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INHOMOGENEOUS ROTATING UNIVERSES WITH MATTER, ELECTROMAGNETIC FIELD AND SCALAR FIELD
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7.1: INTRODUCTION

Cosmological solutions of the Einstein's field equations, in which the matter content of the model rotates relative to the compass of inertia have been discussed by Van Stockum (1937), Godel (1949), Wright (1965), Maitra (1966) and Ozsvath and Schucking (1969). Chakraborty and Bandyopadhyay (1983) have given a stationary generalisation of the Godel's universe with perfect fluid and a scalar field. Here it should be noted that all the above-said solutions are spatially homogeneous. Some Inhomogeneous rotating universes with closed time-like geodesics of the matter have been discussed by Damiao Soares (1980). The material distribution, filling these models is a charged incoherent matter. In the present chapter we wish to generalise the above solutions to include a scalar field.

7.2: THE GEOMETRY AND THE RICCI TENSOR

We consider a space-time whose geometry is described by the line-element
\[ ds^2 = A_0^2 \left[ dt + 4m^2 d\phi \right]^2 - dr^2 - B^2 K^2 \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right] \] ...(7.2.1)

Where \( A_0 \) is a constant and \( B \) is an undetermined function of \( r \). \( m \) and \( K \) are functions of \( \theta \) satisfying the differential equations

\[ \frac{4m}{K^2 \sin \theta} \frac{dm}{d\theta} = \lambda_1 \] ...(7.2.2)

and

\[ \frac{d^2 K}{d\theta^2} - \frac{1}{K} \left( \frac{dK}{d\theta} \right)^2 + \cot \theta \frac{dK}{d\theta} - K = \lambda K^3 \] ...(7.2.3)

Here \( \lambda_1 \) and \( \lambda \) are constants. \( \lambda \) is proportional to the curvature of the two spheres.

\[ d\Sigma^2 = K^2(\theta) \left[ d\theta^2 + \sin^2 \theta \, d\phi^2 \right] \]

Let us introduce the following basic 1-forms in the space-time manifold described by the metric (7.2.1)

\[ \theta^1 = dr, \quad \theta^2 = BKd\theta \]

\[ \theta^3 = BK \sin \theta \, d\phi, \quad \theta^4 = A_0 \left[ dt + 4m^2 d\phi \right] \] ...(7.2.4)

The metric (7.2.1) can now be expressed in the Cartan's frame (7.2.4) as

\[ ds^2 = (\theta^4)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 = g_{(ab)} \theta^a \theta^b \] ...(7.2.5)
Here and in what follows the bracketed indices denote the tetrad components with respect to the tetrad (7.2.4). Using Cartan's structure equations

\[ d\omega^a + \omega^a \wedge \omega^b = 0, \quad d\omega^a + \omega^a \wedge \omega^c = \frac{1}{2} R^a_{bcd} \theta^d \]

and the formula \( R_{(ab)} = R_{abc} \) one can find the connection 1-forms \( \omega^a \) and the tetrad components \( R_{(ab)} \) of the Ricci tensor for the metric (7.2.1) and the tetrad (7.2.4).

The surviving \( \omega^a \) and \( R_{(ab)} \) are listed below for ready reference.

\[ \omega_1^1 = - \omega_2^2 = - \frac{B^*}{B} \theta^2 \]
\[ \omega_2^2 = - \omega_3^3 = - \frac{B^*}{B} \theta^3 \]
\[ \omega_3^3 = - \omega_4^4 = - \frac{1}{BK} \left[ \frac{1}{K} \frac{dK}{d\theta} + \text{Cot} \theta \right] \theta^3 + \frac{A_{00} \lambda_1}{B^2} \theta^4 \]
\[ \omega_4^4 = \omega_2^2 = \frac{A_{00} \lambda_1}{B^2} \theta^3 \]
\[ \omega_3^3 = \omega_4^4 = - \frac{A_{00} \lambda_1}{B^2} \theta^2 \]

and

\[ R_{(ab)} = \frac{2B^*}{B} \]

...(7.2.6)
Here and in what follows the overhead dash denotes derivative with respect to $r$.

