some rotating non-static Bianchi type IX cosmological models whose source is perfect fluid with non-zero heat flow.

Vaidya and Patel (1986) have developed a general scheme for the derivation of exact Bianchi type IX solutions of Einstein's equations corresponding to a mixture of matter and a pure radiation field. They have also obtained some simple Bianchi type IX rotating world models filled with such material distributions. Pandya (1986) has discussed some Godel-type universes filled with a mixture of matter and a pure radiation field.
The present chapter deals with the generalizations of Pandya's results to the case in which there is also present a time-independent zero mass meson field. Here it should be noted that the scalar field stationary generalization of Godel's universe is discussed by Chakraborty and Bandyopadhyay (1983).

8.2 : THE GEOMETRY

We consider a space-time defined by a Godel-type line-element

\[ ds^2 = \left[ dt + Ne^x dy \right]^2 - \alpha^2 e^{2x} dy^2 - dx^2 - dz^2 \]  \hspace{1cm} \text{(8.2.1)}

Where N and \( \alpha \) are functions of the time t. It is not hard to see that when \( N = 1 \) and \( \alpha^2 = 1/2 \), (8.2.1) becomes the metric of the rotating Godel cosmos. Here it should be noted that the metric form (8.2.1) has been discussed by Koppar and Patel (1988) in connection with viscous fluid cosmologies with non-zero heat flux. Patel and Koppar (1989) have also discussed the metric (8.2.1) in connection with rotating universes whose material content is a mixture of viscous fluid, heat flux and a scalar field.

Introducing the basic 1-forms \( \theta^a \) defined by

\[ \theta^1 = dx, \theta^2 = ae^x dy, \theta^3 = dz, \theta^4 = dt + Ne^x dy \]  \hspace{1cm} \text{(8.2.2)}
the metric (8.2.1) can be expressed in terms of Cartan's frame (8.2.2) as

\[ ds^2 = [\theta^4]^2 - [\theta^1]^2 - [\theta^2]^2 - [\theta^3]^2 = g_{ab} \theta^a \theta^b \]  

...(8.2.3)

Using the Cartan's equation of structure, one can find the connection 1-forms \( \omega^a_b \) and the tetrad components \( R^a_{bcd} \) of the curvature tensor for the metric (8.2.1). For the sake of brevity, we shall not give the expressions for \( \omega^a_b \) and \( R^a_{bcd} \) here. Using these components one can quickly compute the tetrad components \( R_{(ab)} = R_{abc} \) of the Ricci tensor for the metric (8.2.1) and the tetrad (8.2.2). The surviving \( R_{(ab)} \) are listed below for ready reference:

\[
R_{(12)} = \frac{N^2}{2\alpha} \left[ \frac{\dot{N}}{N} - \frac{\dot{\alpha}}{\alpha} \right]
\]

\[
R_{(14)} = \frac{\dot{\alpha}}{\alpha} + \frac{N^2}{2\alpha^2} \left[ \frac{\dot{N}}{N} - \frac{3\dot{N}}{N} \right]
\]

\[
R_{(11)} = 1 - \frac{N^2}{2\alpha^2}
\]

\[
R_{(22)} = \frac{N^2}{\alpha^2} \left[ \frac{\dot{N}^2}{N^2} + \frac{\dot{N}^2}{N^2} - \frac{\dot{N} \dot{\alpha}}{N \alpha} \right] - \frac{\dot{\alpha}}{\alpha} + 1 - \frac{N^2}{2\alpha^2}
\]
Here and in what follows an overhead dot denotes the differentiation with respect to \( t \).

