4. COMPUTATION OF EIGENVALUES VIA GERSCHGORIN CIRCLES

4.1 Introduction

An algorithm has been developed to compute the eigenvalues of the system matrix of order \((n \times n)\) in which a suitable step length has been considered and Bisection method has been applied repeatedly in that step length to compute the eigenvalues of the system matrix.

Bisection method, false position method and Secant method are also applied in Gerschgorin bound (without using step length ) and eigenvalues are computed in all the cases and the total computation taken by each methods are compared and the results are presented in tabular form.

All types of eigenvalues are computed using the Gerschgorin circles when the trace of the system is not zero. Similarly eigenvalues at the origin, repeated eigenvalues, eigenvalues which are equal and opposite in sign are computed using Gerschgorin circles when the trace of the system matrix is zero.

An algorithm to compute the common eigenvalues between the two system matrices are developed and his is applied to find the fixed modes of the Decentralized Control systems.

4.2 Determination of suitable step length to compute the eigenvalues of the system matrix accurately using Gerschgorin circles.

Given a system matrix \(A\) of order \((n \times n)\), the Gerschgorin circles are drawn explained in chapter 1.. Let \([-a, b]\) be the Gerschgorin bound. In this Gerschgorin bound, various step lengths are taken to compute the eigenvalues of the system matrix. Step lengths considered are as follows

\[ ab \quad \text{– Length of the Gerschgorin bound.} \]
\[ R_{\text{min}} \quad \text{– Minimum of sum of each row – wise elements.} \]
\[ R_{\text{max}} \quad \text{– Maximum of sum of each row – wise elements.} \]
The values of the above ratios are taken as step length. For examples consider the step length \( h = \frac{ab}{\text{trace}} \). For this value of \( h \), in the interval \([-a, a+h]\) changes in sign of the determinant \( (\lambda_i I - A) \) are computed. If there exist changes in sign of the determinant \( (\lambda_i I - A) \), then Bisection method are applied until determinant \( (\lambda_i I - A) = 0 \) for some values of \( \lambda_i \) in that interval. Then this value of \( \lambda \) becomes one of the eigenvalues of the system matrix. We are repeating the above procedure in the \([a+h, a+2h]\) and so on. This process is continued till all the eigenvalues of the system matrix are obtained.

Algorithm:
Step 1: Read a system matrix of order \((n \times n)\).
Step 2: for \((i=0 ; i< n ; i++)\)
\[
C_i \leftarrow a_{ii}
\]
Step 3: for \((i=0 ; i< n ; i++)\)
\[ r_i \leftarrow \sum_{j=0}^{n} \text{abs}(a_{ij}) \]

Step 4: Take union of row wise circles

Step 5: for (j=0 ; j < n ; j++)

\[ r_j \leftarrow \sum_{i=0}^{n} \text{abs}(a_{ij}) \]

Step 6: Take union of column circles

Step 7: Intersection of row wise circles and column wise circles gives the Gerschgorin bound.

Step 8: Compute \( h \leftarrow \frac{ab}{\text{trace}} \) where ab is the length of the Gerschgorin bound.

Step 9: In \([-a, b]\) at the point of intersection of the circles.

Step 10: Is determinant \((\lambda_i I - A) > 0\) and determinant \((\lambda_i I - A) < 0\)?

Step 11: If it is true for the step length h, compute determinant \((\lambda_i I - A)\)

Step 12: Is determinant \((\lambda_i I - A) = 0\)?

Step 13: True, Print \(\lambda\) is the eigenvalue, else go to step 10.

Step 14: Print, Total Computations.

Various types of matrices from \([57, 58, 59, 60, 61, 62, 64, 65, 83]\) are taken to compute the eigenvalues in our experimentation.

The above algorithm is applied for each example for all the step lengths mentioned above and the result are tabulated as shown in table 4.1
Table 4.1: List of matrices and various step lengths which gives minimum computations to obtain the eigenvalues of the system matrix.

<table>
<thead>
<tr>
<th>Sl.No.</th>
<th>Matrices</th>
<th>Step length with minimum computations</th>
<th>Total computations</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Ordinary</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Gauss elimination method.</td>
</tr>
<tr>
<td>1.</td>
<td>$A_1$</td>
<td>$\frac{\text{trace}}{c_{\text{max}}} = \frac{\text{trace}}{R_{\text{max}}} = 4.5$</td>
<td>300</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2840</td>
</tr>
<tr>
<td>2.</td>
<td>$A_2$</td>
<td>$\frac{ab}{c_{\text{max}}} = \frac{ab}{r_{\text{max}}} = \frac{c_{\text{max}}}{ab} = \frac{R_{\text{max}}}{ab} = 1.0$</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>399</td>
</tr>
<tr>
<td>3.</td>
<td>$A_3$</td>
<td>$\frac{ab}{R_{\text{min}}} = 3.5$</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1000</td>
</tr>
<tr>
<td>4.</td>
<td>$A_4$</td>
<td>$\frac{ab}{\text{trace}} = 2.142857$</td>
<td>195</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1300</td>
</tr>
<tr>
<td>5.</td>
<td>$A_5$</td>
<td>$\frac{ab}{c_{\text{max}}} = \frac{ab}{c_{\text{min}}} = \frac{ab}{\text{trace}} = 1.0$</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>364</td>
</tr>
<tr>
<td>6.</td>
<td>$A_6$</td>
<td>$\frac{ab}{\text{Sum}} = 0.592543$</td>
<td>537</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3568</td>
</tr>
<tr>
<td>7.</td>
<td>$A_7$</td>
<td>$\frac{\text{trace}}{\text{Sum}} = \frac{\text{trace}}{c_{\text{min}}} = \frac{\text{trace}}{\text{trace}} = 1.0$</td>
<td>150</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>959</td>
</tr>
<tr>
<td>8.</td>
<td>$A_8$</td>
<td>$\frac{ab}{\text{trace}} = \frac{\text{trace}}{ab} = 1.0$</td>
<td>102</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1491</td>
</tr>
<tr>
<td>9.</td>
<td>$A_9$</td>
<td>$\frac{\text{trace}}{\text{Sum}} = 0.5$</td>
<td>372</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2485</td>
</tr>
<tr>
<td>10.</td>
<td>$A_{10}$</td>
<td>$\frac{ab}{\text{Avg}} = 5.6$</td>
<td>435</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2900</td>
</tr>
<tr>
<td>11.</td>
<td>$A_{11}$</td>
<td>$\frac{\text{trace}}{R_{\text{min}}} = 2.5$</td>
<td>471</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>3157</td>
</tr>
<tr>
<td>12.</td>
<td>$A_{12}$</td>
<td>$\frac{\text{Sum}}{\text{trace}} = \frac{\text{trace}}{\text{trace}} = 1.0$</td>
<td>147</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1787</td>
</tr>
<tr>
<td>13.</td>
<td>$A_{13}$</td>
<td>$\frac{ab}{\text{Sum}} = 0.5$</td>
<td>282</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>1300</td>
</tr>
<tr>
<td>14.</td>
<td>$A_{14}$</td>
<td>$\frac{\text{sl}}{\text{trace}} = \frac{\text{trace}}{\text{sl}} = 1.0$</td>
<td>66</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>387</td>
</tr>
<tr>
<td>15.</td>
<td>$A_{15}$</td>
<td>$\frac{ab}{r_{\text{max}}} = 1.0$</td>
<td>408</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>2707</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>16.</td>
<td>$A_{16}$</td>
<td>$\frac{ab}{S_{l}} = \frac{ab}{S_{u}} \frac{ab}{R_{min}} = 2.0$</td>
<td>93</td>
</tr>
<tr>
<td>17.</td>
<td>$A_{17}$</td>
<td>$\frac{R_{min}}{ab} = \frac{ab}{R_{min}} \frac{ab}{c_{min}} \frac{ab}{c_{min}} = 1.0$</td>
<td>30</td>
</tr>
<tr>
<td>18.</td>
<td>$A_{18}$</td>
<td>$\frac{ab}{c_{max}} = 0.8$</td>
<td>225</td>
</tr>
<tr>
<td>19.</td>
<td>$A_{19}$</td>
<td>$\frac{ab}{Sum} = 1.0$</td>
<td>48</td>
</tr>
<tr>
<td>20.</td>
<td>$A_{20}$</td>
<td>$\frac{R_{min}}{trace} = 0.857143$</td>
<td>768</td>
</tr>
<tr>
<td>21.</td>
<td>$A_{21}$</td>
<td>$\frac{detA}{ab} = 1.0$</td>
<td>339</td>
</tr>
<tr>
<td>22.</td>
<td>$A_{22}$</td>
<td>$\frac{trace}{ab} = \frac{trace}{S_{l}} = 4.00$</td>
<td>400</td>
</tr>
<tr>
<td>23.</td>
<td>$A_{23}$</td>
<td>$h=0.5 \quad \text{all step lengths with}$</td>
<td>372</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>24.</td>
<td>$A_{24}$</td>
<td>$\frac{ab}{norm} = \frac{c_{max}}{ab} \frac{norm}{ab} = \frac{R_{max}}{ab} = \frac{ab}{r_{max}} \frac{c_{max}}{c_{max}} = 1.0$</td>
<td>300</td>
</tr>
<tr>
<td>25.</td>
<td>$A_{25}$</td>
<td>$\frac{ab}{c_{min}} = \frac{c_{max}}{c_{min}} \frac{c_{min}}{ab} = 0.5$</td>
<td>444</td>
</tr>
<tr>
<td>26.</td>
<td>$A_{26}$</td>
<td>$\frac{ab}{trace} = 2.1081$</td>
<td>417</td>
</tr>
</tbody>
</table>