7.3 THE SOURCE FIELDS

The material distribution filling the models is assumed to be a mixture of a perfect fluid, source-free electromagnetic fields and a scalar field. The energy momentum tensor for such distribution is given by

$$T_{ik} = (p + \rho \nu_i \nu_k - p g_{ik} + E_{ik} + \phi_i \phi_k - \frac{1}{2} g_{ik} \phi_m \phi^m)$$

...(7.3.1)

with $\nu_i \nu_i = 1$, $\phi_i = \frac{\partial \phi}{\partial x^i}$

...(7.3.2)

Here $p$, $\rho$, $\nu_i$ and $\phi$ are respectively the fluid pressure, the material density, the flow vector of the fluid and the scalar field. $\phi$ satisfies the equation

$$\frac{\partial}{\partial x^i} \left[ \sqrt{-g} g^{ik} \frac{\partial \phi}{\partial x^k} \right] = 0$$

...(7.3.3)
Here $E_{ik}$ are the components of the electromagnetic energy tensor. The Einstein's field equations are

$$R_{ik} - \frac{1}{2} g_{ik} R + \Lambda g_{ik} = -8\pi T_{ik} \quad \ldots(7.3.4)$$

where $T_{ik}$ are given by (7.3.1). It is easy to see that the field equations (7.3.4) can be expressed in the tetrad form as

$$R_{(ab)} = -8\pi \left[ (p+\rho)\nu_{(a)}\nu_{(b)} - \frac{1}{2} (\rho-p)g_{(ab)} \right] + \Lambda g_{(ab)} - 8\pi E_{(ab)} - 8\pi \phi_{(a)} \phi_{(b)} \quad \ldots(7.3.5)$$

Here $\Lambda$ is a cosmological constant, $\nu_{(a)}$, $E_{(ab)}$, and $\phi_{(a)}$ are the tetrad components of $\nu$, $E_{ik}$ and $\phi_i$ with respect to the tetrad (7.2.4) respectively.

We now consider the electromagnetic field given by the following surviving tetrad components:

$$F_{(14)} = -F_{(41)} = E(r) \quad \ldots(7.3.6)$$

$$F_{(23)} = -F_{(32)} = H(r) \quad \ldots(7.3.6)$$

We know that the source-free Maxwell equations are

$$dF = 0, \quad dF^* = 0 \quad \ldots(7.3.7)$$
where \( F = E \theta^1 \wedge \theta^4 + H \theta^2 \wedge \theta^3 \)

\[
F^* = -H \theta^1 \wedge \theta^4 + E \theta^2 \wedge \theta^3 \quad \text{ ...(7.3.8)}
\]

In the above equations \( d \) denotes the exterior derivative and \( F^* \) denotes the dual of the electromagnetic 2-form \( F \). From the equations (7.3.7) and (7.3.8) it is easy to see that \( E \) and \( H \) satisfy the equations:

\[
\begin{align*}
H'' + 2H \frac{B'B'}{B} - \frac{2A_o \lambda_1}{B^2} E &= 0 \\
E'' + 2E \frac{B'B'}{B} + \frac{2A_o \lambda_1}{B^2} H &= 0 
\end{align*} \quad \text{ ...(7.3.9)}
\]

The differential equations (7.3.9) admit the solution

\[
E = \frac{q}{B^2} \cos 2\lambda_1 \bar{r}, \quad H = \frac{q}{B^2} \sin 2\lambda_1 \bar{r} \quad \text{ ...(7.3.10)}
\]

where \( \bar{r} \) is defined by the differential relation

\[
d\bar{r} = \frac{A_o}{B^2} dr \quad \text{ ...(7.3.11)}
\]

and \( q \) is a constant of integration.

The tetrad components \( E_{(ab)} \) of the electromagnetic energy tensor are given by
If we assume scalar field \( \phi \) is a function of \( r \) only, the equation (7.3.3) gives

\[
\phi' = \frac{a}{B^2}
\]

where \( a \) is a constant of integration. The tetrad components \( \phi_{(a)} \) can be very easily obtained from the relations \( \phi_{(a)} = e_{(a)i}^i \phi \), \( e_{(a)}^i \theta^a = dx^i \), \( \phi_{(a)} \) are determined as

\[
\phi_{(1)} = \phi', \quad \phi_{(2)} = \phi_{(3)} = \phi_{(4)} = 0
\]

We use the co-moving co-ordinates. Therefore the tetrad components \( \nu_{(a)} \) can be taken as

\[
\nu_{(a)} = (0, 0, 0, 1)
\]

7.4: THE FIELD EQUATIONS AND THEIR SOLUTIONS

In view of the results (7.3.12), (7.3.14) and (7.3.15), the field equations (7.3.5) imply the following relations:

\[
8\pi p = \Lambda + \left[ \frac{B \phi'}{B^2} + \frac{\phi'}{B} - \frac{\alpha/2}{B^4} \right]
\]
\[ 8\pi p = \frac{1}{B^4} + \left[ 4\lambda^2 A^2 - 8nq^2 \right] - \Lambda - 3 \left[ \frac{B'}{B^2} + \frac{\lambda}{B^2} - \frac{\alpha/2}{B^4} \right] \] \quad \ldots(7.4.2)

and

\[ \frac{B''}{B^2} - \frac{B'}{B^2} - \frac{\lambda}{B^2} + \frac{\alpha}{B^4} = 0 \] \quad \ldots(7.4.3)

where the constant \( \alpha \) is defined by

\[ \alpha = 2A_o^2 \lambda^2 + 8na^2 - 8nq^2 \] \quad \ldots(7.4.4)

Thus if we can solve the differential equation (7.4.3) for the metric function \( B \), then \( p \) and \( \rho \) can be determined from (7.4.1) and (7.4.2).

The first integral of the (7.4.3) is given by

\[ \left[ \frac{B'}{B} \right]^2 = C + \frac{\alpha}{2} B^{-4} - \lambda B^{-2} \] \quad \ldots(7.4.5)

provided \( \frac{B'}{B} \) is not identically zero. Using the result (7.4.5) we have

\[ 8\pi p = C + \Lambda \] \quad \ldots(7.4.6)

\[ 8\pi \rho = C + \frac{1}{B^4} \left[ 4A_o^2 \lambda^2 - 8nq^2 \right] - \Lambda \] \quad \ldots(7.4.7)
From (7.4.6) and (7.4.7) we can see that the models have a matter singularity where B=0. It should also be noted that singularity in the electromagnetic field and in the scalar field always occur for B = 0.

Some possible solutions of the differential equation (7.4.5) are listed below:

**SOLUTION 1:** \( \lambda = 0, \alpha = 0, C > 0 \)

\[
B^2 = \frac{1}{2} \left[ Ke^{2l} - \frac{\alpha}{2Kl^2} e^{-2l} \right], \quad l^2 = C
\]

\[8\pi\rho = \Lambda + l^2\]

\[8\pi\rho = -\Lambda - 3l^2 + \frac{1}{B^4} \left[ 4\Lambda^2 \lambda^2 - 8\pi q^2 \right]\]

where \( K \) is an arbitrary constant.