8.3: THE SOURCE AND THE FIELD EQUATIONS

In the present chapter we intend to discuss some exact solutions of the field equations

\[
R_{ik} - \frac{1}{2} g_{ik} R + \Lambda g_{ik} = -8\pi \left[ (p + \rho) \nu_i \nu_k - pg_{ik} + \sigma \omega_i \omega_k \right] \\
-8\pi \left[ \phi_i \phi_k - \frac{1}{2} g_{ik} g^{lm} \phi_l \phi_m \right] \tag{8.3.1}
\]

where

\[
\nu^i \nu_i = 1, \quad \omega^i \omega_i = 0, \quad \nu^i \omega_i = 1 \tag{8.3.2}
\]

and

\[
\left( \sqrt{-g} g^{ik} \phi_k \right)_{,i} = 0 \tag{8.3.3}
\]

Here comma indicates partial derivative. In the above \( p, \rho \) and \( \Lambda \) are the respectively the fluid pressure, matter density and the cosmological constant. \( \phi \) is a scalar field and \( \sigma \omega_i \omega_k \) is the tensor arising out of flowing null radiation.
We assume that the scalar field $\phi$ is a function of $z$ only. The result (8.3.3) leads us to

$$\phi = az$$

...(8.3.4)

Where $a$ is an arbitrary constant. It is easy to see that the tetrad components $\phi_{(\omega)}$ of $\phi_i$ are given by

$$\phi_{(\omega)} = (0,0,a,0)$$

...(8.3.5)

It is easy to see that the field equations (8.3.1) can be expressed in $\theta$-basis as

$$R_{(ab)} = -8\pi \left[ (p + \rho)\nu_{(a)}\nu_{(b)} - \frac{1}{2}(\rho - p)g_{(ab)} \right] + \Lambda g_{(ab)} - 8\pi \sigma_{(\omega)}\omega_{(b)} - 8\pi \phi_{(a)}\phi_{(b)}$$

...(8.3.6)

Where $\nu_{(\omega)}$ and $\omega_{(\omega)}$ are the tetrad components of $\nu_i$ and $\omega_i$ respectively.

For the metric (8.2.1) and the tetrad (8.2.2) we choose

$$\nu_{(a)} = \begin{bmatrix} \text{Sinh}\lambda, 0, 0, \text{Cosh}\lambda \end{bmatrix}$$

$$\omega_{(a)} = \begin{bmatrix} -e^{-\lambda}, 0, 0, e^{-\lambda} \end{bmatrix}$$

...(8.3.7)

Where $\lambda$ is a function of co-ordinates to be determined from the
field equations. We have verified that $\nu$ and $\omega$ given by (8.3.7) satisfy the conditions (8.3.2).

The results (8.3.5), (8.3.6) and (8.3.7) imply the following relations:

\[ R_{(12)} = 0 \quad \ldots (8.3.8) \]

\[ R_{(22)} = 8\pi a^2 \quad \ldots (8.3.9) \]

\[ 8\pi \rho = \Lambda + \frac{1}{2} \left[ R_{(11)} - R_{(44)} \right] \quad \ldots (8.3.10) \]

\[ 8\pi \rho = -\Lambda - 16\pi a^2 + \frac{1}{2} \left[ R_{(11)} - R_{(44)} \right] \quad \ldots (8.3.11) \]

\[ e^{2\lambda} = \frac{R_{(11)} + R_{(44)} + 2R_{(14)}}{16\pi a^2 + R_{(44)} - R_{(11)}} \quad \ldots (8.3.12) \]

\[ 8\pi \sigma = \frac{R_{(14)}^2 - R_{(11)}R_{(44)} + 8\pi a^2 \left[ 8\pi a^2 + R_{(44)} - R_{(11)} \right]}{16\pi a^2 + R_{(44)} - R_{(11)}} \quad \ldots (8.3.13) \]

Where $R_{(ab)}$ are given by (8.2.4).

8.4: THE SOLUTION OF THE FIELD EQUATIONS

The equation (8.3.8) can be easily integrated. The solution can be expressed as
\[ N = K\alpha \quad \ldots (8.4.1) \]

Where \( K \) is an integration constant.

Using (8.4.1) in the equation (8.3.9), we get the differential equation

\[ N + N \left[ 1 - \frac{K^2/2 - 8\pi a^2}{K^2 - 1} \right] = 0 \quad \ldots (8.4.2) \]

The solution of (8.4.2) depends upon the value of the constant \( K \). Therefore the following cases arise:

Case(i) \( : 1 - 8\pi a^2 - K^2/2 = 0 \)

Case(ii) \( : \frac{1 - 8\pi a^2 - K^2/2}{K^2 - 1} = m^2 \)

Case(iii) \( : \frac{1 - 8\pi a^2 - K^2/2}{K^2 - 1} = -n^2 \)

Here \( m \) and \( n \) are constants. We shall now list the expressions for \( N, \rho, p, e^{2\lambda} \) and \( \sigma \) for the above-mentioned three cases.

Case(i):

In this case we have

\[ N = At + B \quad \ldots (8.4.3) \]
\[ 8n\rho = \Lambda + 1/2 \] \hspace{1cm} \ldots (8.4.4)

\[ 8n\rho = -\Lambda + 1/2 - 16na^2 \] \hspace{1cm} \ldots (8.4.5)

\[ e^{2\lambda} = 1 + \frac{2A}{(At+B)} \] \hspace{1cm} \ldots (8.4.6)

\[ 8n\sigma = \left[16na^2 - 1\right] A^2 \left[At+B\right]^{-2} \] \hspace{1cm} \ldots (8.4.7)

Where \( A \) and \( B \) are the constants of integration. The geometry of the solution of this case is described by the metric

\[ ds^2 = \left[dt + (At+B)e^xdy\right]^2 - \frac{1}{2} \left[At + B\right]^2 e^{2x}\left[1 - 8na^2\right]^{-1} dy^2 - dx^2 - dy^2 \] \hspace{1cm} \ldots (8.4.8)

When \( a=0 \), the scalar field vanishes and the metric (8.4.8) reduces to the metric discussed by Novello and Rebaucas (1978) in connection with the rotating universes with heat flow. Thus we have another physical interpretation of the metric of Novello and Rebaucas in the case \( a = 0 \). If \( A = 0 \), the metric (8.4.8) becomes static. In this case we get the solution discussed by Chakraborty and Bandyopadhyay (1983). The choice \( A=0, a = 0 \) gives us the Godel solution. Thus the metric (8.4.8) describes a non-static generalization of Godel universe with matter, pure radiation and scalar field.
Case: (ii)

In this case we have

\[ 1 - 8\pi a^2 - K^2/2 = m^2 \left( K^2 - 1 \right) \]

\[ \text{ie } K^2 = 2 \left[ 1 - 8\pi a^2 + m^2 \right] \left[ 1 + 2m^2 \right]^{-1} \] \hspace{1cm} \ldots (8.4.9)

The various parameters for this case are given by

\[ N = C_1 \cos \omega t \] \hspace{1cm} \ldots (8.4.10)

\[ 8\pi p = \Lambda + \frac{1}{2} \frac{\left[ 1 + m^2 + 16\pi a^2 m^2 \right]}{\left[ 2m^2 + 1 \right]} \] \hspace{1cm} \ldots (8.4.11)

\[ 8\pi \rho = -\Lambda - 16\pi a^2 + \frac{1}{2} \frac{\left[ 1 + m^2 + 16\pi a^2 m^2 \right]}{\left[ 2m^2 + 1 \right]} \] \hspace{1cm} \ldots (8.4.12)

\[ e^{2\lambda} \left[ 16\pi a^2 \left( 1 + m^2 \right) - 1 - m^2 \right] = (16\pi a^2 - 1) \left[ 1 - m^2 - 2m \tan(\omega t) \right] \] \hspace{1cm} \ldots (8.4.13)

\[ 8\pi \sigma \left[ 16\pi a^2 - 1 \right] \left[ m^2 + 1 \right] \left[ 2m^2 + 1 \right] = \]

\[ m^2 \left[ 16\pi a^2 - 1 \right]^2 \tan^2 \omega t - m^2 \left[ 8\pi a^2 \left[ 1 - 2m^2 + 16\pi a^2 \right] - 1 \right] - 8\pi a^2 \left[ 1 - 8\pi a^2 \right] + \]

\[ 8\pi a^2 \left[ 2m^2 + 1 \right] + \left[ m^2 \left[ 32\pi a^2 + 1 \right] + 8\pi a^2 - 1 \right] \] \hspace{1cm} \ldots (8.4.14)

