Chapter 5

TOTAL RESTRAINED DOMINATION NUMBER IN GRAPHS
5.1 Introduction

For every vertex $v \in V$, the open neighborhood $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and its close neighborhood is the set $N[v] = N(v) \cup \{v\}$. The open neighborhood of the set $S \subseteq V$ is the set $N(S) = \bigcup_{v \in S} N(v)$, and the close neighborhood of $S$ is the set $N[S] = N(S) \cup S$.

In a graph a stem is a vertex adjacent to at least one end-vertex. The sets of all end-vertices and all stems are denoted by $\Omega(G)$ and $\Omega_1(G)$, respectively. In $K_2$ a vertex is both an end-vertex and a stem. A tree is an acyclic connected graph. A trivial tree is $K_1$. A tree $T$ is a double star if it contains exactly two vertices that are not end-vertices. A double star with respectively $p$ and $q$ end-vertices attached at each stem vertex is denoted by $S_{p,q}$.

For two vertices $u$ and $v$ in a connected graph $G$, the distance $d_G(u, v)$ between $u$ and $v$ is the length of a shortest $u - v$ path in $G$. For a set $S \subseteq V$ and a vertex $v \in V$, the distance $d_G(v, S)$ between $v$ and $S$ is the minimum distance between $v$ and a vertex of $S$. If a vertex $u$ is adjacent to a vertex $v$, we write $u \sim v$, while if $u$ and $v$ are nonadjacent, we write $u \not\sim v$. If $v$ is adjacent to no vertex in a set $A \subseteq V(G)$ then we write $v \sim A$ and if $v$ is adjacent to every vertex in $A$ then we write $v \sim A$. A plane graph is a planar graph together with an embedding in the plane. From the Jordan Closed
Curve Theorem, we know that a cycle $C$ in a plane graph separates the plane into two regions, the interior of $C$ and the exterior of $C$. For more detail see [20, 56].

A set $D \subseteq V(G)$ is a dominating set of $G$ if for every vertex $v \in V(G) - D$, there exists a vertex $u \in D$ such that $v$ and $u$ are adjacent. The minimum cardinality of a dominating set in $G$ is the domination number denoted $\gamma(G)$. The literature on domination has been surveyed in the two books by Haynes et al. [25, 26]. Also, we refer to [55].

A set $D \subseteq V(G)$ is a total dominating set (TDS) of a graph $G$ if each vertex of $G$ has a neighbor in $D$. Equivalently, a set $D \subseteq V(G)$ is a TDS of a graph $G$ if $D$ is a dominating set of $G$ and $\langle D \rangle$ does not contain an isolate vertex. The cardinality of a minimum TDS in $G$ is the total domination number and is denoted by $\gamma_t(G)$. A minimum TDS of a graph $G$ is called a $\gamma_t(G)$-set. The notion of total domination in graphs was introduced by Cockayne et al. [6] in 1980 and for a survey of total domination in graphs see [23], for more detail see [17, 27, 53].

A subset $S$ of vertices of $G$ is a restrained dominating set if $N[S] = V$ and the subgraph induced by $V - S$ has no isolated vertex. The restrained domination number $\gamma_r(G)$ is the minimum cardinality of a restrained dominating set of $G$. The restrained
domination number was introduced by Domke et al. [12] and has been studied by several author (see for example [10, 11]).

A set $D_{tr} \subseteq V(G)$ is a total restrained dominating set (TRDS) of a graph $G$ if it is a dominating set and the induced subgraphs $D_{tr}$ and $V(G) - D_{tr}$ do not contain an isolated vertex. The cardinality of a minimum TRDS in $G$ is the total restrained domination number and is denoted by $\gamma_{tr}(G)$. A minimum TRDS of a graph $G$ is called a $\gamma_{tr}(G)$-set. Thus, the total restrained dominating set of a graph combines the properties of both a total dominating set and a restrained dominating set. We assume that every graph without an isolated vertex has a TRDS and $D_{tr} = V(G)$ is such a set. Moreover, the above definitions imply that for any graph $G$ without an isolated vertex every TRDS is a TDS, so $\gamma_t(G) \leq \gamma_{tr}(G)$. The total restrained domination number of a graph was defined by De-Xiang Ma et al. [57] in 2005.

We state the following result which is due to Goddard and Henning [18]:

![Fig. 5: The graph $G_9$](image)
Theorem L. [18] If $G$ is a planar graph with $\text{diam}(G) = 2$, then
$\gamma(G) \leq 2$ or $G = G_9$, where $G_9$ is the graph of Fig. 5.

In this chapter, we first give some results on total restrained domination number of graphs. Also, we characterize all graphs $G$ of order $n$ for which
(i) $\gamma_{tr}(G) = n$;
(ii) $\gamma(G) = 1$ and $\gamma_{tr}(G) = 3$;
(iii) $\gamma_{tr}(G) = 2$.

Further, we give some bounds on total restrained domination number of graphs with diameter 3. Finally, we present some bound for total restrained domination number of some graphs with diameter 2 and $\gamma$-set of cardinality 2 and total restrained domination number 3.

5.2 Results

We begin with the following observation that has a straightforward proof.

Observation 5.2.1. Let $G$ be a nontrivial connected graph of order $n$. Then
(i) $\gamma(G) \leq \gamma_t(G) \leq \gamma_{tr}(G)$ and $\gamma_r(G) \leq \gamma_{tr}(G)$. Further, $\gamma_{tr}(G) \geq \max\{\gamma_r(G), \gamma_t(G)\}$;

(ii) $\Omega(G) \cup \Omega_1(G) \subseteq D_{tr}$;

(iii) $2 \leq \gamma_{tr}(G) \neq n - 1$.

**Observation 5.2.2.** $\gamma_{tr}(K_n) = 2$, where $n \neq 3$ and for $n = 3$, $\gamma_{tr}(K_n) = 3$.

**Lemma 5.2.3.** Let $T$ be a tree of order $n \geq 3$ with $\text{diam}(T) = 2$ or $3$, then $\gamma_{tr}(T) = n$.

**Proof.** Clearly, $\text{diam}(T) = 2$ if and only if $T = K_{1,n-1}$. Also, $\text{diam}(T) = 3$ if and only if $T$ is a double star graph, say $S_{p,q}$. By these facts and by Observation 5.2.1, it follows that $\gamma_{tr}(K_{1,n-1}) = n$ and $\gamma_{tr}(S_{p,q}) = |V(S_{p,q})|$.

In the following theorem, we show that there exists a connected graph $G$ of order $n \geq 4$ and $\text{diam}(G) = \gamma_{tr}(G) = 2$.

**Proposition 5.2.4.** Let $G$ be a planar complete bipartite graph of order $n$. Then $\gamma_{tr}(G) = 2$ if and only if either $G = K_2$ or
\( G = K_{2,n-2} \).

**Proof.** It is well-known that a complete bipartite graph \( G \) is planar if and only if \( G = K_{m,n-m} \), where \( m = 1, 2 \). On the other hand, \( \gamma_{tr}(K_{1,n-1}) = n \). Now, by combining of the stated assumptions the desired result follows. \(\square\)

**Observation 5.2.5.** \( \gamma_{tr}(K_{m,n-m}) = 2 \), where \( n-m \geq m \geq 2 \).

As a consequence of Proposition 5.2.4 and Observation 5.2.5 we have the following corollary:

**Corollary 5.2.6.** Let \( G \) be a graph of order \( n \geq 4 \) which contains \( K_{m,n-m} \) as subgraph, where \( n-m \geq m \geq 2 \). Then \( \gamma_{tr}(G) = 2 \).

**Proposition 5.2.7.** Let \( G \) be a nontrivial connected graph of order \( n \) with \( diam(G) = 2 \) and \( \Omega(G) \neq \emptyset \). Then \( \gamma_{tr}(G) = |\Omega(G)| + 1 \).

**Proof.** Since \( G \) is connected with \( diam(G) = 2 \) and \( \Omega(G) \neq \emptyset \), then \( G \) contains a \( K_{1,n-1} \) as a spanning subgraph. Let \( x \) be the unique stem vertex of \( G \). Then all of the non-endvertices must be adjacent to \( x \). It follows that \( \Omega(G) \cup \{x\} \) is the unique TRDS of \( G \). Hence the result follows. \(\square\)
5.3 Characterizations

In the following result, we characterize all graphs $G$ of order $n$ with $\gamma_{tr}(G) = n$.

**Proposition 5.3.1.** Let $G$ be a connected graph of order $n \geq 2$. Then $\gamma_{tr}(G) = n$ if and only if either $G = K_3$ or $G$ is a graph such that $\Omega(G) \cup \Omega_1(G) \cup S = V(G)$ and $S = \{v | v \notin \Omega(G) \cup \Omega_1(G) \text{ and } N(v) \subseteq \Omega_1(G) \text{ for all } v\}$

**Proof.** Necessity: Observation 5.2.1(ii) asserts that $\Omega(G) \cup \Omega_1(G) \subseteq D_{tr}$. On the other hand, since $N(v) \subseteq \Omega_1(G)$ for all $v \notin \Omega(G) \cup \Omega_1(G)$ and $G - (\Omega(G) \cup \Omega_1(G))$ is union of isolated vertices. These implies that $\gamma_{tr}(G) = n$.

Sufficiency: Observation 5.2.2, shows that $\gamma_{tr}(K_3) = 3$. Now, we assume that, $G \neq K_3$. Let $\gamma_{tr}(G) = n$. Assume, to the contrary, that $N(v) \notin \Omega(G) \cup \Omega_1(G)$. Then there is a vertex $u \in V(G) - (\Omega_1(G) \cup \Omega(G))$ such that $uv \in E(G)$. Hence, there are $w, r \in V(G)$ such that $uw \in E(G)$ and $vr \in E(G)$, where $w$ and $r$ are not vertices of degree 1. Therefore $S = V - \{u, v\}$ is a $\gamma_{tr}(G)$-set, a
contradiction with $\gamma_{tr}(G) = n$. \hfill \square

**Proposition 5.3.2.** Let $G$ be a connected graph of order $n$ and $\gamma(G) = 1$. Then (i) $\gamma_{tr}(G) = \Omega(G) \cup \Omega_1(G)$ while $G$ has a pendant edge. (ii) $\gamma_{tr}(G) \leq 3$ while $G$ has no pendant edge, with equality if and only if $G$ is union of $k = \frac{n-1}{2}$ copies of complete graph $K_3$ such that all of them have a common vertex and $n$ is an odd integer.

**Proof.** It is well-known that $\gamma(G) = 1$ if and only if $G$ has a vertex of degree $n-1$. Now, if $G$ has a pendant edge, then $\Omega(G) \cup \Omega_1(G)$ is a TRDS of $G$, this completes the part (i). Finally, if $G$ has no pendant edge, then clearly we have $\gamma_{tr}(G) \leq 3$.

If $G$ is union of $k = \frac{n-1}{2}$ copies of complete graph $K_3$ such that all of them have a common vertex and $n$ is an odd integer, then $\gamma_{tr}(G) = 3$.

Conversely, since $\gamma(G) = 1$, then $G$ contains a $K_{1,n-1}$ as spanning subgraph. We claim that $G$ is union of $k = \frac{n-1}{2}$ copies of complete graph $K_3$ such that all of them have a common vertex. Assume, to the contrary, that a copy, say $G_i \neq K_3$, and so $|V(G_i)| \geq 4$. 86
Then there exists a vertex \( w \in V(G_i) \) such that \( G_i - \{u, w\} \) has no isolated vertex and \( \{u, w\} \) is a \( \gamma_{tr} \)-set of \( G \), where \( u \) is the common vertex of all copies, a contradiction with \( \gamma_{tr}(G) = 3 \). This completes the proof.

In the following result, we characterize all graphs \( G \) for which \( \gamma_{tr}(G) = 2 \).

**Proposition 5.3.3.** Let \( G \) be a connected graph of order \( n \). Then \( \gamma_{tr}(G) = 2 \) if and only if either (i) \( G \) has a vertex \( v \) of degree \( n - 1 \) such that \( |\Omega(G)| = 1 \), or \( G \) has no pendant edge and a component of \( G[N(v)] \) is of order at least 3. or (ii) \( G \) has \( S_{p,q} \) as spanning subgraph such that \( G \) has no pendant edge and each component of \( G[V(G) - \{u, v\}] \) is of order at least 2 and \( p + 1 \leq \deg_G(u) \leq n - 2 \), \( q + 1 \leq \deg_G(v) \leq n - 2 \) for some \( u \) and \( v \).

**Proof.** Necessity: is obvious.

Sufficiency: Since \( \gamma_{tr}(G) = 2 \), then \( \gamma(G) \leq 2 \). We consider the following two cases:

**Case 1:** If \( \gamma(G) = 1 \), then \( G \) contains a \( K_{1,n-1} \) as spanning sub-
graph. Using Observation 5.2.1(ii) Which implies that $|\Omega(G)| \leq 1$, otherwise, a contradiction with $\gamma_{tr}(G) = 2$. If $|\Omega(G)| = 1$, and so $\Omega(G) \cup \Omega_1(G)$ is the unique TRDS of $G$, hence the desired result follows. Finally, if $|\Omega(G)| = 0$. Since $G$ contains a $K_{1,n-1}$ as spanning subgraph. Therefore, there exists a vertex $v \in V(G)$ such that $deg(v) = n - 1$. It is easy to see that $G[N(v)]$ is union of some connected graph such that each vertex of these graphs are adjacent to $v$ in $G$. We deduce at least one of them must be of order at least 3, otherwise a contradiction with $\gamma_{tr}(G) = 2$.

Case 2: If $\gamma(G) = 2$, then $G$ must be have a $S_{p,q}$ as spanning subgraph. Observation 5.2.1(ii) implies that $G$ has no pendant edge and there exist two vertices $u$ and $v$ such that $G[V(G) - \{u, v\}]$ is union of some connected graph of order at least 2 and $p + 1 \leq deg_G(u) \leq n - 2$, $q + 1 \leq deg_G(v) \leq n - 2$, otherwise, a contradiction with $\gamma_{tr}(G) = 2$. \qed
5.4 Bounds on total restrained domination number

**Theorem 5.4.1.** Let \( G \) be a nontrivial connected graph with \( \text{diam}(G) = 3 \) and \( |\Omega_1(G)| = 2 \). Let \( T = \{ u | d(u, w) = 3, w \in \Omega(G) \} \) and \( |T| = m \). Let \( v \) be a vertex in \( N(\Omega_1(G)) \) such that \( |N_T(v)| = k \) and \( k \) be the maximum number between vertices such as \( v \) and \( k \) be the maximum number between vertices such as \( v \). Then
\[
|\Omega(G) \cup S| + 2 \leq \gamma_{tr}(G) \leq |\Omega(G) \cup S| + m - k + 3,
\]
where \( S = \{ v \mid N(v) \subseteq \Omega_1(G) \} \). These bounds are sharp.

**Proof.** It is clear to see that \( |\Omega(G) \cup S| + 2 \leq \gamma_{tr}(G) \). Now, we show that \( \gamma_{tr}(G) \leq |\Omega(G) \cup S| + 2 + (1 + m - k) \). Let \( v \) be a vertex such that \( |N_T(v)| = k \). Let the other vertices in \( T \) is total dominated by at most \( m - k \) vertices in \( N(\Omega_1(G)) \cap N(T) \). Let \( W \) be at most these \( m - k \) vertices. Then \( D = \Omega(G) \cup S \cup \Omega_1(G) \cup W \cup \{ v \} \) is a total dominating set and \( D \) and \( G - D \) have no isolated vertex. So \( \gamma_{tr} \leq |\Omega(G) \cup S| + 2 + (1 + m - k) \).

The sharpness of the lower bound is trivial. To show for the
sharpness of the upper bound, we define the graph $G_{2,m,k}$ as shown in Fig. 6, where $m = 6$ and $k = 4$.

![Graph G_{2,m,k}](image)

Fig. 6: The graph $G_{2,m,k}$

It is easy to check that $S = \{u_1, u_3\}$, $\Omega(G) = \{x, y\}$, $T = \{w_2, w_3, z_1, z_2, z_3, z_4\}$, $N(\Omega_1(G)) = \{u_1, u_2, u_3, w_1, w_4, a, b, x, y\}$ and $N_T(u_2) = \{z_1, z_2, z_3, z_4\}$ such that $|N_T(u_2)|$ is the maximum number between vertices in $N(\Omega_1(G))$. Then $\gamma_{tr}(G) = |\Omega(G) \cup S| + m - k + 3$. Further, $\{a, b, x, y, w_1, w_4, u_1, u_2, u_3\}$ is a $\gamma_{tr}$-set for $G$. 

**Theorem 5.4.2.** Let $G$ be a nontrivial connected graph with $\text{diam}(G) = 3$ and $|\Omega_1(G)| = 1$. Then $|\Omega(G)| + 2 \leq \gamma_{tr}(G) \leq \text{deg}(v) + |S| + 1$, where $v \in \Omega_1(G)$ and $S = \{u \mid N(u) \subseteq N(v)\} - \{u \mid V(G[N[u] - \{v\}] \subseteq N[v]\}$. These bounds are sharp.

**Proof.** Let $w, v, s, t$ be a diametral path in $G$ and $w \in \Omega(G)$ and
$v \in \Omega_1(G)$. Clearly, every vertex from $V(G) - N[v]$ will be joined to a vertex from $N[v] - \Omega(G)$. Let $V(G) - N[v] = S \cup \overline{S}$, where $S = \{z | N(z) \subseteq N[v]\} - \{z | V(G[N[z] - \{v\}] \subseteq N[v]\}$. It is easy to check that for every vertex $r \in \overline{S}$ there exist a vertex $x \in \overline{S}$ such that $x \sim r$. Therefore, $N[v] \cup S$ is a total restrained dominating set of $G$, This completes the proof.

![Graph](image)

**Fig. 7.**

The sharpness of the lower bound is trivial. To show for the sharpness of the upper bound, we consider the constructed graph Fig. 7. Further, $S = \{u\}$ and $\{w_1, w_2, \ldots, w_t, v, a, b, u\}$ is a $\gamma_{tr}$-set of the graph.
5.5 Total restrained domination number of some planar graphs

If \( G \) is a planar graph of diameter 2. Then by Theorem L, we have \( G = G_9 \) or \( \gamma(G) \leq 2 \). It is straightforward to see that \( \gamma_{tr}(G_9) = 3 \). Now, suppose \( \gamma(G) \leq 2 \). If \( \gamma(G) = 1 \), then we can apply Proposition 5.3.2 to obtain total restrained domination number of \( G \). Now, in the following results, we discuss total restrained domination number of planar graph \( G \) of diameter 2 and \( \gamma(G) = 2 \).

**Theorem 5.5.1.** Let \( G \) be a planar graph of diameter 2 and with a \( \gamma(G) \)-set \( \{a, b\} \subseteq V(G) \), \( d(a, b) = 1 \) and \( |N(a) \cap N(b)| \leq 1 \). Then \( \gamma_{tr}(G) \leq 3 \).

*Proof.* Let \( \gamma(G) = 2 \) and \( d(a, b) = 1 \), then \( G \) has no pendant edge, otherwise, a contradiction with \( diam(G) = 2 \) or \( \gamma(G) = 2 \). If \( |N(a) \cap N(b)| = 0 \), then it is easy to see that \( \{a, b\} \) is a \( D_{tr} \). If \( |N(a) \cap N(b)| = 1 \), we may assume that \( N(a) \cap N(b) = \{u\} \). If \( deg(u) = 2 \), then \( \{a, b, u\} \) is a total restrained domination number of \( G \), otherwise \( \{a, b\} \) is a \( D_{tr} \). This completes the proof. \( \square \)
Theorem 5.5.2. Let $G$ be a planar graph of diameter 2 and with a $\gamma(G)$-set $\{a,b\} \subseteq V(G)$, $d(a,b) = 2$ and $|N(a) \cap N(b)| \leq 2$. Then $\gamma_{tr}(G) \leq 3$.

Proof. Let $\gamma(G) = 2$ and $d(a,b) = 2$, then $G$ has no pendant edge. Assume, to the contrary, let $G$ has a pendant edge. Then it must be at $t$, where $t$ lies on $a - t - b$ path, a contradiction by the stated $\gamma$-set. Let $A$, $B$ and $C$ be three sets such as $N(a) \cap N(b) = C$, $N(a) - C = A$ and $N(b) - C = B$. Since $d(a,b) = 2$, so $|C| \geq 1$.

We process the following cases:

Case 1: $|C| = 1$ and so $C = \{c\}$. If $c \sim (A \cup B)$, then contradicting with $\gamma(G) = 2$. Hence, there is a vertex from $A \cup B$ such that is not adjacent to $c$. Without loss of generality, we may assume that $x \in A$ and $x \not\sim c$. Let $B_1$ and $B_2$ be partitions of $B$, such that $B_1$ is the set of those vertices which has a neighbor in $A \cup B$ and $B_2$ is the set of those vertices that are adjacent to $c$ and has no neighbor in $A \cup B$.

Claim 1. $B_2 = \emptyset$.  

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Proof. Assume, to the contrary, that $B_2 \neq \emptyset$, and so $y \in B_2$. It is easy to check that $d(x, y) \geq 3$, a contradiction with $diam(G) = 2$. Hence, the desired result follows.

Thus, each vertex of $B$ must be adjacent to $A \cup B$. Now, we consider the following two cases:

(i): $\deg(z) = 2$ for some $z \in A$ such that $z \sim \{a, c\}$. It is easy to check that $c \sim B$. Assume, to the contrary, that there exists a vertex $b' \in B$ such that $c \sim b'$, then $d(z, b') \geq 3$, a contradiction with $diam(G) = 2$. Certainly, planarity of $G$ and $diam(G) = 2$ show that $|B| \leq 2$, otherwise, a contradiction with $diam(G) = 2$.

On the other side, all vertices of $A$, except those vertices such as $z$’s, are adjacent to $c$ or only a vertex of $B$, say $b_1$, otherwise, a contradiction with $diam(G) = 2$. It implies that $\{b_1, c, b\}$ is a TRDS of $G$, where $|B| = 1$, and $\{b_1, c\}$ is a TRDS of $G$, where $|B| = 2$. Hence the result follows.

(ii): $\deg(z) \geq 3$ for all $z \in A$. Hence, $\{a, b, c\}$ is a TRDS of $G$. Hence the result follows.
Case 2: If \(|C| = 2\), and let \(C = \{c_1, c_2\}\). We have the following easy claim.

**Claim 2.** Every vertex \(x \in A \cup B\) has a neighbor in \(A \cup B\).

Now, we continue to complete of the proof by the followings:

(i) If \(c_1 \sim y\) or \(c_2 \sim y\), where \(y \in A \cup B\). Without loss of generality, we may assume that \(c_1 \sim y\). Then by using Claim 2, \(\{a, c_2, b\}\) is a TRDS of \(G\).

(ii) If \(c_1 \sim y\) and \(c_2 \sim y\) for every \(y \in A \cup B\). Therefore, every vertex of \(A\) must be adjacent to a vertex of \(B\) and converse. Otherwise, if there exist two vertices \(a_1 \in A\) and \(b_1 \in B\) such that \(a_1 \sim b_1\), a contradiction with \(diam(G) = 2\) (Because, \(d(a_1, b_1) \geq 3\), \(d(a_1, b) \geq 3\) or \(d(a, b_1) \geq 3\)). Now, we consider the following claim.

**Claim 3.** If \(c_i \sim A \cup B\) for \(i = 1, 2\), then \(|A| \not\leq 4\) and \(|B| \not\leq 4\).

**Proof.** Assume, to the contrary, that \(|A| \geq 4\) and \(|B| \geq 4\). Since, \(c_i \sim A \cup B\), and by assumption \(diam(G) = 2\), it implies that, there exists a vertex \(x \in A \cup B\), say \(x \in A\), such that \(d(x, y) \geq 3\) for some \(y \in B\), otherwise a contradiction with \(diam(G) = 2\). Hence
the result follows.

Claim 3 and our assumptions imply that one of the following holds.

(ii-1) $|A| = 1$ and $|B| \geq 1$. Let $A = \{a_1\}$. The vertex $a_1$ must be adjacent to all vertices of $B$, otherwise, a contradiction with $diam(G) = 2$. It is easy to check that $\{a_1, a, c_1\}$ is a TRDS of $G$. Hence the result follows.

(ii-2) $|A| = 2$ and $|B| \geq 2$. We simply imply that there exists a vertex in $A$ or $B$, without loss of generality we may assume that $x \in A$, such that $x \sim A - \{x\}$. Otherwise, a contradiction with $diam(G) = 2$. Thus, there exists a vertex $y \in B$ with $x \sim y$ such that $\{x, b, y\}$ is a TRDS of $G$.

(ii-3) $|A| = 3$ and $|B| \geq 3$. An argument similar to that described in the proof of Case (ii-2) shows that the result holds. $\square$
The following shows that Theorem 5.5.2 is not true for $|C| = 3$.

Let $C = \{c_1, c_2, c_3\}$, $A = N(a) - C = \{a_1, a_2, a_3\}$, $B = N(b) - C = \{b_1, b_2, b_3\}$. Let $E(G) = \{ac_i, bc_i\ i = 1, 2, 3\} \cup \{aa_i, c_1a_i, c_1b_i, bb_i\ i = 1, 2, 3\}$ (see Fig. 8.). Then $G$ is a planner graph with $diam(G) = 2$, but $\gamma_{tr}(G) = 4$.

We conclude this chapter with the following corollary.

As an consecutive of the Theorems 5.5.1 and 5.5.2, we have the following corollary:

**Corollary 5.5.3.** Let $G$ be a planar graph of diameter 2 and with a $\gamma(G)$-set $\{a, b\} \subseteq V(G)$ and $|N(a) \cap N(b)| \leq t$. Then $\gamma_{tr}(G) \leq t + 2$.

This bound is sharp.
To sharpness of Corollary 5.5.3, we may construct the Fig. 9.

It is easy to check that \( \{a, b\}, \{a, c\} \) and \( \{b, c\} \) are \( \gamma \)-sets of the constructed graph. Further, \( d(a, b) = d(a, c) = 1 \) and \( d(b, c) = 2 \). Also, \( |N(a) \cap N(b)| = t_1 \), \( |N(a) \cap N(c)| = t_2 \) and \( |N(b) \cap N(c)| = t_3 + 1 \), where \( t_1 \leq t_2 \leq t_3 - 1 \) and \( t = \min\{t_1, t_2, t_3 - 1\} \).