**SOLUTION 2:** \( \lambda = 0, C > 0, \alpha = 0 \)

\( B = B_0 e^{1r} \), where \( B_0 \) is a constant and \( l^2 = C \)

\[8\pi\rho = \Lambda + l^2\]

\[8\pi\rho = -\Lambda - 3l^2 + \frac{8\pi}{B^4} \left[ q^2 - 2a^2 \right]\]
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SOLUTION 3: \( \lambda = 0, \ C > 0, \ \alpha < 0 \)

\[ B^2 = \frac{1}{2} \left[ Ke^{21r} - \frac{\alpha}{2K^1} e^{-21r} \right] \]

where \( K \) is constant and \( I^2 = C \)

\[ 8\pi p = \Lambda + I^2 \]

\[ 8\pi \rho = -\Lambda - 3I^2 + \frac{1}{B^4} \left[ 4A^2 \lambda^2 - 8\pi q^2 \right] \]

SOLUTION 4: \( \lambda = -1, \ C > 0, \ \alpha < 0 \)

\[ B^2 = \frac{1}{2} \left[ A_0 e^{21r} + \frac{K^2}{A_0 e^{21r}} - \frac{1}{I^2} \right] \]

where \( A_0 \) is constant, \( I^2 = C \) and \( K^2 = \frac{1}{41^4} - \frac{\alpha}{21^2} \)

\[ 8\pi p = \Lambda + I^2 \]

\[ 8\pi \rho = -\Lambda - 3I^2 + \frac{1}{B^4} \left[ 4A^2 \lambda^2 - 8\pi q^2 \right] \]

SOLUTION 5: \( \lambda = -1, \ C = 0, \ \alpha < 0 \)

\[ B^2 = -\frac{\alpha}{2} + [r + A_0]^2, \ \text{where} \ A_0 \ \text{is constant.} \]

\[ 8\pi p = \Lambda \]
8πρ = -Λ + \frac{1}{B^4} \left[ 4A_o^2λ^2_1 - 8πq^2 \right]

**SOLUTION 6:** \quad \lambda = -1, \quad C > 0, \quad \alpha = 0

\[
B = \frac{1}{21} \left[ K_o e^{1r} - \frac{1}{K_o e^{1r}} \right]
\]

where \( K_o \) is constant and \( l^2 = C \)

8πρ = Λ + l^2

8πρ = -Λ - 3l^2 + \frac{1}{B^4} \left[ 4A_o^2λ^2_1 - 8πq^2 \right]

**SOLUTION 7:** \quad \lambda = 1, \quad C > 0, \quad \alpha > 0

\[
B^2 = \frac{1}{2} \left[ α_o e^{21r} + \frac{K^2}{α e^{21r}} + \frac{1}{l^2} \right]
\]

where \( α_o \) is constant, \( l^2 = C \) and \( K^2 = \frac{α}{2l^2} - \frac{1}{4l^4} \)

8πρ = Λ + l^2

8πρ = -Λ - 3l^2 + \frac{1}{B^4} \left[ 4A_o^2λ^2_1 - 8πq^2 \right]
SOLUTION 8:  \( \lambda = 1, C > 0, \alpha = 0 \)

\[
B^2 = \frac{1}{4l^2} \left[ K_0 e^{1r} + \frac{1}{K_0 e^{1r}} \right]^2
\]

where \( K_0 \) is constant, \( l^2 = C \)

\[
8\pi p = \Lambda + l^2
\]

\[
8\pi \rho = -\Lambda - 3l^2 + \frac{8\pi}{4} \left[ q^2 - 2a^2 \right]
\]

7.5: DISCUSSION

The isometry groups acting transitively on the \( r = \) constant sections are Bianchi type II, VIII or IX respectively for \( \lambda = 0, +1, -1 \). For \( \lambda = 0 \), the equation (7.2.3) for \( K \) have the general solution

\[
K(\theta) = \frac{1}{\sin \theta} \left[ \frac{1-\cos \theta}{1+\cos \theta} \right]^{q_0/2}
\]  ...(7.5.1)

where \( q_0 \) is an arbitrary constant of integration. For \( q_0 \neq 0 \), the function \( K(\theta) \) defines a projective mapping of the sphere on the whole plane and for \( q_0 = 0 \) the projected sphere has the structure of a cylinder, although in both the cases the local curvature is \( \lambda = 0 \). The equation (7.2.2) for the function \( m(\theta) \) can be easily integrated for \( \lambda = 0 \). The solution can be expressed as
\[
m^2(\theta) = \frac{\lambda_1}{4q_o} \left[ \frac{1 - \cos \theta}{1 + \cos \theta} \right]^{q_o^2} \text{ if } q_o \neq 0 \quad (7.5.2)
\]

or

\[
m^2(\theta) = \frac{\lambda_1}{2} \log \left( \tan \frac{\theta}{2} \right) \text{ if } q_o = 0 \quad (7.5.3)
\]

Defining new co-ordinates \( \overline{\theta} = \frac{1}{q_o} \left[ \frac{1 - \cos \theta}{1 + \cos \theta} \right]^{q_o^2} \) with \( q_o \neq 0 \) and

\[
0 \leq \overline{\theta} \leq \infty \text{ and } \overline{\theta} = -\log \left( \tan \frac{\theta}{2} \right) \text{ with } q_o = 0 \text{ and } -\infty \leq \overline{\theta} \leq \infty,
\]

the manifolds may be extended with the line element.

\[
ds^2 = A_o^2 \left[ dt - 2\lambda_1 \overline{\theta} d\phi \right]^2 - dr^2 - B^2(r) \left[ d\overline{\theta}^2 + d\varphi^2 \right] \quad \text{for } q_o = 0, \text{ and}
\]

\[
ds^2 = A_o^2 \left[ dt + \lambda_1 q_o \overline{\theta}^2 d\phi \right]^2 - dr^2 - B^2(r) \left[ d\overline{\theta}^2 + q_o \overline{\theta}^2 d\varphi^2 \right] \quad \text{for } q_o \neq 0.
\]

For \( \lambda = +1 \), the solution of (7.2.3) can be taken as \( K(\theta) = \tan \theta \). In this case the equation (7.2.2) yields the solution

\[
m^2 = \frac{\lambda_1}{2} \left[ \sec \theta + \cos \theta \right].
\]

Defining the co-ordinate \( \overline{\theta} \) by

\[
\overline{\theta} = -\log(\cos \theta) \quad \theta \leq \overline{\theta} \leq \infty
\]

the manifold may be extended to one with the metric
\[ ds^2 = A_0^2 \left( dt - \frac{\lambda}{2} \cos \Theta \, d\phi \right)^2 - dr^2 - B^2(r) \left( d\Theta^2 + \sin^2 \Theta \, d\phi^2 \right) \]

For \( \lambda = -1 \), we have \( K = 1 \) and \( m^2 = -\frac{\lambda}{2} \cos \Theta \). The metric, in this case can be put into the form

\[ ds^2 = A_0^2 \left( dt - 2\lambda \cos \Theta \, d\phi \right)^2 - dr^2 - B^2(r) \left( d\Theta^2 + \sin^2 \Theta \, d\phi^2 \right) \]

This completes the detail discussion of the solutions of the defining differential equations for the metric functions \( m(\Theta) \) and \( K(\Theta) \).

We have verified that the expansion and shear of the matter velocity field given by (7.3.15) are zero. We have found that the rotation of the fluid velocity is given by

\[ \Omega = \frac{1}{2A_0^2 \lambda} + \frac{A_0^2 \lambda}{B^2} \]

Thus the solutions discussed in this chapter represent inhomogeneous rotating universes with vanishing shear and expansion. Here it should also be noted that the introduction of scalar field affects the geometry and the physics of the models. Closed time-like lines, of the type present in Godel and other rotating universes, are also present here. The curves \( t = \text{constant}, \Theta = \text{constant}, r = \text{constant} \), with
are time-like for values of 
(r,θ) such that \(4A_o^2\lambda^2 \cos^2 \theta - B^2(r) \sin^2 \theta > 0\).

7.6 : CONCLUDING REMARKS

In this chapter we have discussed some stationary rotating inhomogeneous universes whose matter consists of perfect fluid and scalar field.

In the next chapter we shall discuss another rotating system in general relativity which will describe the field of a radiating Kerr particle embedded in a homogeneous universe with an electromagnetic field.
REFERENCES: