Chapter 3

Similarity Methods for PDEs

In this chapter we present a brief summary of the similarity techniques that are one of the few general techniques for obtaining exact solutions of partial differential equations. Some of them are explained with the help of illustrative examples.

3.1 Introduction

Many problems of the physical phenomena are frequently formulated in terms of ordinary differential equations or partial differential equations. There are various techniques developed by scientists for obtaining exact solutions of a variety of linear and nonlinear models in physics and mechanics such as exact methods, numerical methods, approximate methods, similarity methods etc. But the most of them work only for very limited classes of problems. Many special cases have yielded to appropriate changes of variable. In fact, a similarity technique has yielded solution to non-linear partial differential equation which has been intractable to more standard solution technique. It is perhaps one of the most useful methods available for the treatment of such equations of mathematical physics and engineering boundary value problems. The most general criterion for a transformation is simply to convert given problem into a simpler problem in some sense or the other, that is, either transform the system under consideration to a one for which solution is known; or linearize it; or even transform it to algebraic equations.
3.2 Direct Methods

One of the best and the most applications of symmetry is similarity reduction of the partial differential equations. The symmetry methods are simple and broadly applicable, so they can play vital role in solving partial differential equations, especially nonlinear ones. In this thesis we restrict our attention to symmetries of partial differential equations. The key words are,

**Similarity transformations:** The transformations which reduce number of independent variables of a system of partial differential equations at least one less than that of the original equation are designated *Similarity transformations.*

**Similarity method:** The methods which search of such as transformations (Similarity transformations) is known as *similarity method.*

**Similarity equations:** The system of partial differential equations, $\phi = 0$ (say), in $n - 1$ independent variables which are obtained from transform a given partial differential equations , $\Phi = 0$ (say), in $n$ independent variables is known as *similarity equations or similarity representation* of the system of $\Phi$.

**Similarity solutions:** The solutions that have been obtained by employing similarity transformations are generally designated as *similarity solutions.*

Mathematically speaking, similarity solutions of partial differential equations appear when the number of independent variables can be reduced at least one less of original problems by transformation these variables.

In recent years, there are many methods developed by mathematicians and physicists in order to obtain the similarity transformations of reduction of a PDEs, those methods can be classified into two categories, namely *Direct methods* and *Group theoretic methods.*

### 3.2 Direct Methods

Direct methods such as dimensional analysis method, free parameter method by Gies (1955), separation of variables methods Abbott and Kline (1960) and general direct method Clarkson and Kruskal [38] don’t invoke group invariance. They are the simplest and most straightforward methods of determining similarity transformations. These methods are discussed below.
3.2.1 Dimensional Analysis Method

Dimensional analysis method is a method of reducing the number and complexity of variables required to describe a given physical situation by making use of the information implied by the units of the physical quantities involved. In other words, it is a technique for restructuring the original variables measured using dimensions of a problem into a set of dimensionless products (non-dimensional variables) using the constraints imposed upon them by their dimensions, such non-dimensional variables are fewer than the originals and may have a more appropriate interpretation.

These nondimensional quantities can be deduced directly from the governing differential equations if these equations are known or by performing a simple dimensional analysis on the variables involved if the governing differential equations are known or unknown. The Buckingham $\pi$-theorem provides a method for computing sets of dimensionless parameters from the given variables, even if the form of the equation is still unknown (see e.g. Fischer et al. [65]).

Consider a given physical situation that can be described by $n$ dimensional variables, and consider further that each of these dimensional variables are expressible in terms of certain fundamental dimensional quantities (mass $M$, length $L$, time $T$, temperature $\theta$, etc.) which be $m$ in number. The Buckingham $\pi$-theorem states that there are, then, $n - r$ independent non-dimensional variables (called $\pi$-variables) where $r$ is rank of the $m \times n$ matrix (called dimensional matrix) formed from the dimensional of the these $n$-governing variables.

3.2.2 Free Parameter Method

Free parameter method for determining similarity transformations are fairly straightforward and simple to apply. This method depends upon the introduction of some unknown transformation function(s) of dependent variable occurring in a particular partial differential equation as the product of functions suggested by a possible similarity. One function of this product is a function of all of the independent variables except one. The other function is supposed to depend on a single parameter, say $\eta$. 
where $\eta$ is a variable obtained from a transformation of variables involving independent variables, especially which not appearing in the first function. This method is used also for the problems with boundary conditions.

Let $u$ be the dependent variable of a particular partial differential equation and the independent variables are $x_1, x_2, ..., x_n, y$. We can express $u$ as:

$$u = \Phi(x_1, x_2, ..., x_n)F(\eta)$$

where

$$\eta = \eta(x_1, x_2, ..., x_n, y)$$

such that the boundary conditions on $u$ can be transformed to constant boundary conditions $F(\eta)$.

### 3.2.3 Similarity via Separation of Variables

A classical technique known as the method of separation of variables is perhaps one of the oldest systematic methods for solving partial differential equations which is used in the work of Daniel Bernoulli (1700-1782), Leonhard Euler (1707-1783), and Joseph Louis Lagrange (1736-1813) who attempted to solve the wave equation in the mid-1700’s by this method, also called the Bernoulli product method. It is commonly associated with the name of Joseph Fourier (1768-1830), who developed it for his research on conductive heat transfer. Birkhoff [25] was the first to suggest an alternative method for obtaining similarity solutions. Later on, it was named *Separation of Variables method*.

The explicit statement of the relationship between symmetry and separation of variables appeared for the first time in the year by Winternitz and Fris [152] and extensively developed by Abbott and Kline[7] in same year. In general, Separation of Variables method is quite similar to free parameter method. The similarity lies in invoking the similarity transformation at the outset of the analysis. The difference is that, in the separation of variables method, $\eta$ is assumed to be separable and initially specified while in the free parameter method we proceed without specifying what $\eta$ is.
3.2.4 CK’s Direct method by Clarkson and Kruskal

Clarkson and Kruskal (1989), developed a direct method for finding similarity reductions of partial differential equations, and the novel features of this method are entirely straightforward without group analysis. Clarkson et al [38, 39] proposed also open problems to develop the direct method for seeking symmetry reductions of nonlinear PDEs with arbitrary functions.

The basic idea is to seek a solution of a given PDEs in the form

\[ u^j = W_j(x_1, ..., x_n, F_j(\eta_1, ..., \eta_{n-r})), \]

\[ ((\eta_i = \eta_i(x_1, ..., x_n), \ i = 1, 2, ..., n-r \text{ and } j = 1, 2, ..., m) \quad (3.1) \]

which is the most general form for a similarity transformation. It is interesting that for most equations, it is enough to seek the symmetry reductions in a simple form

\[ u^j(x_1, x_2, ..., x_n) = \alpha_j(x_1, x_2, ..., x_n) + \beta_j(x_1, x_2, ..., x_n)F_j(\eta_1, \eta_2, ..., \eta_q), \ q = n - r \quad (3.2) \]

instead of the general form (3.1). In this method, we need only substituting (3.1) into a given partial differential equation and demanding that the result be another differential equation with \( n - r \) independent variables, \( \eta_q(q = 1, 2, ..., n - r) \), and \( F' \)s are the new dependent variables. This done by imposing conditions on \( F' \)s, \( \alpha' \)s, \( \beta' \)s, \( \eta' \)s and their derivatives enable one to solve for \( F' \)s and \( \eta' \)s.

Most recently, CK’s direct method of finding similarity reductions of partial differential equations has been modified by Lou and Ma [81, 83], through the so-called general direct method. In the general direct method, both the Lie point symmetry groups and the non-Lie symmetry groups can be obtained for some PDEs.

Nevertheless, it is clear that the direct method introduces general form for a similarity transformation without specifying what \( \eta' \)s are. This is leading to algebraic manipulations and tedious calculations especially in case a huge and highly nonlinear systems of partial differential equations which represent deficiency of the method.
3.3  Group Theoretic Methods

The group theoretic methods are the similarity techniques which seek to investigate the group of transformations which leave a given problem invariant, these groups enable us to derive a proper transformations (similarity transformations) which reduce a given differential equations. In general, group theoretic methods are further classified into two parts:

(i) Methods which are searching for *finite groups of transformations* which leave a given partial differential equations invariant such as:


(ii) Methods which are searching for *infinitesimal groups of transformations* which leave a given partial differential equations invariant such as:


3.3.1  Birkhoff and Morgan’s Method

This method was the first application of Lie’s theories of continuous transformation group of symmetry analysis. In this method, special form of simple group is initially assumed, and then invariance of the partial differential equation without auxiliary conditions is invoked. Next step is determining the absolute invariants by inspection and/or trial and exploiting the conditions under which the given problem is invariant. Finally, we check whether the auxiliary conditions are appropriate with the absolute invariants (which refer here to the similarity transformations) and then transforming the problem.
Example 3.3.1 Consider the one-parameter linear group

\begin{align*}
  t &= A^{\alpha_1} \bar{t} \quad (3.3a) \\
  x &= A^{\alpha_2} \bar{x} \quad (3.3b) \\
  u &= A^{\alpha_3} \bar{u} \quad (3.3c)
\end{align*}

where \( A \) is a numerical parameter and \( \alpha_1, \alpha_2, \alpha_3 \) are real numbers and the cylindrical Korteweg-deVries equation

\[ u_t + uu_x + u_{xxx} + \frac{u}{2t} = 0 \quad (3.4) \]

which is said to describe the cylindrical solitons. The first step in Morgan’s Method is to establish the invariant analysis of given equation only (without auxiliary conditions in case boundary value problems). To perform an invariance of equation (3.4) under the transformations (3.3a-3.3c). It is required to calculate the extended groups up to the third order. Namely, \( u_t, u_x \) and \( u_{xxx} \).

Make use of definition 2.3.2 to carry out the transformation on the differential equation (3.4) and show whether or not they are invariant under this transformations. This is, by substitute of (3.3a-3.3c) into (3.4), we got

\[ u_t + uu_x + u_{xxx} + \frac{u}{2t} = A^{\alpha_3-\alpha_1} \bar{u}_t + A^{2\alpha_3-\alpha_2} \bar{u}_x + A^{3\alpha_3-3\alpha_2} \bar{u}_{xxx} + A^{\alpha_3-\alpha_1} \bar{u} \quad (3.5) \]

which is absolute invariant under (3.3a-3.3c) if

\[ \alpha_3 - \alpha_1 = 2\alpha_3 - \alpha_2 = \alpha_3 - 3\alpha_2 \quad (3.6) \]

From which we have

\[ \frac{\alpha_2}{\alpha_1} = \frac{1}{3}, \quad \frac{\alpha_3}{\alpha_1} = -\frac{2}{3}. \quad (3.7) \]

The next step in this method is to find the so-called “absolute invariants” under this group of transformations by inspection and/or trial. By eliminating the parameter \( A \) from (3.3a-3.3c), and make use (3.7) we obtain,

\[ \eta = \frac{x}{t^{\frac{1}{3}}}, \quad \xi = \frac{u}{t^{\frac{2}{3}}}. \quad (3.8) \]
as two absolute invariants of $(3.3a-3.3c)$.

By Morgan’s theorem [2.5.1], the partial differential equation $(3.4)$ can be reduced to ordinary differential equation, with the invariant $\eta$ of the subgroup $(3.3a,3.3b)$ as independent variable and $F(\eta)$ as dependent variable, where $F(\eta)$ formed by the transformation of the second independent absolute $\xi$, i.e., $F(\eta) = \xi$. In light of this, the transformations take the form:

$$\eta = \frac{x}{t^3}, \quad u = t^{-\frac{5}{2}}F(\eta).$$  \hspace{1cm} (3.9)

the cylindrical KortewegdeVries equation $(3.4)$ become

$$F''' + (F - \frac{1}{3}\eta)F'' - \frac{1}{6}F = 0.$$ \hspace{1cm} (3.10)

In addition to that assuming a simple group of transformation at the outset of analysis limits the generality of the results. The other drawbacks of this method represented in using inspection and/or trial in finding the absolutes invariants and in the case the boundary value problems, one needs to check whether the auxiliary conditions are appropriate to the transformations or not, if not, there is no similarity solution; and this leads to a wastage of efforts.

For instance in Timol and Timol [146], the resulting similarity transformations by $\Gamma_1$ are disproportionately with the boundary conditions for the case (II) and also the resulting similarity transformations by $\Gamma_2$ are disproportionately with the boundary conditions for the case (I).

### 3.3.2 Moran and Gaggioli Method (Deductive Similarity Method)

In 1968 Moran and Gaggioli [91] presented a systematic group formulation for similarity analysis. A formulation is nothing but modifications of Moran’s methods with a very general class of transformation groups of the form

$$G_\tau = \begin{cases} S_\tau : \{ \bar{x}_i = f_i(x_1, ..., x_n, a_1, ..., a_\tau) \ i = 1, ..., n, \ n \geq 2 \\ \bar{u}^j = g^j(u^1, ..., u^n, a_1, ..., a_\tau) \ j = 1, ..., m, \ m \geq 1 \end{cases}.$$ \hspace{1cm} (3.11)
These modifications concentrate in performing the invariant analysis of partial differential equations as well as auxiliary conditions and determining the absolute invariants by a systematic procedure.

In the same year, Morgan et al. [94] modified their method and proposed a group in the form

\[
S_r : \begin{cases} 
\bar{x}_i &= C^r_i(a_1, a_2, \ldots, a_r)x_i + K^r_i(a_1, a_2, \ldots, a_r) \\
\bar{u}^j &= C^r_u(a_1, a_2, \ldots, a_r)u^j + K^r_u(a_1, a_2, \ldots, a_r)
\end{cases} \quad (3.12)
\]

The method is illustrated in the following example

**Example 3.3.2** Consider the one-parameter group of transformations \( G_1 \) in the form

\[
S_1 : \begin{cases} 
\bar{t} &= C^i(t) + K^i(t) \\
\bar{x} &= C^x(x) + K^x(x) \\
\bar{u} &= C^u(a)u + K^u(a)
\end{cases} \quad (3.13)
\]

where \( a \) a numerical parameter and the porous media equation

\[
u_t + (u^n)_{xx} = 0 \quad (3.14)
\]

with

\[
u(x, t) = 0, \quad \text{at } t = 0, \quad (3.15a)
\]

\[
u(x, t) = U, \quad \text{at } x = 0 \quad (3.15b)
\]

\[
u(x, t) = 0 \quad \text{as } x \to \infty, 0 < t \quad (3.15c)
\]

By definition, it is straightforward to show that (3.14) and (3.15a)-(3.15c) are invariant under the one-parameter group (3.13).

Make use of definition 2.5.1, we get

\[
u_t - (u^n)_{xx} = \bar{\nu}_t - n(n-1)\bar{u}^{n-2}\bar{u}_x^2 - n\bar{u}^{n-1}\bar{u}_{xx} = C^u_n \frac{C^u}{C^n} (n(n-1)u^{(n-2)}u_x^2 - nu_{xx}) + R \quad (3.16)
\]

where \( R = (n-2)(C^u_n)^{n-3}K^u + \ldots + (K^u)^{n-2} + (n-1)(C^u)_{n-2}K^u + \ldots + (K^u)^{n-1} \).

The conformal invariance of (3.14) implies

\[
R = 0 \quad \text{and} \quad \frac{C^u}{C^n} = \frac{(C^u)^n}{(C^n)^2} = H(a) \quad (3.17)
\]
Further, the vanishing of $R$ and invariant of boundary conditions (3.15a)-(3.15c), implying that

$$K'' = K^x = K^t = 0, \quad C'' = C''_{(0)} = 1 \quad (3.18)$$

In view of (3.18), and invoking the results (3.17), we get

$$C^t = (C^x)^2 \quad (3.19)$$

By substituting from (3.18) and (3.19) into (3.13), we get class $G'$ of one-parameter group in the form

$$G_1 = \begin{cases} 
S_1 : & \bar{t} = (C^x)^2 (a) t \\
& \bar{x} = C^x (a) x \\
& \bar{u} = u
\end{cases} \quad (3.20)$$

under which the system of partial differential equations (3.14) and auxiliary conditions (3.15a)-(3.15c) are transformed invariantly. The second step in this method is to find the absolute invariants of group (3.20), via systematic technique depend on invoked a basic theorem from group theory as following, owing to Theorem 2.3.1, $\eta(x,t)$ is an absolute invariant of subgroup $S$ if it satisfies

$$2\alpha_2 t \frac{\partial \eta}{\partial t} + \alpha_2 x \frac{\partial \eta}{\partial x} = 0 \quad (3.21)$$

which has a solution in the form $\eta = \frac{x}{t^{1/2}}$. By a similar analysis the absolute invariant of the dependent variable $u$ is $\xi = u$.

Therefore, using the similarity transformation

$$\eta = \frac{x}{t^{1/2}}, \quad u = F(\eta) \quad (3.22)$$

(3.14) becomes

$$\eta F + n(n - 1) F^{n-2} (F')^2 + n F^{n-1} F'' = 0 \quad (3.23)$$

with

$$F(0) = U, \quad F(\infty) = 0.$$
3.3.3 Hellums-Churchill Procedure

Hellums and Churchill [54] were introduced new procedure in the similarity analysis later renamed Hellums-Churchill Procedure. This method is essentially similar to the Birkhoff and Morgan’s Method, albeit it uses a simple affine group of transformations. A mass, length and time arise as fundamental dimensional in the procedure, this is imply that the method is very suitable for physical problems. It is showing results unobtainable by using traditional dimensional analysis method.

A major drawback of this method and the two previous methods is that a simple group of transformation is assumed at the outset of analysis which limits the generality of the results. (for details see [54] and [130]).

3.3.4 Lie Classical Method

Lie classical method was first introduced by Lie [80]. This method starts out with a general infinitesimal group of transformations. By invocation of invariance under the infinitesimal group, determining equations are derived. Solving such equations gives the infinitesimals and therefore the deduced infinitesimal group. We assume again a system \( E \) of PDEs (2.2). The classical method only makes use of equations \( E_\mu u^i(x) = 0 \) and thus involves setting \( \tilde{E}_\mu u^i(x) \) proportional to \( E_\mu u^i(x) \). This provides a set of conditions on \( \xi_i, \eta^j \) without the use of the invariant surface equation. Thus, determining those infinitesimals \( \xi_i, \eta^j \) enable us to derive similarity transformations.

3.3.5 Non-Classical Method

The nonclassical procedure was introduced by Bluman and Cole [29] as a generalization of the classical method which was presented above. This method is based on invocation of invariance under the infinitesimal group and makes use of the invariant surface condition (2.15), determining equations are derived. Solving such determining equations give the infinitesimals and therefore the deduced infinitesimal groups. i.e., the basic idea of the method is to require that both the given system of PDEs and the
surface condition must be invariant under the infinitesimal generator. This method has been generalized by Olver and Rosenau \cite{107, 108}.

### 3.3.6 Characteristic Function Method

The characteristic function method of Na and Hansen \cite{100, 99} is convenient and more systematic than the classical and nonclassical methods. It is also based on invocation the invariance of infinitesimal group but it express the infinitesimals of the group in terms of characteristic functions, $W's$, (2.16). Therefore the procedure for finding the infinitesimals reduces to the determination of the characteristic functions. More details are to be found in Seshadri and Na \cite{130} and Na and Hansen \cite{100}.

The last three methods exploit group invariance under the infinitesimal groups of transformations contain systematic algebraic manipulations but lengthy and tedious. So the tremendous amount of work necessary to derived a solution to a given differential equations. This work increases more if partial differential equations are huge and highly non-linear. This is the main deficiency of such methods.