**Average of all step length which takes minimum computations**

Consider the average of all step lengths which takes minimum computations for finding the eigenvalues

Average = \( \frac{(h_{1}+h_{2}+h_{3}+\ldots+h_{21})}{21} \)

= \( \frac{(1*12)+(0.5*4)+(4.5*2)+2.14+2.5)}{21} \) = 1.4
Table 4.2: List of matrices which gives computations for the various step lengths

<table>
<thead>
<tr>
<th>Matrices</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
<th>1.1</th>
<th>1.2</th>
<th>1.3</th>
<th>1.4</th>
<th>1.5</th>
</tr>
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<tbody>
<tr>
<td>$D_1$</td>
<td>198</td>
<td>540</td>
<td>522</td>
<td>204</td>
<td>495</td>
<td>108</td>
<td>477</td>
<td>234</td>
<td>459</td>
<td>459</td>
<td>198</td>
</tr>
<tr>
<td>$D_2$</td>
<td>54</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>36</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$D_3$</td>
<td>234</td>
<td>450</td>
<td>423</td>
<td>231</td>
<td>387</td>
<td>126</td>
<td>486</td>
<td>–</td>
<td>477</td>
<td>468</td>
<td>–</td>
</tr>
<tr>
<td>$D_4$</td>
<td>162</td>
<td>–</td>
<td>–</td>
<td>231</td>
<td>90</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$D_5$</td>
<td>90</td>
<td>–</td>
<td>123</td>
<td>–</td>
<td>54</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$D_6$</td>
<td>450</td>
<td>756</td>
<td>702</td>
<td>657</td>
<td>630</td>
<td>234</td>
<td>585</td>
<td>576</td>
<td>558</td>
<td>549</td>
<td>–</td>
</tr>
<tr>
<td>$D_7$</td>
<td>1656</td>
<td>548</td>
<td>1371</td>
<td>1260</td>
<td>1661</td>
<td>963</td>
<td>1017</td>
<td>972</td>
<td>927</td>
<td>1656</td>
<td>729</td>
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<tr>
<td>$D_8$</td>
<td>882</td>
<td>1107</td>
<td>1008</td>
<td>927</td>
<td>873</td>
<td>450</td>
<td>783</td>
<td>756</td>
<td>720</td>
<td>882</td>
<td>–</td>
</tr>
<tr>
<td>$D_9$</td>
<td>855</td>
<td>1469</td>
<td>1188</td>
<td>867</td>
<td>1116</td>
<td>666</td>
<td>1935</td>
<td>936</td>
<td>891</td>
<td>855</td>
<td>450</td>
</tr>
<tr>
<td>$D_{10}$</td>
<td>3420</td>
<td>3231</td>
<td>2817</td>
<td>2520</td>
<td>2277</td>
<td>1719</td>
<td>2256</td>
<td>1809</td>
<td>1701</td>
<td>3420</td>
<td>1152</td>
</tr>
<tr>
<td>$D_{11}$</td>
<td>1984</td>
<td>4336</td>
<td>4096</td>
<td>3952</td>
<td>3808</td>
<td>1024</td>
<td>2256</td>
<td>2192</td>
<td>3472</td>
<td>1984</td>
<td>1392</td>
</tr>
<tr>
<td>$D_{12}$</td>
<td>1984</td>
<td>4336</td>
<td>4096</td>
<td>3504</td>
<td>3808</td>
<td>1408</td>
<td>3616</td>
<td>3568</td>
<td>3472</td>
<td>2848</td>
<td>1392</td>
</tr>
<tr>
<td>$D_{13}$</td>
<td>414</td>
<td>387</td>
<td>495</td>
<td>390</td>
<td>–</td>
<td>342</td>
<td>459</td>
<td>–</td>
<td>450</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$D_{14}$</td>
<td>792</td>
<td>720</td>
<td>675</td>
<td>639</td>
<td>–</td>
<td>576</td>
<td>576</td>
<td>558</td>
<td>540</td>
<td>513</td>
<td>522</td>
</tr>
<tr>
<td>$D_{15}$</td>
<td>378</td>
<td>693</td>
<td>648</td>
<td>612</td>
<td>594</td>
<td>198</td>
<td>558</td>
<td>540</td>
<td>531</td>
<td>522</td>
<td>513</td>
</tr>
<tr>
<td>$D_{16}$</td>
<td>531</td>
<td>513</td>
<td>495</td>
<td>477</td>
<td>460</td>
<td>459</td>
<td>459</td>
<td>450</td>
<td>450</td>
<td>441</td>
<td>441</td>
</tr>
<tr>
<td>$D_{17}$</td>
<td>–</td>
<td>4048</td>
<td>–</td>
<td>3712</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>3424</td>
<td>–</td>
<td>–</td>
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</tr>
<tr>
<td>$D_{18}$</td>
<td>544</td>
<td>1872</td>
<td>2400</td>
<td>1456</td>
<td>2368</td>
<td>576</td>
<td>2304</td>
<td>1744</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$D_{19}$</td>
<td>–</td>
<td>30260</td>
<td>–</td>
<td>–</td>
<td>–</td>
<td>18360</td>
<td>17595</td>
<td>17085</td>
<td>16575</td>
<td>16320</td>
<td>15810</td>
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<tr>
<td>$D_{20}$</td>
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<td>4048</td>
<td>3856</td>
<td>2384</td>
<td>–</td>
<td>2192</td>
<td>–</td>
<td>3424</td>
<td>–</td>
<td>3328</td>
<td>–</td>
</tr>
<tr>
<td>$D_{21}$</td>
<td>432</td>
<td>450</td>
<td>441</td>
<td>333</td>
<td>432</td>
<td>306</td>
<td>–</td>
<td>423</td>
<td>–</td>
<td>423</td>
<td>–</td>
</tr>
<tr>
<td>$D_{22}$</td>
<td>3164</td>
<td>450</td>
<td>441</td>
<td>441</td>
<td>432</td>
<td>292</td>
<td>–</td>
<td>423</td>
<td>–</td>
<td>423</td>
<td>–</td>
</tr>
<tr>
<td>$D_{23}$</td>
<td>1264</td>
<td>2400</td>
<td>2352</td>
<td>3040</td>
<td>2304</td>
<td>688</td>
<td>2256</td>
<td>2256</td>
<td>–</td>
<td>3184</td>
<td>–</td>
</tr>
</tbody>
</table>

For various matrices set of values have been tabulated which computes the eigenvalues with values less than 1.4 and also for the values greater than 1.4 in the above table. For some set of matrices, we compute the eigenvalues of the matrices at
step length \( h = 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.1, 1.2, 1.3, 1.4, 1.5 \) and total computations taken by all the step lengths have been tabulated in the above table.

**Conclusions:** - From the above table we observe that for all the matrices, all the step lengths except \( h = 1.0 \) takes more computations whereas at \( h = 1.0 \) it takes less computations compared to the other step lengths to compute the eigenvalues of the matrices. Hence we can conclude that for few class of matrices, to compute the eigenvalues very accurately with minimum computations \( h = 1.0 \) is preferred (based on our exhaustive experimentations) compared to the other step lengths. We have not obtained a constant step length since the step length depends upon on the numerical values of the system matrix and also the spread of the eigenvalues are not constant.

4.3 Determination of eigenvalues of the system matrix by applying the Bisection method, false position method and secant method at Gerschgorin bound without using steplength.

Consider a system matrix of order \((3\times3)\)

\[
A = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -1 & 2
\end{pmatrix}
\]

The Gerschgorin circle of the above matrix is drawn and is shown in figure 4.1

Fig 4.1: Gerschgorin bound \([0,4]\)
Eigenvalues obtained by applying the Bisection method at the Gerschgorin bound are
\[ \lambda_1 = 0.585786 \]
\[ \lambda_2 = 2 \]
\[ \lambda_3 = 3.414214 \]

Eigenvalues obtained by applying the false position method at the Gerschgorin bound are
\[ \lambda_1 = 0.585787 \]
\[ \lambda_2 = 3 \]
\[ \lambda_3 = 4 \]

Eigenvalues obtained by applying the Secant method at the Gerschgorin bound are
\[ \lambda_1 = 0.584961 \]
\[ \lambda_2 = 2 \]
\[ \lambda_3 = 3.145039 \]

Table: 4.3 Results based on Bisection method, false position method and Secant method applied at Gerschgorin bound.

<table>
<thead>
<tr>
<th>Methods</th>
<th>Eigenvalues</th>
<th>Total computations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bisection method</td>
<td>[ \lambda_1 = 0.585786 ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \lambda_2 = 2 ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \lambda_3 = 3.414214 ]</td>
<td></td>
</tr>
<tr>
<td>False position</td>
<td>[ \lambda_1 = 0.585787 ]</td>
<td></td>
</tr>
<tr>
<td>method</td>
<td>[ \lambda_2 = 3 ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \lambda_3 = 4 ]</td>
<td></td>
</tr>
<tr>
<td>Secant method</td>
<td>[ \lambda_1 = 0.584961 ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \lambda_2 = 2 ]</td>
<td></td>
</tr>
<tr>
<td></td>
<td>[ \lambda_3 = 3.145039 ]</td>
<td></td>
</tr>
</tbody>
</table>
Conclusions: We observe from the above, Secant method applied at the Gerschgorin bounds computes the eigenvalues accurately with minimum computations compared with the other two methods.

To compute the various types of eigenvalues of the matrix, the detailed theoretical studies with illustrative examples are given in Appendices I and II.

4.4 Common eigenvalues between two matrices

4.4.1 Computation of common eigenvalues between two matrices using Gerschgorin circles.

Given the linear matrix equation

$$AX + XB = C$$

Where $A$ and $B$ are square matrices of order is $(m \times m)$ and $(n \times n)$ respectively, this equation has a unique solution, if $A$ and $-B$ should not have common eigenvalues. Then this unique solution is given by

$$X = \int_0^\infty e^{At} C e^{Bt} dt$$

:. We have to identify the common eigenvalues between $A$ and $-B$. The existing method has computational problems. Hence we use Gerschgorin circles technique to identify the common eigenvalues.

In the Gerschgorin circles method, we draw the Gerschgorin circles of the matrix $A$ and also the Gerschgorin circles of the matrix $B$. The area bounded by the Gerschgorin circles drawn from matrix $A$ are superimposed with the area that is bounded with that of matrix $B$. The resulting area obtained is the common area where the common eigenvalues of both the matrices $A$ and $B$ lie. To check whether or not the common eigenvalues exist, compute determinant $(\lambda_i I - A)$ and also determinant $(\lambda_i I - B)$ at the end of the Gerschgorin bounds. If there exist change in sign of both determinants, then there exists common eigenvalues between $A$ and $B$. To compute the common eigenvalues Bisection method has to be applied.
Examples are given to illustrate the above theory:

Example 4.4.1:

Consider the system matrix of $A$ of order (5x5)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(5x5)

Gerschgorin circle of the above matrix $A$ is drawn and is shown in figure 4.2.

Fig 4.2: Gerschgorin bound [-1, 1]

Eigenvalues of the above matrix $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 0$, $\lambda_4 = 0$, $\lambda_5 = 1$

Consider the system matrix of $B$ of order (5x5)
Gerschgorin circle of the above matrix is drawn and is shown in figure 4.3

\[ A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 2 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 5 & 0 \\
0 & 0 & 0 & 4 & 0 \\
\end{pmatrix} \text{ (5x5)} \]

Eigenvalues of the above matrix \( \lambda_1 = -1, \ \lambda_2 = 0, \ \lambda_3 = 1, \ \lambda_4 = 1, \ \lambda_5 = 2 \)

Gerschgorin circles of the common area of the Gerschgorin circles of matrix \( A \) and matrix \( B \) are drawn as shown in figure 4.36

Fig 4.3: Gerschgorin bound \([-2, 5]\)
The common eigenvalues in both matrices are $\lambda_1 = 0, \lambda_2 = 1$

**Conclusions:** When we superimpose the areas of the individual matrices $A$ and $B$, common area is reduced and hence the bound also. Therefore it becomes easier to find the common eigenvalues in the smaller bound.

**4.4.2 Applications:** To find the fixed modes in Decentralized Control systems.

**Definition 1:** A linear time invariant multivariable system described by

$$x(t) = A x(t) + \sum_{i=1}^{n} B_i U_i (t) \quad \Rightarrow (4.4.2.1)$$

$$y_i (t) = C_i x(t) \text{ where } i = 1, 2, ..., n \quad \Rightarrow (4.4.2.2)$$

called N station decentralized system.

**Definition 2:** Given the above system (4.4.2.1) and (4.4.2.2) if we define a set of $\mathcal{F}$ of block diagonal matrices.

$$\mathcal{F} = \left\{ F* \in \mathcal{F} = \text{block diagram ( } F_1, F_2, \ldots, F_n ) \mid F_i \in \mathbb{R} \right\}$$
Then the set of decentralized fixed modes of (1) w.r.t \( \mathcal{F} \) defined as
\[
\Lambda(A,B_iC_iF) = \bigcap_{F \in \mathcal{F}} \sigma(A + \sum_{i=1}^{n} B_iC_iF_i)
\]

In 1970’s and 1980’s there had been considerable interest in the study of decentralized control of large scale linear multivariable systems. The structure of a decentralized controller is an important issue in the control of large systems. This is because of existence of decentralized fixed modes proposed by Wang and Davidson in 1973. Decentralised fixed modes are those nodes of the system which are invariant under the implementation of all decentralizes controllers having a particular structure. In stabilization of decentralized control systems, computations of fixed modes is playing important role. Since the fixed modes should belong to left open half of the s-plane for the system to be stable.

Here, our object is to compute the eigenvalues of \( A \) and \( A_F = A + BFC \) where \( A \) is the system matrix, \( B \) is the input matrix and \( C \) is the output matrix of decentralized systems. And where \( F \) is a Decentralised feedback matrix and is chosen randomly. We draw the Gerschgorin circles for both the matrices \( A \) and \( A_F \) respectively. Then we superimpose the Gerschgorin circles of \( A \) and \( A_F \) to get the common area. In that common area we check whether the eigenvalues belong to \( A \) and \( A_F \) if it is common value. That eigenvalue will be fixed mode. Since we use a graphical method it saves lots of computation over the existing method. Another advantage of using the graphical method is that, generally fixed modes are few in numbers one or two for large scale decentralized systems. So our method will be easier to do this computation of eigenvalues. Currently, to compute accurate fixed modes of a decentralized system, EISPACK subroutines were used to compute the eigenvalues of \( A \) and \( A_F \) with the single precision and double precision accurately and then find the common eigenvalues between the matrices \( A \) and \( A_F \). But this method will take lot of computations. [32].
Illustrative examples:

Application 1:

Consider the following two Channel scalar Decentralized control system whose state variable equations are

\[
\dot{X} = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix} X + \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 1
\end{pmatrix} U
\]

\[
Y = \begin{pmatrix}
0 & 0 & 1 \\
-1 & 1 & 0
\end{pmatrix} X
\]

Choose Feedback matrix \( F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \)

Select randomly \( f_1 = 2 \) and \( f_2 = 1 \)

Then

\[
A_F = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 2 \\
-1 & 1 & 1
\end{pmatrix}
\]

Now we shall draw the Gerschgorin circle for the matrix \( A \)
Gerschgorin circle of above is drawn and is as shown in figure 4.5

Eigenvalues of the above matrix are $\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 1$

Now consider the $A_F$ matrix

$$A_F = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix}$$

Gerschgorin circle of the above matrix is drawn and is shown in figure 4.6
Eigenvalues of the above matrix are $\lambda_1 = 1.4142$  $\lambda_2 = 1$  $\lambda_3 = -1.4142$

Common area obtained by overlapping above two circles are drawn is shown in figure 4.7

Fig 4.7: Gerschgorin bound $[-1,1]$
Common eigenvalues of $A$ and $A_f$ is $\lambda = 1$

**Application 2:**

Consider a two channel decentralized control system with single input single output for the first channel and single input and two outputs for the second channel. whose state variable equations are:

$$
\dot{X} = 
\begin{pmatrix}
1 & 2 & 0 \\
3 & 4 & 3 \\
0 & 2 & 1
\end{pmatrix}
X + 
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
1 & 0
\end{pmatrix}
U
$$

$$
Y = 
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
X
$$

Where the feedback matrix

$$
f = 
\begin{pmatrix}
f_{11} & 0 & 0 \\
0 & f_{22} & f_{23}
\end{pmatrix}
$$

Select $F = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 2 & 3
\end{pmatrix}
$

$$
A + BFC = 
\begin{pmatrix}
1 & 2 & 0 \\
f_{22} + 3 & 4 & f_{23} + 3 \\
0 & f_{11} + 2 & 1
\end{pmatrix}
$$
Now consider the matrix $A$

$$
A = \begin{pmatrix}
1 & 2 & 0 \\
3 & 4 & 3 \\
0 & 2 & 1
\end{pmatrix}
$$

Gerschgorin circle of the above matrix is drawn and is shown in figure 4.8

Fig 4.8: Gerschgorin bound [-2, 8]

$$
A_F = \begin{pmatrix}
1 & 2 & 0 \\
5 & 4 & 6 \\
0 & 3 & 1
\end{pmatrix}
$$

Gerschgorin circles of the above matrix are drawn as shown in figure 4.9
Common area obtained by overlapping above two circles are drawn as shown in figure 4.10.

Common eigenvalues of $A$ and $A_F$ are $\lambda = 1$

**Conclusion:** We observe that the Gerschgorin circles to identify the common eigenvalues if the overlapping area is the Gerschgorin area of one of the circles of the matrix $A$ or matrix $B$. and in sometimes, the bound reduces which takes less computation.