Where \( C_1 \) is an arbitrary constant.
From the above results it is clear that when $m = 0$, the radiation density vanishes. Thus the non-static character of the solution is intimately linked with the radiation density. In the case $m = 0$, we get the solution discussed by Chakraborty and Bandyopadhyay (1983). The choice $m = 0$, $a = 0$ leads us to the usual Godel's universe. The geometry of the solution of this case is described by the line element

$$ds^2 = \left[dt + C_1 \cos mt \, e^\lambda dy\right]^2 - dx^2 - dz^2 - \frac{C_1^2 \cos^2 mt (1+2m^2)}{2(1+m^2-8\pi a^2)} \, e^{2\lambda} dy^2$$

...(8.4.15)

Here we have assumed that $8\pi a^2 \neq 1+m^2$ i.e $K \neq 0$. The metric (8.4.15) represents non-static generalization of Godel's universe with a scalar field and a pure radiation field.

Case (iii):

In this case we have $1 - 8\pi a^2 - k^2/2 = -n^2[k^2 - 1]$

i.e $k^2 = 2[1 - n^2 - 8\pi a^2] \left[1 - 2n^2\right]^{-1}$

...(8.4.16)

The values of $N$, $p$, $\rho$, $e^{2\lambda}$ and $\sigma$ are given by

$$N = C_2 e^{nt}$$
\( 8\pi p = \Lambda + \frac{1}{2} \left[ \frac{1-n^2(1+16\pi a^2)}{1-2n^2} \right] \) \hspace{1cm} \ldots (8.4.17)

\( 8\pi \rho = -\Lambda - 16\pi a^2 + \frac{1}{2} \left[ \frac{1-n^2(1+16\pi a^2)}{1-2n^2} \right] \) \hspace{1cm} \ldots (8.4.18)

\[ e^{2\lambda} = \frac{[1+n^2][16\pi a^2-1] + 2n[16\pi a^2-1]}{16\pi a^2[1-2n^2] - [1-n^2][16\pi a^2+1]} \] \hspace{1cm} \ldots (8.4.19)

\[ 8\pi \sigma[1-n^2][16\pi a^2-1][1-2n^2] = \]

\[ n^2[16\pi a^2-1]-8\pi a^2[8\pi a^2-1]+8\pi a^2[8\pi a^2-1+n^2]-n^2[8\pi a^2[16\pi a^2-2n^2-1]+1] \] \hspace{1cm} \ldots (8.4.20)

Where \( C_2 \) is a constant of integration.

The metric of the solution of this case can be expressed in the form

\[ ds^2 = \left[ dt + C_2 e^{(x+nt)} dy \right]^2 - dx^2 - dz^2 - \frac{C_2 e^{2(x+nt)} [1-2n^2]}{2[1-n^2-8\pi a^2]} dy^2 \] \hspace{1cm} \ldots (8.4.21)

When \( n = 0, \) \( \sigma \) becomes zero and the metric (8.4.21) becomes stationary. In this case we recover the solution of Chakraborty and Bandyopadhyay (1983). The Choice \( a = 0, n = 0 \) leads to the
usual Godel metric. Therefore the metric (8.4.21) represents a non-static generalization of Godel's universe with a scalar field and a pure radiation field.

We have verified that the flow vector $\nu_i$ given by (8.3.7) has non-zero rotation in all the three above-mentioned cases. We have also verified that $\nu_i$ given by (8.3.7) has non-zero expansion and shear.

8.5: CONCLUDING REMARKS

In the above analysis we have obtained three non-static generalizations of Godel's rotating universe. The source of these generalizations consists of a mixture of perfect fluid, a pure radiation field and a time-independent scalar field. The flow vector of the fluid has non-zero expansion, shear and rotation.

In the next chapter we shall discuss the field of a radiating Kerr particle embedded in a rotating homogeneous universe with a source-free electromagnetic field.
REFERENCES: