5. COMPARITIVE STUDY OF THE COMPUTATIONS OF THE EIGENVALUES OF A MATRIX BETWEEN THE EXISTING METHODS AND GERSCHGORIN CIRCLES METHOD.

5.1 Introduction

In the existing literature, several methods are available to compute the eigenvalues of the given matrix [1]. In some methods eigenvalues are computed for the symmetric matrix. The symmetric matrix is transformed to tri diagonal matrix using plane rotations. The rotations have to be applied for several numbers of iterations until all the lower and upper triangular elements become zero. In power method, it computes only the largest and smallest eigenvalues. Hence no methods directly computes all the eigenvalues of the matrix.

In this chapter, an attempt is made to apply the Gerschgorin circles method, to compute the eigenvalues of the matrix and are compared, the results with the existing methods. This simple graphical approach which does not require any transformation of the given matrix to any form and it is applicable for any given arbitrary matrix. Even though there are a few computational burdens in the Gerschgorin circles method, it computes the eigenvalues very accurately and also it helps in the identifications of the eigenvalues without computations which are discussed in the next chapters.

5.2 Computation of eigenvalues using existing methods and Gerschgorin circles

5.2.1 Bounds for eigenvalues of tri diagonal matrix using Strum sequences and Gerschgorin circles.

The concept of stability plays very important role in the analysis of systems. A system matrix can be used [1] for the stability studies. In the existing literature, there exists Given’s method for the computation of bounds of the eigenvalues and it takes lots of computation [56]. It has been found that by using Gerschgorin technique the bounds of the eigenvalues can be obtained graphically, which takes no computations.

In the following the given’s method and Gerschgorin method have been explained to compute the intervals under which the roots of the given matrix $A$ lie and both the methods are compared.
Given technique

In [56], the following tridiagonal matrix is considered to compute range of intervals for the eigenvalues.

\[
A = \begin{pmatrix}
4.000 & 3.000 & 0.000 & 0.000 \\
3.000 & 3.334 & 1.666 & 0.000 \\
0.000 & 1.666 & -1.220 & 0.907 \\
0.000 & 0.000 & 0.907 & 1.907
\end{pmatrix}
\]

The following procedure has been adopted in the given method

Firstly, the strum sequences has been computes as given below

\[P_0 (\lambda) = 1\]
\[P_1 (\lambda) = 4 - \lambda\]
\[P_2 (\lambda) = (3.334 - \lambda)P_1 (\lambda) - 9\]
\[P_3 (\lambda) = -(1.32 + \lambda)P_2 (\lambda) - 2.776 P_1 (\lambda)\]
\[P_4 (\lambda) = (1.987 - \lambda)P_3 (\lambda) - 0.823 P_2 (\lambda)\]

Table 5.1: Bound obtained by Strum sequences in the existing method

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>(P_0)</th>
<th>(P_1)</th>
<th>(P_2)</th>
<th>(P_3)</th>
<th>(P_4)</th>
<th>(V(\lambda))</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>+1</td>
<td>7</td>
<td>35.338</td>
<td>39.396</td>
<td>170.077</td>
<td>0</td>
</tr>
<tr>
<td>-2</td>
<td>+1</td>
<td>6</td>
<td>23.004</td>
<td>-1.0133</td>
<td>-18.945</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>+1</td>
<td>3</td>
<td>-1.998</td>
<td>-3.693</td>
<td>-2.0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>+1</td>
<td>2</td>
<td>-6.337</td>
<td>15.47</td>
<td>5.01</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>+1</td>
<td>1</td>
<td>-8.667</td>
<td>34.661</td>
<td>-27.98</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>+1</td>
<td>-2</td>
<td>-3.668</td>
<td>32.402</td>
<td>-127.01</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>+1</td>
<td>-3</td>
<td>1.989</td>
<td>-8.22</td>
<td>39.57</td>
<td>4</td>
</tr>
</tbody>
</table>
Then the range under which the roots lie are calculated as follows

\[ V(-2) - V(-3) = 1 \]  one root lies between \(-3 < \lambda < -2\)

Similarly other root ranges are \(1 \leq \lambda_2 \leq 2, 2 \leq \lambda_3 \leq 3\) and \(6 \leq \lambda_4 \leq 7\)

We now obtain \(V(-2.5)\) as 1 and comparing with \(V(-3)\) and \(V(-2)\) it is concluded that \(-3 \leq \lambda_1 \leq -2.5\)

This procedure can be continued. The range \(-3 \leq \lambda_1 \leq -2.5\) can be further bisected and a lighter bound for \(\lambda_1\) can be determined. Continued application of the bisection technique will keep reducing in the interval in which \(\lambda_1\) lies until a desired accuracy of \(\lambda_1\) is attained. However in this given method the calculation of the strum sequences need a lot of computation. Hence Gerschgorin technique is used to do the same and it is described as follows.

**Gerschgorin circle technique**

Consider the same tridiagonal matrix (5.2.1) and the Gerschgorin circle of the above matrix is drawn as shown in figure 5.1

![Gerschgorin bound [-3.9, 8]](image-url)

**Fig 5.1: Gerschgorin bound [-3.9, 8]**
From the Gerschgorin circle diagram it is easy to decide the bound and the interval in which the eigenvalues lie. The Gerschgorin bound is \([3.9, 8]\). Intervals are as follows.

\[-3.9 < \lambda < -1.4, -1.4 < \lambda < 0, 0 < \lambda < 0.9, 0.9 < \lambda < 1, 1.1 < \lambda < 3, 3 < \lambda < 3.9, 3.9 < \lambda < 7, 7 < \lambda < 8\]

**Conclusion:** In the given technique we observe that Strum sequences has to be computed and using this sequences the bound for which the eigenvalues are found.

In the Gerschgorin circles technique, bound and also the interval are computed graphically without any computations.

### 5.2.2 Eigenvalues of the tridiagonal matrix by Givens method and Gerschgorin circles method.

The eigenvalues of the tridiagonal matrix have been computed by the Givens method in [1]. For this method, Gerschgorin circles method has been applied.

**Existing method:** [1]

Consider the tridiagonal matrix of order \((3 \times 3)\)

\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\rightarrow (5.2.2)
\]

The strum sequence is

\[
f_0 = 1, \quad f_1 = \lambda - 2, \\
f_2 = (\lambda - 2) f_1 - f_0 = (\lambda - 2)^2 - 1 \\
f_3 = (\lambda - 2) f_2 - f_1 = (\lambda - 2)^3 - 2(\lambda - 2)
\]

We have
Table 5.2: Bound for the existence of eigenvalues in given’s method using Strum sequences

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$V(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0</td>
</tr>
</tbody>
</table>

$f_3(2) = 0$, so that $\lambda = 2$ is an eigenvalue. There is an eigenvalue in the intervals (0, 1) and (3, 4). We now find the better estimate of the eigenvalues by repeated Bisection methods.

We have

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$f_0$</th>
<th>$f_1$</th>
<th>$f_2$</th>
<th>$f_3$</th>
<th>$V(\lambda)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>0.75</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>0.625</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>2</td>
</tr>
<tr>
<td>0.5625</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>3</td>
</tr>
<tr>
<td>0.59375</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>3</td>
</tr>
</tbody>
</table>

The eigenvalue is located in (0.5, 1). Again

We repeat this procedure until the required accuracy is obtained. The exact value of this eigenvalue is $2 - \sqrt{2} = 0.585786$. 120
Total computation for existing Bisection method: 855

Proposed method:

The Gerschgorin circle of the matrix (5.2.2) is drawn and is shown in figure 5.2

![Gerschgorin circle](image)

Fig 5.2 : Gerschgorin bound [0, 4]

Eigenvalues have been computed by applying Bisection method, false position method, Secant method in the Gerschgorin bound

Table 5.3: Results showing the comparisons of total computations obtained by Bisection, false and Secant method.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Eigen values</th>
<th>Total computations</th>
</tr>
</thead>
</table>
| Bisection method    | $\lambda_1 = 0.585786$  
                      | $\lambda_2 = 2$  
                      | $\lambda_3 = 3.144214$  | 855                       |
| False position method | $\lambda_1 = 0.5857$  
                           | $\lambda_2 = 3$  
                           | $\lambda_3 = 4$  | 522                       |
| Secant method       | $\lambda_1 = 0.584961$  
                      | $\lambda_2 = 2$  
                      | $\lambda_3 = 3.145039$  | 225                       |
Conclusions: It is observed that in the existing given method uses the Strum sequence to compute the bound in which the eigenvalues lie which requires computations. It also computes the eigenvalues by applying Bisection method at each bound obtained by strum sequences.

In the proposed method the Gerschgorin bound are obtained graphically which requires no computations and also the eigenvalues have been computed using the Bisection method, false position method and Secant method applied at the Gerschgorin bound. In each case eigenvalues have been computed and the values are compared. The secant method computes the eigenvalues accurately and it takes few computations compared with the other methods and also with the existing method.

5.2.3 Eigenvalues of a system matrix using Jacobi method and Gerschgorin circles

The eigenvalues of the symmetric matrix have been computed by the Jacobi method in [1]. For this method, Gerschgorin circles method has been applied.

Existing method: [1]

Consider the symmetric matrix of order (3x3)

\[
A = \begin{pmatrix}
1 & \sqrt{2} & 2 \\
\sqrt{2} & 1 & \sqrt{2} \\
2 & \sqrt{2} & 1 \\
\end{pmatrix}
\rightarrow (5.2.3)
\]

(3x3)

The largest half diagonal elements is \(a_{12} = 2\) \(a_{31} = 2\). The other two elements in the (2x2) sub matrix is 1.

\[
\theta = \frac{\pi}{2} \tan^{-1} \frac{4}{0} = \frac{\pi}{4}
\]

\[
S_1 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\
0 & 1 & 0 \\
\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\
\end{pmatrix}
\]

The first rotation gives
The largest half diagonal element in magnitude in $B_1$ is $a_{12} = 2$. The other elements are 3.

$$\theta = \frac{\pi}{4} \tan^{-1} \frac{1}{4} = \frac{\pi}{4}$$

$$S_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$S_2$ is the matrix of eigenvectors $S = S_1 S_2$.
The eigenvalues of the matrix are 5, 1,-1 which are on the leading B and and the corresponding eigenvectors are S.

**Proposed method:**

Gershgorin circle of the matrix (5.2.3) and is drawn as shown in figure 5.3

![Gershgorin Circle](image)

**Fig 5.3: Gershgorin bound [-2, 5]**

Eigenvalues of the above matrix are obtained by applying $h = \frac{\text{Avg}}{ab} = 2.3$ at the Gershgorin bound. Here \( \text{Avg} \) = average of all the elements in the matrix, \( ab \) = length of the Gershgorin bound.

\[
\begin{align*}
\lambda_1 &= -1.00223 \\
\lambda_2 &= 1.387613 \\
\lambda_3 &= 4.613004
\end{align*}
\]

Total Computation: 417

Total iteration: 139
Conclusion: We observe that in the existing method to compute the eigenvalues $\theta$ is computed and in the first rotation the matrix $B_1$ is computed since the upper and lower elements are not zero again matrix $B_2$ is computed in the second rotation and the process is continued until the elements in the upper and lower elements are not zero. The elements in the principle diagonal are the eigenvalues of the given matrix.

In the proposed method, eigenvalues are computed by taking $h = 2.3$ at the Gerschgorin bound, though the computations are more the eigenvalues are obtained which does not require much steps.

5.2.4 Rutihauer method and Gerschgorin circles method to compute eigenvalues of arbitrary matrices.

The eigenvalues of the arbitrary matrix have been computed by the Rutihauer method in [1]. For this method, Gerschgorin circles method has been applied.

Existing method [1]:

Consider the symmetric matrix of order (3x3)

$$A = \begin{pmatrix}
1 & 1 & 1 \\
2 & 1 & 2 \\
1 & 3 & 2 \\
\end{pmatrix} \rightarrow (5.2.4)$$

Using the Rutishauer method. Iterate till the elements in the lower triangular part is less than 0.05 in magnitude.

We have

$$A_1 = A = \begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
1 & -2 & 1 \\
\end{pmatrix}$$

$$A_1 = U_1L_1 = \begin{pmatrix}
1 & 1 & 1 \\
0 & -1 & 0 \\
0 & 0 & 1 \\
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
2 & 1 & 0 \\
1 & -2 & 1 \\
\end{pmatrix} = \begin{pmatrix}
4 & -1 & 1 \\
-2 & -1 & 0 \\
1 & -2 & 1 \\
\end{pmatrix}$$
\[ A_1 = U_2 L_2 = \begin{pmatrix}
1 & 0 & 0 \\
-\frac{1}{2} & 1 & 0 \\
\frac{1}{4} & \frac{7}{8} & 1
\end{pmatrix}
\begin{pmatrix}
4 & -1 & 1 \\
0 & -\frac{3}{2} & \frac{1}{2} \\
0 & 0 & \frac{1}{6}
\end{pmatrix} = L_2 U_2 \]

\[ A_4 = U_3 L_3 = \begin{pmatrix}
4.789474 & -0.037037 & 1 \\
-0.171745 & -1.011606 & 0.315789 \\
0.001949 & -0.045267 & 0.222222
\end{pmatrix} \]

\[ A_5 = U_4 L_4 = \begin{pmatrix}
4.791209 & 0.007633 & 1 \\
0.036469 & -0.997316 & 0.351648 \\
0.000084 & 0.009207 & 0.206107
\end{pmatrix} \]
\[
\begin{pmatrix}
1 & 0 & 0 \\
0.007623 & 1 & 0 \\
.000018 & -0.009231 & 1
\end{pmatrix}
\begin{pmatrix}
4.791209 & 0.007633 & 1 \\
0 & -0.997374 & 0.344036 \\
0 & 0 & 0.208265
\end{pmatrix} = L_5 \; U_5
\]

\[
A_6 = U_5 \; L_5 =
\begin{pmatrix}
4.791209 & 0.007633 & 1 \\
0.036469 & -0.997316 & 0.351648 \\
0.000084 & 0.009207 & 0.206107
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 0 & 0 \\
-0.001583 & 1 & 0 \\
.000001 & 0.001931 & 1
\end{pmatrix}
\begin{pmatrix}
4.791265 & -0.001598 & 1 \\
0 & -0.1000553 & 0.345619 \\
0 & 0 & 0.208597
\end{pmatrix} = L_6 \; U_8
\]

\[
A_7 = U_6 \; L_6 =
\begin{pmatrix}
4.791289 & 0.000333 & 1 \\
0.001584 & -0.999886 & 0.345619 \\
0.000084 & 0.000403 & 0.208597
\end{pmatrix}
\]

Hence the eigenvalues of \( A \) are approximately 4.791289, 0.999886 and 0.208579

Exact eigenvalues are

\(
\lambda = 4.791288 \\
\lambda = -1 \\
\lambda = 0.208712.
\)
Proposed method:

Gerschgorin circle is drawn for the matrix (5.2.4) and is as shown in figure 5.4

![Gerschgorin bound](image)

Fig 5.4: Gerschgorin bound [-3, 5]

Eigenvalues of the above by applying step length $h = 1.0$ and Bisection method are

$$
\lambda_1 = 4.791992 \\
\lambda_2 = -1 \\
\lambda_3 = 0.28008
$$

Total computation: 339

Conclusions: We can observe that the existing method converts the given matrix as the product of lower and upper triangular for seven times which is a tedious process to compute the eigenvalues and the values are closer to the exact value.

In the proposed method the computations of eigenvalues is very simple and the Bisection technique is applied at the Gerschgorin bounds with the step length one. The eigenvalues compared with the exact values are very closer to the exact and one of the eigenvalues is exactly same. Hence Gerschgorin technique is simpler and calculates the eigenvalues accurately compared to the existing method.
5.2.5 Householder QL-Method and Gerschgorin circles method to compute the eigenvalues of the tridiagonal matrices.

The eigenvalues of the tridiagonal matrix have been computed by the Householder method in [54]. For this method, Gerschgorin circles method has been applied.

Existing method: [54]

Consider the tridiagonal matrix of order (3x3)

\[ T_1 = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix} \] \rightarrow (5.2.5)

The above matrix is obtained by applying Householder method to some symmetric matrix.

The first step is to evaluate \( L_1 \)

\[ L_1 = S_2 \ S_3 \ T_1 \] Note \( n = 3 \), two matrices since \( k = 2,3,\ldots,n \)

First evaluate \( S_2 \ T_1 = T'_1 \)

For \( k = 3 \)

\[ S_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_2 & -\sin \theta_2 \\ 0 & \sin \theta_2 & \cos \theta_3 \end{pmatrix} \]

Where \( a_3 \) and \( b_3 \) are taken from \( T_1 \)

\[ \cos \theta_2 = \cos \theta_3 = \frac{a_3}{\sqrt{b_3^2 + a_3^2}} = \frac{2}{\sqrt{(1)^2 + (2)^2}} = 0.894 \]

\[ \sin \theta_3 = \frac{a_3}{\sqrt{b_3^2 + a_3^2}} = 0.447 \]
Hence
\[
S_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.894 & -0.447 \\
0 & -0.447 & 0.894 \\
\end{pmatrix}
\begin{pmatrix}
4 & 2 & 0 \\
2 & 3 & 1 \\
0 & 1 & 2 \\
\end{pmatrix}
\]

Notice that the elements above the \(n^{th}\) diagonal element or \((n-1, n) - th\) element are reduced to zero. The \((n, n-2)\) element have non zero.

Next step is to multiply \(S_2 T'_1 = T''_1 = L_1\)

For \(k = 2\)

\[
S_2 = \begin{pmatrix}
\cos \theta_2 & - \sin \theta_2 & 0 \\
\sin \theta_2 & \cos \theta_2 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
\cos \theta_1 = \cos \theta_2 = \frac{a_3}{\sqrt{b_3^2 + a_3^2}} = \frac{0.745}{\sqrt{(2.236)^2 + (2.236)^2}} = 0.745
\]

\[
\sin \theta_2 = \frac{b_3}{\sqrt{b_3^2 + a_3^2}} = \frac{2}{\sqrt{(2.236)^2 + (2.236)^2}} = 0.745
\]

Hence
\[
S_2 = \begin{pmatrix}
0.745 & -0.667 & 0 \\
0.667 & 0.745 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\]

\[
L_1 = S_2 T'_1 = T''_1
\]

\[
\begin{pmatrix}
0.745 & -0.667 & 0 \\
1.788 & 2.235 & 0 \\
0 & 0 & 1 \\
\end{pmatrix}
\begin{pmatrix}
4 & 2 & 0 \\
0.667 & 0.745 & 0 \\
0.894 & 2.235 & 2.235 \\
\end{pmatrix}
\]

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\[ L_1 = \begin{pmatrix} 1.788 & 0 & 0 \\ 4.0 & 3.0 & 0 \\ 0.893 & 2.235 & 2.235 \end{pmatrix} \]

\[ Q_1 = S_3^T * S_2^T \]
\[ Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.894 & -0.447 \\ 0 & -0.447 & 0.894 \end{pmatrix} \begin{pmatrix} 0.745 & -6.67 & 0 \\ 0.667 & 0.745 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ \begin{pmatrix} 0.745 & 0.667 & 0 \\ -0.596 & 0.666 & 0.447 \\ 0.298 & -0.333 & 0.894 \end{pmatrix} \]

Now a check is performed to verify \( T_1 = Q_1 L_1 \)
\[ T_1 = \begin{pmatrix} 0.745 & 0.667 & 0 \\ -0.596 & 0.666 & 0.447 \\ 0.298 & -0.333 & 0.894 \end{pmatrix} \begin{pmatrix} 1.788 & 0 & 0 \\ 4.0 & 3.0 & 0 \\ 0.894 & 2.235 & 2.235 \end{pmatrix} \]

\[ \begin{pmatrix} 4.0 & 2.0 & 0 \\ 2.0 & 3.0 & 1.0 \\ 0 & 1.0 & 2.0 \end{pmatrix} \]
Now the first transformation can be performed.

\[ T_2 = L_1 \, Q_1 \]

\[
T_2 = \begin{pmatrix}
1.788 & 0 & 0 \\
4.0 & 3.0 & 0 \\
0.894 & 2.235 & 2.235 \\
\end{pmatrix}
\begin{pmatrix}
0.745 & 0.667 & 0 \\
-0.596 & 0.666 & 0.447 \\
0.298 & -0.333 & 0.894 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
1.33 & 1.19 & 0 \\
1.19 & 4.667 & 1.341 \\
0 & 1.341 & 3.0 \\
\end{pmatrix}
\]

Now that the off diagonal elements of \( T_2 \) are reduced compared to those in \( T_1 \)

\( T_2 \) is now factorized into \( L_2 \) and \( Q_2 \) in the following way

\[ L_2 = S_2 \ast S_1 \ast T_2 \]

Evaluate \( S_3 \, T_2 = T_2' \) first

For \( k = 3 \)

\[
S_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_2 & -\sin \theta_2 \\
0 & \sin \theta_2 & \cos \theta_2 \\
\end{pmatrix}
\]

Where \( a_3 \) and \( b_3 \) are taken from \( T_2 \)

\[
\cos \theta_2 = \cos \theta_3 = \frac{a_3}{\sqrt{b_3^2 + a_3^2}} = \frac{3.0}{\sqrt{(1.341)^2 + (3)^2}} = 0.913
\]

\[
\sin \theta_3 = \frac{b_3}{\sqrt{b_3^2 + a_3^2}} = \frac{1.341}{\sqrt{(1.341)^2 + (3)^2}} = 0.408
\]

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Hence

\[
S_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & -0.913 & -0.408 \\
0 & 0.408 & 0.913
\end{pmatrix}
\]

\[
S_3 T_2 = T_2'
\]

\[
= \begin{pmatrix}
1 & 0 & 0 \\
0 & -0.913 & -0.408 \\
0 & 0.408 & 0.913
\end{pmatrix}
\begin{pmatrix}
1.33 & 1.19 & 0 \\
1.19 & 4.67 & 1.34 \\
0 & 1.34 & 3.0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1.33 & 1.19 & 0 \\
1.090 & 3.72 & 0 \\
0.490 & 3.13 & 3.29
\end{pmatrix}
\]

Notice that the element in \( T_2' \) reduce to zero above the \( k^{th} \) diagonal element.

The next step is to multiply \( S_2 T_2' = T_2'' = L_2 \)

For \( k = 2 \)

\[
S_2 = \begin{pmatrix}
\cos \theta_1 & -\sin \theta_2 & 0 \\
\sin \theta_2 & \cos \theta_2 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Where \( a_2 \) and \( b_2 \) are taken from \( T_2' \)

\[
\cos \theta_1 = \cos \theta_2 = \frac{a_3}{\sqrt{b_3^2 + a_3^2}} = \frac{3.72}{\sqrt{(1.19)^2 + (3.72)^2}} = 0.972
\]
\[
\sin \theta_2 = \frac{b_3}{\sqrt{b_3^2 + a_3^2}} = \frac{1.19}{\sqrt{(1.19)^2 + (3.72)^2}} = 0.305
\]

\[
S_2 = \begin{pmatrix}
0.952 & -0.305 & 0 \\
0.305 & 0.952 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
L_2 = T_2'' = S_2 \cdot T_2'
\]

\[
= \begin{pmatrix}
0.952 & -0.305 & 0 \\
0.305 & 0.952 & 0 \\
0 & 0 & 1
\end{pmatrix} \cdot \begin{pmatrix}
1.33 & 1.19 & 0 \\
1.090 & 3.71 & 0 \\
0.490 & 3.13 & 3.29
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0.93 & 0 & 0 \\
1.44 & 5.89 & 0 \\
0.490 & 3.13 & 3.29
\end{pmatrix}
\]

The orthogonal matrix \( Q_2 \) is formed by multiplying the transpose of the S matrices

\[
Q_2 = S_3^T \cdot S_2^T
\]

\[
Q_2 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.913 & 0.408 \\
0 & 0.408 & 0.913
\end{pmatrix} \cdot \begin{pmatrix}
0.952 & 0.305 & 0 \\
-0.305 & 0.952 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0.952 & 0.305 & 0 \\
-0.278 & 0.869 & 0.408 \\
0.124 & -0.388 & 0.913
\end{pmatrix}
\]
Now a check is performed

\[ T_2 = L_2 Q_2 \]

\[
\begin{pmatrix}
0.952 & 0 & 0 \\
-2.78 & 0.869 & -0.408 \\
0.124 & -0.388 & 0.913
\end{pmatrix}
\begin{pmatrix}
0.93 & 0 & 0 \\
1.44 & 3.89 & 0 \\
0.490 & 3.13 & 3.29
\end{pmatrix}
\]

Now the next transform is performed

\[ T_3 = L_2 Q_2 \]

\[
\begin{pmatrix}
0.93 & 0 & 0 \\
1.44 & 3.89 & 0 \\
0.490 & 3.13 & 3.29
\end{pmatrix}
\begin{pmatrix}
0.952 & 0.305 & 0 \\
-2.78 & -0.869 & 0.408 \\
0.124 & -3.88 & 0.913
\end{pmatrix}
\]

The procedure is continued until the off diagonal elements are smaller than a specified amount, and then the diagonal elements give the eigenvalues. After 3 iterations we see that

\[ \lambda_1 = 0.89 \]
\[ \lambda_2 = 3.83 \]
\[ \lambda_3 = 4.28 \]

However the off diagonal elements are still quite large. Therefore more iteration are needed to find a more accurate answer. After 7 iterations we find that

\[ \lambda_1 = 0.86 \]
\[ \lambda_2 = 2.48 \]
\[ \lambda_3 = 5.65 \]
Exact eigenvalues are

\[ \lambda_1 = 0.85 \]
\[ \lambda_2 = 2.48 \]
\[ \lambda_3 = 5.67 \]

**Proposed method:**

Gerschgorin circle is drawn for matrix (5.2.5) and is as shown in figure 5.5

![Gerschgorin bound](image)

**Fig 5.5: Gerschgorin bound [0, 6]**

Eigenvalues of the above matrix obtained by applying the Secant method at the Gerschgorin bound are

\[ \lambda_1 = 0.854897 \]
\[ \lambda_2 = 2.476024 \]
\[ \lambda_3 = 5.669079 \]

Total computation: 270

**Conclusion:** The existing method as we observe from above requires transformation of matrices, hence the method proposed is a simple graphical approach where no transformation is required and the eigenvalues obtained are very exact.
5.2.6 Householder QR-Method and Gerschgorin circles method to compute the eigenvalues of the tridiagonal matrices.

The eigenvalues of the tridiagonal matrix have been computed by the Householder method in [54]. For this method, Gerschgorin circles method has been applied.

Existing method: [54]

Consider the tridiagonal matrix of order (3x3)

\[
A = \begin{pmatrix}
1 & 2 & 3 \\
1 & 4 & 9 \\
1 & 8 & 27
\end{pmatrix}
\Rightarrow (5.2.6)
\]

The first step is to apply Householder method to get A into upper Hessenberg configuration \((n-2)\) iterations will be required to perform this

We have \( P = \frac{2ww^T}{\|w\|^2} \)

Where w is a vector containing the following elements

\[
W = [ A_{k1}, A_{k2}, \ldots, A_{k,k-1}, -s, 0, \ldots ]^T
\]

\[
S = \pm \sqrt{A_{k1}^2 + A_{k2}^2 + \ldots + A_{k,k-1}^2}
\]

\[
V_k = [ A_{k1}, A_{k2}, \ldots, A_{kn} ]^T
\]

\(K\) is the row in matrix A being deflated

\(V_k\) is the row vector in A being deflated

Start with \(k = 2\) and decrease until \(k = 2\) \(k = 3\)

\[
V_3 = \{ 1, 8, 27 \}^T
\]

\[
S = \pm \sqrt{(1)^2 + (8)^2} = \pm8.06
\]

\[
W_1 = \{1.8 \pm 8.06, 0\}
\]
Note: +8.06 is chosen so that the absolute value of the appropriate value to \( w \) will be a maximum.

\[
P = \frac{2w w^T}{\|w\|_2^2}
\]

\[
w w^T = \begin{pmatrix} 1 \\ 0 \end{pmatrix} * \begin{bmatrix} 1 & 16.06 & 0 \end{bmatrix} = \begin{bmatrix} 16.06 & 257.9 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\|w\|_2^2 = (1)^2 + (16.06)^2 + (0)^2 = 258.9
\]

\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{2}{258.9} = \begin{bmatrix} 1 & 16.06 & 0 \\ 16.06 & 257.9 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\[
\begin{pmatrix} 128.45 & -16.06 & 0 \\ -16.06 & -128.45 & 0 \\ 0 & 0 & 129.45 \end{pmatrix}
\]

Now the first iteration can be performed

\[
A_{new} = P_1^T A_{old} P_1
\]

\[
\begin{pmatrix} 128.45 & -16.06 & 0 \\ -16.06 & -128.45 & 0 \\ 0 & 0 & 129.45 \end{pmatrix} \frac{2}{258.9} * \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & 27 \end{bmatrix}
\]
Now this is in upper Hessenberg form.

Now the $Q,R$ method can be applied.
The matrix A can be factorized into upper triangular and orthogonal matrix. This can be done if A is in Hessenberg form. For clarity, let this form of A be denoted by H.

The upper triangular matrix is formed using the following procedure.

\[ R= (\text{upper triangular}) = S_{n-1}^T \ast S_{n-2}^T \ast \ldots \ast S_1^T \ast H, \text{since } n = 3 \]

\[ R = S_2^T \ast S_1^T \ast H \]

For \( k = 1 \)

\[
S_1 = \begin{pmatrix}
\cos \theta_1 & -\sin \theta_1 & 0 \\
\sin \theta_1 & \cos \theta_1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
\cos \theta_1 = \cos \theta_2 = \frac{a_1}{\sqrt{b_1^2 + a_1^2}} = \frac{0.68}{\sqrt{(-0.58)^2 + (0.68)^2}} = 0.761
\]

\[
\sin \theta_1 = \frac{b_1}{\sqrt{b_1^2 + a_1^2}} = \frac{-0.58}{\sqrt{(-0.58)^2 + (0.68)^2}} = -0.649
\]

The value of \( a_k \) and \( b_k \) are taken from the matrix H.

\[
S_1^T = \begin{pmatrix}
0.761 & -0.649 & 0 \\
0.649 & 0.761 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

\[
S_2^T \ast H \ast H' = \begin{pmatrix}
0.761 & 0.649 & 0 \\
0.649 & 0.761 & 0 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0.68 & -1.58 & 1.86 \\
-0.58 & 4.32 & -9.3 \\
0 & -8.06 & 27
\end{pmatrix}
\]
\[
H' = \begin{pmatrix}
0.894 & -4.01 & 7.45 \\
0 & 2.26 & -5.87 \\
0 & -8.06 & 27
\end{pmatrix}
\]

For \( k = 2 \)

\[
S_3 = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta_2 & -\sin \theta_2 \\
0 & \sin \theta_2 & \cos \theta_2
\end{pmatrix}
\]

Where

\[
\cos \theta_2 = \cos \theta_3 = \frac{a_3}{\sqrt{b_2^2 + a_3^2}} = \frac{2.26}{\sqrt{(-8.06)^2 + (2.26)^2}} = 0.269
\]

\[
\sin \theta_2 = \frac{b_3}{\sqrt{b_2^2 + a_3^2}} = \frac{-8.06}{\sqrt{(-8.06)^2 + (2.26)^2}} = -9.63
\]

The values of \( a_2 \) and \( b_2 \) are taken from \( H' \)

\[
S_2^r = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.269 & -0.963 \\
0 & 0.963 & 0.269
\end{pmatrix}
\]

\[
S_2^r * H = R = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.269 & -0.963 \\
0 & 0.963 & 0.269
\end{pmatrix}
\]
\[
Q^T = S_n^T \ast S_{n-1}^T \ast \ldots \ast S_1^T
\]

In this case
\[
Q^T = S_2^T \ast S_1^T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0.269 & -0.963 \\
0 & 0.963 & 0.269
\end{pmatrix}
\begin{pmatrix}
0.781 \\
0.649 \\
0
\end{pmatrix}
\begin{pmatrix}
0.781 & -0.649 & 0 \\
0.649 & 0.761 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0.761 & -0.649 & 0 \\
0.175 & 0.205 & -0.963 \\
0.625 & 0.733 & 0.269
\end{pmatrix}
\begin{pmatrix}
0.761 \\
0.175 \\
0.625
\end{pmatrix}
\begin{pmatrix}
0.761 & 0.175 & 0.625 \\
0.175 & -0.205 & 0.733 \\
0 & -0.963 & 0.269
\end{pmatrix}
\]

Hence
\[
Q = \begin{pmatrix}
0.761 & 0.175 & 0.625 \\
0.175 & -0.205 & 0.733 \\
0 & -0.963 & 0.269
\end{pmatrix}
\]

Now a check is performed to verify \( Q \) and \( R \) are the correct factor of \( H \)
\[
H = Q \ast R = \begin{pmatrix}
0.761 & 0.175 & 0.625 \\
0.175 & -0.205 & 0.733 \\
0 & -0.963 & 0.269
\end{pmatrix}
\begin{pmatrix}
0.894 & -4.01 & 7.45 \\
0 & 8.37 & -27.6 \\
0 & -8.06 & 1.61
\end{pmatrix}
\]

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This is true, thus $Q$ and $R$ are valid.

Now the reduction process can be performed.

$$H_{\text{new}} = R_{\text{old}} \ast Q_{\text{old}}$$

$$H_{\text{new}} = \begin{pmatrix}
0.68 & -1.58 & 1.86 \\
-0.58 & 4.32 & -9.3 \\
0 & -8.06 & 27
\end{pmatrix}$$

$$= \begin{pmatrix}
0.894 & -4.01 & 7.45 \\
0 & 8.37 & -27.6 \\
0 & 0 & 1.61
\end{pmatrix} \begin{pmatrix}
0.761 & 0.175 & 0.625 \\
-0.849 & 0.205 & 0.733 \\
0 & -0.963 & 0.269
\end{pmatrix}$$

$$= \begin{pmatrix}
3.28 & -7.84 & -0.38 \\
-5.43 & 28.3 & -1.28 \\
0 & -1.55 & 0.433
\end{pmatrix}$$

Now let $H_{\text{new}} = H_{\text{old}}$ and repeat the factorization to get the next $Q_{\text{old}}$ and $R_{\text{old}}$. Then multiply $R_{\text{old}} \ast Q_{\text{old}}$ to get the $H_{\text{new}}$. When a $H_{\text{new}}$ is formed with a sufficiently low half diagonal elements stop the iterative process. The elements along the diagonal of this $H_{\text{new}}$ represent a close approximation to the system's eigenvalues. After fourth iteration

$\lambda_1 = 0.22$

$\lambda_2 = 1.84$

$\lambda_3 = 29.84$
**Condition for \( Q - R \) method:**

The \( Q, R \) method will be assured to converge on the system eigenvalues if the sum of the squares of the elements below the main diagonal of the matrix \( H_{new}^{(i)} \) is smaller than of its predecessor \( H_{new}^{(i-1)} \). For instance in the above example.

\[(5.93)^2 + (-4.8)^2 < (0.58)^2 + (-8.06)^2\]

Hence the eigenvalues are converging.

**Proposed method:**

Gerschgorin circle of the matrix (5.2.6) is drawn and is as shown in figure 5.6

![Image of Gerschgorin bound](image.png)

**Fig 5.6: Gerschgorin bound [-6, 36]**

Eigenvalues of the above matrix obtained by applying the Bisection method for the step length \( h = \text{avg} / \text{ab} \) where \( \text{avg} \) is the average of all the elements in the matrix and \( \text{ab} \) is the Gerschgorin bound.

\[\lambda_1 = 0.219434\]
\[\lambda_2 = 1.841113\]
\[\lambda_3 = 29.944238\]

Total Computation: 552
Total iteration: 184
**Conclusions:** In the existing method, we observe that the given matrix is first converted into Hessenberg form and in turn this matrix is converted to tri diagonal matrix which is a lengthy process and if the given condition above is satisfied this method can be applied.

The proposed method is simple graphical technique where no conditions are required. In this method, it computes the eigenvalues of any matrix, i.e. it does not the convert the given matrix to any form. Also the eigenvalues obtained are very accurate. The number of steps involved to compute the eigenvalues are minimum compared to the existing method.

**5.2.7 Computation of the largest Eigenvalues using Power method and Gerschgorin circles method:**

The concept of stability plays very important role in the analysis of the system. A system matrix can be used from [1] for the computation of largest eigenvalue. In the literature, there exists power method for the computation of largest eigenvalue and it takes lot of computation [1]. It has been found that by using Gerschgorin technique that the largest eigenvalue can be computed, which takes less computations compared to the power method.

**Existing method [1]:**

Consider the system matrix of order (3x3)

\[ A = \begin{pmatrix} -2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \] \[
\rightarrow (5.2.7)
\]

Choose the initial vector \( X_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)
Then \( X_1 = A X_0 = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} \)

\( X_2 = A X_1 = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix} \)

\( X_3 = A X_2 = \begin{pmatrix} 2 & 0 & -1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \)

Continuing these iterations

\( X_4 = A X_3 = \begin{pmatrix} -4 \\ 0 \\ -4 \end{pmatrix} \begin{pmatrix} 12 \\ 4 \\ 4 \end{pmatrix} = \begin{pmatrix} -28 \end{pmatrix} \)

\( X_5 = A X_4 = \begin{pmatrix} 60 \\ -62 \\ 4 \end{pmatrix} \begin{pmatrix} -14 \\ 126 \\ -4 \end{pmatrix} = \begin{pmatrix} 30 \\ 252 \end{pmatrix} \)

\( X_6 = A X_5 = \begin{pmatrix} -124 \\ 126 \\ 4 \end{pmatrix} \begin{pmatrix} -4 \\ -4 \end{pmatrix} = \begin{pmatrix} 252 \\ -254 \end{pmatrix} \)

Put \( X = X_8 \) \quad Y = X_6 \) Then
The largest eigenvalue is

\[
\frac{m_0}{m_1} = \frac{-63268}{31268} = -2.0234105 \, (9^{th} \text{ iteration})
\]

**Total computations taken for above method:** 82.

**Gerschgorin circle method:**

Gerschgorin circle of the above (5.2.7) is drawn and is as shown in figure 5.7
**Procedure:** From the above figure we observe that Gershgorin bound \([-4, 2]\) so to test the maximum eigenvalue we start with \(-4\). First we check whether or not the eigenvalue lie in \([-4, -3]\) by computing the determinant \((\lambda I - A)_{\lambda = -4}\) and determinant \((\lambda I - A)_{\lambda = -3}\) which implies that there is no eigenvalue in the interval \([-4, -3]\). Then we check in the interval \([-3, -2]\) and we found the determinant \((\lambda I - A)_{\lambda = -2} = 0\) at \(\lambda = -2\). Hence the maximum eigenvalue is \(-2\).

**Total computation:** 16

From the above we observe that the Gershgorin circles method takes less computation compared to the power method.

**Conclusions:** The method of computing eigenvalues via Gershgorin circles works for all the examples when the Gershgorin bound are unequal. We have presented the examples were the method does not work.

Counter example to the above method:

Consider the system matrix of order (4x4)

\[
A = \begin{bmatrix}
0 & -1 & 1 & 0 \\
-1 & 0 & 1 & 1 \\
1 & -1 & 0 & -1 \\
1 & 1 & -1 & 0
\end{bmatrix} \quad \text{(4x4)}
\]

Gershgorin circles of the matrix (5.2.8) are drawn as shown in figure 5.8

**Fig 5.8:** Gershgorin bound \([-3, 3]\)
The eigenvalues are

\[ \lambda_1 = -1.8136 \]
\[ \lambda_2 = 2.3429 \]
\[ \lambda_3 = 0.4707 \]
\[ \lambda_4 = -1.0000 \]

Incidentally we got the following result

(i) The trace = 0 and the bounds are equal.

(ii) When \( \text{trace} \neq 0 \) and the bounds are unequal.

**Case (i):** When the trace = 0 and bounds are equal. Given all the centers of the Gerschgorin circles are at origin, this implies that there exists at least one eigenvalue on the positive real axis of \( s \)-plane. The eigenvalues may be complex conjugate eigenvalues with positive real part.

Consider the system matrix of order (6x6)

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 \\
-1 & 1 & 0 & 1 & 0 & 1 \\
1 & -1 & 1 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & -1 & 1 & 0
\end{pmatrix}
\]

\[ \rightarrow (5.2.9) \]

Gerschgorin circle of the matrix (5.2.9) is drawn and is as shown in figure 5.9
The eigenvalues are
\[ \lambda_1 = -2.4836 \]
\[ \lambda_2 = 1.5488 \]
\[ \lambda_3 = 0.5184 \]
\[ \lambda_4 = 0.0000 \]
\[ \lambda_5 = -0.2077 + 0.6772 \, i \]
\[ \lambda_5 = -0.2077 - 0.6772 \, i \]

**Conclusions:** From the above example we observe that

(a) Centers of the Gerschgorin circles are at origin and hence *trace* = 0

(b) Gerschgorin bounds are equal on either side of the imaginary axis.

(c) There exist a positive real eigenvalue and complex conjugate pairs with negative real part.

Case (ii): when *trace* ≠ 0 and bounds are equal
Consider the system matrix of order (3x3)

\[ A = \begin{pmatrix}
2 & -4 & -4 \\
1 & -4 & -5 \\
-1 & 4 & 5 \\
\end{pmatrix} \rightarrow (5.2.10) \]

Gerschgorin circles of the matrix (5.2.10) are drawn as shown in figure 5.10

![Gerschgorin circles](image)

Fig 5.10: Gerschgorin bound [-10, 10]

Eigenvalues of the above matrix are

\[ \lambda_1 = 2.0000 \]
\[ \lambda_2 = 1.0000 \]
\[ \lambda_3 = 0.0000 \]

**Conclusions:** All the eigenvalues are positive.

Hence finally we conclude for the above two cases there exist at least one eigenvalue (real or complex) on the right hand side of s-plane. Hence the systems are unstable.
5.2.8 Positive definiteness of the real symmetric matrix using Gerschgorin circles.

A system is said to be positive definite if all the eigenvalues of the system matrix are positive. A system matrix [1] has been considered. The leading principle minors are considered and determinants are computed. If all the determinants of the principle minors are positive then the system is said to be positive definite. In the Gerschgorin circle method, the Gerschgorin circles are drawn for the system matrix, since all the circles lie on the open right half of the s-plane, the system is positive definite. Here no computations are required to decide the definiteness of the real symmetric matrix.

Existing method:

All leading minors of A are positive

Consider the system matrix \( A \) of order (3x3) given by

\[
A = \begin{pmatrix}
12 & 4 & -1 \\
4 & 7 & 1 \\
-1 & 1 & 6
\end{pmatrix}
\]

(5.2.11)

Leading minors of \( A \) are \(|\begin{vmatrix} 12 & 4 \end{vmatrix}| = 12\)

\[
| 12 & 4 \\
4 & 7 |
\]

= 68

\[
| 12 & 4 & -1 \\
4 & 7 & 1 \\
-1 & 1 & 6 |
\]

= 364

In the above existing method we need to compute the leading minors which take the total computation of 11.
**Proposed method using Gerschgorin circles:**

Consider the system matrix $A$ of order $(3 \times 3)$ given by

$$
A = \begin{pmatrix}
12 & 4 & -1 \\
4 & 7 & 1 \\
-1 & 1 & 6
\end{pmatrix} \rightarrow (5.2.13)
$$

Gerschgorin circle of the above matrix is drawn and is as shown in figure 5.11

![Gerschgorin circle](image)

**Fig 5.11: Gerschgorin bound [2, 17]**

From the above figure we observe that all the Gerschgorin circles lie on the right half of $s$-plane and also the Gerschgorin bound are positive. This implies that all the eigenvalues of the symmetric matrix are positive. Hence the given symmetric matrix are positive definite.

**Comparison:**

**Total computation: Existing method: 11**

**Proposed method: 0**

**Conclusion:** In the proposed method no computation is required, by observing the Gerschgorin circles and the Gerschgorin bound the positive definiteness of the real symmetric matrix can be identified.
Table 5.4: Results showing the comparisons of total computations of Gerschgorin circles approach and existing methods.

<table>
<thead>
<tr>
<th>SL.NO</th>
<th>Existing Methods</th>
<th>Total Computations</th>
<th>Existing Method</th>
<th>Proposed Method (Gerschgorin circle method)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Given method</td>
<td>855</td>
<td>225</td>
<td></td>
</tr>
<tr>
<td>2.</td>
<td>Jacobi method</td>
<td>36</td>
<td>417</td>
<td></td>
</tr>
<tr>
<td>3.</td>
<td>Rutihauser Method</td>
<td>108</td>
<td>339</td>
<td></td>
</tr>
<tr>
<td>4.</td>
<td>House holder QL-method</td>
<td>190</td>
<td>270</td>
<td></td>
</tr>
<tr>
<td>5.</td>
<td>House holder QR-method</td>
<td>130</td>
<td>552</td>
<td></td>
</tr>
<tr>
<td>6.</td>
<td>Power Method</td>
<td>82</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>7.</td>
<td>Strum sequences</td>
<td>65</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>8.</td>
<td>Positive definiteness</td>
<td>11</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Conclusions:

From the above table, we observe that for the methods 1, 6, 7, 8, the graphical method shows better results compared to the methods 2, 3, 4 and 5 respectively with respect to computations.
5.3 Computation of root of the polynomial by Bisection method, false position method, Secant method and Gerschgorin circles method

The roots of the polynomial are computed using the Bisection method, false position method and Secant method [55]. In this method to compute the roots the initial values have to taken as guess values and then check for the changes in sign by substituting the values in the polynomial. Finding the initial values takes time.

In this thesis an attempt has been made to compute the roots of the polynomial using Gerschgorin circles. For the given polynomial the matrix of the polynomial are considered by taking its companion form. The Gerschgorin circles are drawn for this matrix. The Gerschgorin bounds are obtained. This Gerschgorin bounds are taken as the initial values. The Bisection method, false position method, and secant method are applied at the Gerschgorin bounds and the eigenvalues of the companion matrix are computed which forms the roots of the polynomial.

5.3.1 Computation of the root of the polynomial by Bisection method [55]

Consider the polynomial $x^3 - 2x - 5 = 0$ with the initial values $a = 2$ and $b = 10$
Existing method [55]:

Table 5.5: Root of the polynomial by Bisection method

<table>
<thead>
<tr>
<th>Iteration: 20</th>
<th>Desired roots: 2.094551</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a = 2.000000$</td>
<td>$b = 6.000000$</td>
</tr>
<tr>
<td>$a = 2.000000$</td>
<td>$b = 4.000000$</td>
</tr>
<tr>
<td>$a = 2.000000$</td>
<td>$b = 3.000000$</td>
</tr>
<tr>
<td>$a = 2.000000$</td>
<td>$b = 2.500000$</td>
</tr>
<tr>
<td>$a = 2.000000$</td>
<td>$b = 2.250000$</td>
</tr>
<tr>
<td>$a = 2.000000$</td>
<td>$b = 2.125000$</td>
</tr>
<tr>
<td>$a = 2.062500$</td>
<td>$b = 2.125000$</td>
</tr>
<tr>
<td>$a = 2.093750$</td>
<td>$b = 2.125000$</td>
</tr>
<tr>
<td>$a = 2.093750$</td>
<td>$b = 2.109375$</td>
</tr>
<tr>
<td>$a = 2.093750$</td>
<td>$b = 2.101562$</td>
</tr>
<tr>
<td>$a = 2.093750$</td>
<td>$b = 2.097656$</td>
</tr>
<tr>
<td>$a = 2.093750$</td>
<td>$b = 2.095703$</td>
</tr>
<tr>
<td>$a = 2.093750$</td>
<td>$b = 2.094727$</td>
</tr>
<tr>
<td>$a = 2.094238$</td>
<td>$b = 2.094727$</td>
</tr>
<tr>
<td>$a = 2.094482$</td>
<td>$b = 2.094727$</td>
</tr>
<tr>
<td>$a = 2.094482$</td>
<td>$b = 2.094604$</td>
</tr>
<tr>
<td>$a = 2.094543$</td>
<td>$b = 2.094604$</td>
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<td>$a = 2.094543$</td>
<td>$b = 2.094547$</td>
</tr>
<tr>
<td>$a = 2.094543$</td>
<td>$b = 2.094559$</td>
</tr>
<tr>
<td>$a = 2.094551$</td>
<td>$b = 2.094551$</td>
</tr>
</tbody>
</table>
Proposed method:

The companion matrix of the above matrix

\[
A = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
5 & 2 & 0
\end{pmatrix} \quad \Rightarrow (5.3.1)
\]

Gerschgorin circle of the matrix (5.3.1) is drawn and is as shown in figure 5.12

Fig 5.12: Gerschgorin bound [-5, 5]
Table 5.6 Bisection method applied at the Gerschgorin bound:

<table>
<thead>
<tr>
<th>$x_0$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$f_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.875000</td>
<td>2.187500</td>
<td>2.187500</td>
<td>1.092529</td>
</tr>
<tr>
<td>2.031250</td>
<td>2.187500</td>
<td>2.031250</td>
<td>0.681610</td>
</tr>
<tr>
<td>2.031250</td>
<td>2.109375</td>
<td>2.109375</td>
<td>0.166836</td>
</tr>
<tr>
<td>2.070312</td>
<td>2.109375</td>
<td>2.070312</td>
<td>0.266864</td>
</tr>
<tr>
<td>2.089844</td>
<td>2.109375</td>
<td>2.089844</td>
<td>0.052406</td>
</tr>
<tr>
<td>2.089844</td>
<td>2.099609</td>
<td>2.099609</td>
<td>0.056614</td>
</tr>
<tr>
<td>2.089844</td>
<td>2.094727</td>
<td>2.094727</td>
<td>0.001954</td>
</tr>
<tr>
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<td>2.094727</td>
<td>2.092285</td>
<td>0.025263</td>
</tr>
<tr>
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<td>2.094727</td>
<td>2.093506</td>
<td>0.011664</td>
</tr>
<tr>
<td>2.094116</td>
<td>2.094727</td>
<td>2.094116</td>
<td>0.004857</td>
</tr>
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<td>2.094421</td>
<td>2.094727</td>
<td>2.094421</td>
<td>0.001452</td>
</tr>
<tr>
<td>2.094421</td>
<td>2.094574</td>
<td>2.094574</td>
<td>0.000251</td>
</tr>
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<td>2.094574</td>
<td>2.094498</td>
<td>0.000600</td>
</tr>
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<td>2.094536</td>
<td>2.094574</td>
<td>2.094536</td>
<td>0.000175</td>
</tr>
<tr>
<td>2.094536</td>
<td>2.094555</td>
<td>2.094555</td>
<td>0.000038</td>
</tr>
<tr>
<td>2.094545</td>
<td>2.094555</td>
<td>2.094545</td>
<td>0.000068</td>
</tr>
<tr>
<td>2.094550</td>
<td>2.094555</td>
<td>2.094550</td>
<td>0.000015</td>
</tr>
<tr>
<td>2.094550</td>
<td>2.094553</td>
<td>2.094553</td>
<td>0.000012</td>
</tr>
<tr>
<td>2.094551</td>
<td>2.094551</td>
<td>2.094551</td>
<td>0.000002</td>
</tr>
</tbody>
</table>

Iterations: 19

Desired root: 2.094551
5.3.2 Computation of the root of the polynomial secant method

Table 5.7: Computation of root of the polynomial by Secant method.

<table>
<thead>
<tr>
<th>(x_0)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.00000</td>
<td>2.008197</td>
<td>2.008197</td>
<td>-0.917628</td>
</tr>
<tr>
<td>2.008197</td>
<td>2.015711</td>
<td>2.015711</td>
<td>-0.841402</td>
</tr>
<tr>
<td>2.015711</td>
<td>2.098658</td>
<td>2.098658</td>
<td>0.045942</td>
</tr>
<tr>
<td>2.098658</td>
<td>2.094363</td>
<td>2.094363</td>
<td>0.002098</td>
</tr>
<tr>
<td>2.094363</td>
<td>2.094551</td>
<td>2.094551</td>
<td>0.000004</td>
</tr>
</tbody>
</table>

Root is 2.094551

Iterations: 6

Table 5.8: Secant method applied at Gershgorin bound:

<table>
<thead>
<tr>
<th>(x_0)</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(f_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.000000</td>
<td>0.217391</td>
<td>0.217391</td>
<td>5.424509</td>
</tr>
<tr>
<td>5.000000</td>
<td>0.464406</td>
<td>0.464406</td>
<td>5.828652</td>
</tr>
<tr>
<td>5.000000</td>
<td>0.743373</td>
<td>0.743373</td>
<td>6.075956</td>
</tr>
<tr>
<td>5.000000</td>
<td>1.049687</td>
<td>1.049687</td>
<td>5.942784</td>
</tr>
<tr>
<td>5.000000</td>
<td>1.364897</td>
<td>1.364897</td>
<td>5.187068</td>
</tr>
<tr>
<td>5.000000</td>
<td>1.652453</td>
<td>1.652453</td>
<td>3.792716</td>
</tr>
<tr>
<td>5.000000</td>
<td>1.869572</td>
<td>1.869572</td>
<td>2.204426</td>
</tr>
<tr>
<td>5.000000</td>
<td>1.998130</td>
<td>1.998130</td>
<td>1.018683</td>
</tr>
<tr>
<td>5.000000</td>
<td>2.058046</td>
<td>2.058046</td>
<td>0.399133</td>
</tr>
<tr>
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<td>2.081600</td>
<td>2.081600</td>
<td>0.143508</td>
</tr>
<tr>
<td>5.000000</td>
<td>2.090079</td>
<td>2.090079</td>
<td>0.049796</td>
</tr>
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<td>5.000000</td>
<td>2.093022</td>
<td>2.093022</td>
<td>0.017055</td>
</tr>
</tbody>
</table>
Root is 2.094551

Iteration: 20

5.3.3 Computation of the root by False Position method:

\[ x_0 = 2.008197 \]
\[ x_1 = 10.00000 \]
\[ x_2 = 2.008197 \]
\[ f_2 = -0.917628 \]

Iteration: 2

Root = 2.008197

False position applied at Gerschgorin bound
Table 5.9: False position method applied at Gerschgorin bound.

<table>
<thead>
<tr>
<th>Iteration</th>
<th>x₀</th>
<th>x₁</th>
<th>x₂</th>
<th>f₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2.094164</td>
<td>5.000000</td>
<td>2.094164</td>
<td>0.004323</td>
</tr>
<tr>
<td></td>
<td>2.094278</td>
<td>5.000000</td>
<td>2.094278</td>
<td>0.003049</td>
</tr>
<tr>
<td></td>
<td>2.094359</td>
<td>5.000000</td>
<td>2.094359</td>
<td>0.002150</td>
</tr>
<tr>
<td></td>
<td>2.094416</td>
<td>5.000000</td>
<td>2.094416</td>
<td>0.001516</td>
</tr>
<tr>
<td></td>
<td>2.094456</td>
<td>5.000000</td>
<td>2.094456</td>
<td>0.001069</td>
</tr>
<tr>
<td></td>
<td>2.094484</td>
<td>5.000000</td>
<td>2.094484</td>
<td>0.000754</td>
</tr>
<tr>
<td></td>
<td>2.094504</td>
<td>5.000000</td>
<td>2.094504</td>
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</tr>
<tr>
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<td>5.000000</td>
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</tr>
<tr>
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<tr>
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<td>2.094549</td>
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<td>2.094549</td>
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<td>2.094551</td>
<td>0.000008</td>
</tr>
</tbody>
</table>

Iteration: **20**  
Root is: **2.094551**
5.10 Table: Results of comparisons of computations of the root of the polynomial by Gerschgorin circle approach.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Bisection method and Gerschgorin method</th>
<th>Secant method And Gerschgorin method</th>
<th>False position method and Gerschgorin method</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iterations</td>
<td>20</td>
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<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>20</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Roots</td>
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<td>2.094551</td>
<td>2.008197</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2.094551</td>
</tr>
</tbody>
</table>

Conclusions:

We can observe that,

(a) Bisection method and Gerschgorin circles method compute the eigenvalues very accurately

(b) Secant method takes less iteration where as Gerschgorin circles method takes more iteration but still computes the eigenvalues very accurately

(c)False position method takes less iteration but roots are not nearer to exact value whereas Gerschgorin circles method computes the eigenvalues very accurately but more iteration compared to false position method.