Chapter 4

The $< t >$-property

The question of determining better upper bounds for the clique transversal number dates back to 1990 when Tuza Z. introduced the concept of the clique transversal number [74]. Erdős et.al. [33] determined various upper bounds for the clique transversal number. In an attempt to find graphs which admit a better upper bound, Tuza Z. [74] introduced the concept of the $< t >$-property. Motivated by the open problems mentioned in [33], we studied the $< t >$-property of the cographs, the clique perfect graphs, the perfect graphs, the planar graphs and the trestled graphs of index $k$. In the last section, an open problem on highly clique imperfect graphs is solved.

Some results of this chapter are included in the following paper.

4.1 Clique transversal number

In this section we prove that the domination number is a lower bound for the clique transversal number, but the difference can be arbitrarily large.

**Theorem 4.1.1.** Every clique transversal set is a dominating set.

**Proof.** Let $S$ be a clique transversal set of a graph $G$ and $v \in V(G)$. If $v \in S$ then it is dominated by $S$. If $v \notin S$ then let $C$ be a clique which contains $v$. Since, $S$ is a clique transversal set, there exist a vertex $u \in S \cap C$. But then, $u$ dominates $v$. Therefore, $S$ is a dominating set. 

**Corollary 4.1.2.** Let $G$ be a graph. Then, $\gamma(G) \leq \tau_c(G)$.

**Theorem 4.1.3.** Let $a$ and $b$ be two positive integers such that $2 \leq a \leq b$. There exists a clique perfect graph $G$ such that $\gamma(G) = a$ and $\tau_c(G) = b$.

**Proof.** Let $G$ be the graph obtained from $K_{b,b}$ by attaching $a - 1$ end vertices to $a - 1$ distinct vertices in any one of the partitions of $G$.

To dominate the $a - 1$ end vertices, at least $a - 1$ vertices are required and those vertices cannot dominate the remaining vertices (there exists at least one such vertex, since $b \geq a$) of that partition. Therefore, $\gamma(G)$ is at least $a$. Again, the $a - 1$ distinct neighbors of the $a - 1$ end vertices together with one vertex from the other partition of $K_{b,b}$ dominates $G$. Therefore, $\gamma(G) = a$.

The graph $G$ so constructed is bipartite and hence the only cliques are the edges of $G$. If we take all the $b$ vertices in the partition of $K_{b,b}$ to which end vertices are attached, then that set forms a clique transversal. Therefore, $\tau_c(G) \leq b$. Again, if
we take the $b$ independent edges of $K_{b,b}$, it forms a clique independent set of size $b$. Therefore, $b \leq \alpha_c(G) \leq \tau_c(G)$. Hence, $\tau_c(G) = b$.

Also, since $\alpha_c(G) = \tau_c(G) = b$, $G$ is clique perfect.

Illustration

![Graph Image]

For the graph $G$ in Fig: 4.1, $\gamma(G) = 3$ and $\alpha_c(G) = \tau_c(G) = 4$.

4.2 Cographs and clique perfect graphs

In this section we study the $< t >$-property of cographs and clique perfect graphs. A characterization for cographs and clique perfect graphs which attain maximum value for the clique transversal number is also obtained.

Lemma 4.2.1. If $G = G_1 \lor G_2$ then $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$.

Proof. Any clique in $G$ is of the form $H_1 \lor H_2$ where $H_1$ is a clique in $G_1$ and $H_2$ is a clique in $G_2$. If $V'$ is a clique transversal of $G_1$ (or $G_2$), then any clique of $G$, which contains a clique of $G_1$ (or $G_2$), is covered by $V'$ and hence $V'$ is a clique transversal of $G$ also.
Chapter 4: The $< t >$-property

Now, let $V'$ be a clique transversal of $G$. If possible assume that $V'$ does not cover cliques of $G_1$ and $G_2$. Let $H_1$ and $H_2$ be the cliques of $G_1$ and $G_2$ respectively which are not covered by $V'$. Then $H_1 \lor H_2$ is a clique of $G$ which is not covered by $V'$, which is a contradiction. Hence $V'$ contains a clique transversal of $G_1$ or $G_2$.

Therefore, $\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}$. □

Lemma 4.2.2. The class of all cographs without isolated vertices does not satisfy the $< t >$-property for $t \geq 4$.

Proof. The proof is by construction.

Case 1: $t = 4$

Let $G = G_1 \lor G_2$, where $G_1 = (3K_1 \cup K_2) \lor (3K_1 \cup K_2)$ and $G_2 = (3K_1 \cup K_2)$. Then $n = 15, t = 4$ and $\tau_c(G) = 4$ which implies that $\frac{n}{t} < \tau_c(G)$.

Case 2: $t > 4$

Let $G = G_1 \lor G_2$, where $G_1 = (3K_1 \cup K_{t-3}) \lor (3K_1 \cup K_{t-3})$ and $G_2 = (3K_2 \cup K_{t-2})$. Then $n(G) = 3t + 4$ and $\tau_c(G) = 4$.

Every edge in $G_1$ lies in a complete of size $t$ in $G$ since $G_2$ contains a clique of size $t - 2$. Every edge in $G_2$ lies in a complete of size $t$ for $t \geq 4$ in $G$ since $G_1$ contains a clique of size $2t - 6$. An edge with one end vertex in $G_1$ and the other end vertex in $G_2$ lies in a complete of size $t$ since every vertex in $G_1$ lies in a complete of size $t - 2$ and every vertex of $G_2$ lies in a complete of size 2. Hence $G$ is a cograph in which every edge lies in a clique of size $t$. 
Also, $\frac{n}{t} = 3 + \frac{4}{t}$.

Therefore, $\frac{n}{t} < \tau_c(G)$ for $t > 4$. \hfill $\square$

**Theorem 4.2.3.** The class of clique perfect graphs without isolated vertices satisfies the $< t >$-property for $t = 2$ and $3$ and does not satisfy the $< t >$-property for $t \geq 4$.

**Proof.** Let $G$ be a clique perfect graph in which every edge lies in a complete of size $t$. $G$ being clique perfect, $\tau_c(G) = \alpha_c(G)$.

Case 1: $t = 2$

Since $G$ is without isolated vertices $\alpha_c(G) \leq \frac{n}{2}$. So $\tau_c(G) = \alpha_c(G) \leq \frac{n}{2}$ and hence the class of clique perfect graphs satisfies the $< 2 >$-property.

Case 2: $t = 3$

Every edge of $G$ lies in a clique of size 3. So, the size of the smallest clique of $G$ is 3. Therefore, $\alpha_c(G) \leq \frac{n}{3}$ and $\tau_c(G) = \alpha_c(G) \leq \frac{n}{3}$.

Case 3: $t \geq 4$

The class of cographs is a subclass of clique perfect graphs (Lemma 1.1.8). So by Lemma 4.2.2, the claim follows. \hfill $\square$

**Corollary 4.2.4.** The class of cographs without isolated vertices satisfies the $< t >$-property for $t = 2$ and $3$. Moreover, for the class of connected cographs without isolated vertices, $\tau_c(G)$ is maximum if and only if $G$ is the complete bipartite graph $K_{\frac{n}{2}, \frac{n}{2}}$.

**Proof.** Since the class of cographs is a subclass of clique perfect graphs (Lemma
1.1.8), it satisfies the $< t >$-property for $t = 2$ and 3.

Since the class of cographs satisfy the $< 2 >$-property and $\tau_c(K_{\frac{n}{2}, \frac{n}{2}}) = \frac{n}{2}$, the maximum value of $\tau_c(G)$ is $\frac{n}{2}$. Conversely, let $G$ be a connected cograph with $\tau_c(G) = \frac{n}{2}$. Since $G$ is a connected cograph $G = G_1 \cup G_2$. Therefore, 
\[
\tau_c(G) = \min\{\tau_c(G_1), \tau_c(G_2)\}. 
\]
But, both $\tau_c(G_1)$ and $\tau_c(G_2)$ cannot exceed the number of vertices in $G_1$ and $G_2$ respectively and hence the number of vertices in $G_1$ and $G_2$ must be $\frac{n}{2}$. Again, since $\tau_c(G) = \frac{n}{2}$ all these vertices must be isolated. Therefore, $G = K_{\frac{n}{2}, \frac{n}{2}}$.

**Corollary 4.2.5.** For the class of clique perfect graphs without isolated vertices, $\tau_c(G)$ is maximum if and only if there exists a perfect matching in $G$ in which no edge lies in a triangle.

**Proof.** The class of clique perfect graphs without isolated vertices satisfies the $< 2 >$-property. Therefore, the maximum value that $\tau_c(G)$ can obtain is $\frac{n}{2}$. Let $G$ be a clique perfect graph with $\tau_c(G) = \frac{n}{2}$. $G$ being clique perfect, $\alpha_c(G) = \tau_c(G) = \frac{n}{2}$. Since each clique must have at least two vertices and there are $\frac{n}{2}$ independent cliques, the cliques are of size exactly two. Again, this independent set of $\frac{n}{2}$ cliques forms a perfect matching of $G$ and a clique being maximal complete, the edges of this perfect matching do not lie in triangles.

Conversely, if there exists a perfect matching in which no edge lies in a triangle, the edges of this perfect matching form an independent set of cliques with cardinality $\frac{n}{2}$. Therefore, $\alpha_c(G) \geq \frac{n}{2}$. But, $\alpha_c(G) \leq \tau_c(G) \leq \frac{n}{2}$ and therefore $\tau_c(G) = \frac{n}{2}$.
4.3 Planar graphs

Theorem 4.3.1. The class of planar graphs does not satisfy the $< t >$-property for $t = 2, 3$ and $4$ and $G_t$ is empty for $t \geq 5$.

Proof. Every odd cycle is a planar graph and $\tau_c(C_{2k+1}) = k + 1 > \frac{2k+1}{2}$. Clearly, odd cycles belong to $G_2$ and hence the class of planar graphs does not satisfy the $< 2 >$-property.

Fig : 4.2

The graph in Fig : 4.2 is planar and every edge lies in a triangle. Here, $n = 8$ and the clique transversal number is $3$ which is greater than $\frac{8}{3}$ and hence planar graphs do not satisfy the $< 3 >$-property.

Fig : 4.3
Chapter 4: The $< t >$-property

The graph in Fig: 4.3 is planar and every edge lies in a $K_4$. Here, $n = 15$ and the clique transversal number is 4 which is greater than $\frac{n}{4}$ and hence planar graphs do not satisfy the $< 4 >$-property.

Since $K_5$ is a forbidden subgraph for planar graphs, there is no planar graph $G$ such that all its edges lie in a $K_t$ for $t \geq 5$. Hence, the theorem. \hfill \Box

4.4 Perfect graphs

Theorem 4.4.1. The class of perfect graphs does not satisfy the $< t >$-property for any $t \geq 2$.

Proof. Let $G$ be the cycle of length $3k$, say $v_1v_2, ..., v_{3k}v_1$ where $k > 2$ is odd, in which the vertices $v_1, v_4, ..., v_{3k-2}$ are all adjacent to each other. Then $G$ is perfect and $\tau_c(G) = \lceil \frac{3k}{2} \rceil > \frac{3k}{2}$, since $3k$ is odd. Therefore the class of perfect graphs does not satisfy the $< 2 >$-property.

Now, the class of perfect graphs does not satisfy the $< 3 >$-property since $\overline{C_8}$ is a perfect graph (Lemma 1.1.6) in which every edge lies in a triangle and $\tau_c(\overline{C_8}) = 3 > \frac{8}{3}$.

Since the cographs are a subclass of perfect graphs (Lemma 1.1.7) [27], by Lemma 4.2.2, the class of perfect graphs also does not satisfy the $< t >$-property for $t \geq 4$. \hfill \Box
Chapter 4 : The $< t >$-property

4.5 Trestled graph of index $k$

In this section the clique transversal number and the clique independence number of $T_k(G)$ are determined. A characterization of $G$ for which $T_k(G)$ satisfies the $< 2 >$-property is also given.

**Lemma 4.5.1.** For any graph $G$ without isolated vertices, $\tau_c(T_k(G)) = km + \beta(G)$.

**Proof.** We shall prove the theorem for the case $k = 1$.

Let $V' = \{v_1, v_2, \ldots, v_t\}$ be a vertex cover of $G$. The cliques of $T_1(G)$ are precisely the cliques of $G$ together with the three $K_2$ s formed corresponding to each edge of $G$. Corresponding to each edge $uv$ of $G$ choose the vertex which corresponds to $u$ of the corresponding $K_2$, if $u$ is not present in $V'$. If $u$ is present in $v'$ then, choose the vertex corresponding to $v$, irrespective of $v$ is present in $V'$ or not. Let this new collection together with $V'$ be $V''$. Then $V''$ is a clique transversal of $T_1(G)$ of cardinality $m + \beta(G)$. Therefore, $\tau_c(T_1(G)) \leq m + \beta(G)$.

Let $V' = \{v_1, v_2, \ldots, v_t\}$, where $t = \tau_c(T_1(G))$ be a clique transversal of $T_1(G)$. Let $uv$ be an edge in $G$ and let $u'v'$ be the $K_2$ introduced in $T_1(G)$ corresponding to this $K_2$. At least one vertex from $\{u', v'\}$, say $u'$ must be present in $V'$, since $V'$ is a clique transversal and $u'v'$ is a clique of $T_1(G)$. Remove $u'$ from $V'$. If $V'$ contains $v'$ also then replace $v'$ by $v$. If $v' \notin V'$ then $v \in V'$. Since $V'$ is a clique transversal and $vv'$ is a clique of $T_1(G)$. In either case, one vertex $v$ of the edge $uv$ is present in the new collection. Repeat the process for each edge in $G$ to get $V''$. Clearly, $V''$ is a vertex cover of $G$ with cardinality $\tau_c(T_1(G)) - m$. Hence, $\beta(G) \leq \tau_c(T_1(G)) - m$. Thus, $\tau_c(T_1(G)) = m + \beta(G)$. 
Chapter 4: The $< t >$-property

By a similar argument we can prove that $\tau_c(T_k(G)) = km + \beta(G)$. □

**Notation:** For a given class $\mathcal{G}$ of graphs, let $T_k(\mathcal{G}) = \{ T_k(G) : G \in \mathcal{G} \}$.

**Theorem 4.5.2.** The class $T_k(\mathcal{G})$ satisfies the $< 2 >$-property if and only if $\beta(G) \leq \frac{n}{2}$ $\forall$ $G \in \mathcal{G}$ and $(T_k(\mathcal{G}))_t$ is empty for $t \geq 3$.

**Proof.** Let $G \in \mathcal{G}$. $n(T_k(G)) = n + 2km$ and by Lemma 4.4.1, $\tau_c(T_k(G)) = km + \beta(G)$. Therefore,

$$\tau_c(T_k(G)) \leq \frac{n(T_k(G))}{2} \iff km + \beta(G) \leq \frac{n + 2km}{2} \iff \beta(G) \leq \frac{n}{2}.$$

Hence, $T_k(\mathcal{G})$ satisfies $< 2 >$-property if and only if $\beta(G) \leq \frac{n}{2}$ $\forall$ $G \in \mathcal{G}$.

If $G$ contains at least one edge then $T_k(G)$ has a clique of size 2 and hence $(T_k(\mathcal{G}))_t$ is empty for $t \geq 3$. □

**Lemma 4.5.3.** For any graph $G$ without isolated vertices, $\alpha_c(T_k(G)) = km(G) + \alpha'(G)$.

**Proof.** We shall prove the theorem for the case $k = 1$.

Let $E' = \{ e_1, e_2, ..., e_m \}$ be a maximum matching of $G$ with cardinality $\alpha'(G)$. Let $C_1 = \{ e_{i1}, e_{i2}, e_{i2}, ..., e_{\alpha'1, \alpha'2} \}$ where each $e_{ij}$ for $i = 1, 2, ..., \alpha'$ are the edges which join $e_i$ to the corresponding $K_2$ of $T_1(G)$. Note that each $e_{ij}$ is a clique for $i = 1, 2, ..., \alpha'$ and $j = 1, 2$. Let $C_2 = \{ f_1, f_2, ..., f_{m-\alpha'} \}$ be the $K_2$s in $T_1(G)$ corresponding to the edges of $E - E'$. Also, each $f_i$ is a clique in $T_1(G)$ for $i = 1, 2, ..., m - \alpha'$. Therefore, $C_1 \cup C_2$ is a set of independent cliques of $T_1(G)$ with cardinality $2\alpha'(G) + (m(G) - \alpha'(G)) = m(G) + \alpha'(G)$. Hence, $\alpha_c(T_1(G)) \geq m(G) + \alpha'(G)$.
Chapter 4: The \(<t>-property

Let \( S = \{C_1, C_2, \ldots, C_{\alpha_c}\} \) be a set of independent cliques of \( T_1(G) \) with cardinality \( \alpha_c(T_1(G)) \). Let

\[
S_1 = \{C_i : V(C_i) \subseteq V(G)\};
\]
\[
S_2 = \{C_i : \exists C_j \text{ with } V(C_i) \cap V(G) = \{u\}, V(C_j) \cap V(G) = \{v\} \text{ where } uv \in E(G)\},
\]
\[
S_3 = S - (S_1 \cup S_2)
\]

Note that \(|S_2|\) is always even and the elements of \( S_2 \) can be paired into \((C_i, C_j)\) which satisfy the required property.

Choose one edge from each clique in \( S_1 \) and the edge \( uv \) corresponding to each pair \((C_i, C_j)\) in \( S_2 \) to get an independent set of edges \( E' \subseteq E(G) \). Now, \(|S_3|\) cannot exceed \( m(G) \) and \(|S| = \alpha_c(T_1(G)) \). Therefore, \(|E'| \geq \alpha_c(T_1(G)) - m(G)\). Hence, \( \alpha'(G) \geq \alpha_c(T_1(G)) - m(G) \) and so \( \alpha_c(T_1(G)) \leq m(G) + \alpha'(G) \). Thus, \( \alpha_c(T_1(G)) = m(G) + \alpha'(G) \).

By a similar argument we can prove that \( \alpha_c(T_k(G)) = km(G) + \alpha'(G) \). \( \square \)

**Theorem 4.5.4.** \( T_k(G) \) is a clique perfect graph if and only if \( G \) is a bipartite graph.

**Proof.** Let \( T_k(G) \) be a clique perfect graph. From Lemma 4.5.1 and Lemma 4.5.3, \( \tau_c(T_k(G)) = \alpha_c(T_k(G)) \) if and only if \( \beta(G) = \alpha'(G) \). If \( H \) is an induced subgraph of \( G \) then \( T_k(H) \) is an induced subgraph of \( T_k(G) \) and hence for \( T_k(G) \) to be clique-perfect, \( \beta(H) = \alpha'(H) \) for every induced subgraph \( H \) of \( G \). If \( G \) contains an induced odd cycle of length \( 2k + 1, k \geq 1 \), then \( k + 1 = \beta(C_{2k+1}) \neq \alpha'(C_{2k+1}) = k \), which is a contradiction. Therefore, \( G \) is bipartite.

Now, let \( G \) be bipartite. Then \( T_k(G) \) is bipartite for each \( k \), since \( T_k(G) \) contains an odd cycle if and only if \( G \) contains an odd cycle. For bipartite graphs, the clique
transversal number is same as the minimum number of vertices required to cover all edges and the clique independence number is same as the maximum number of independent edges, since all cliques are of size two. Hence by Lemma 1.1.13 and the fact that each induced subgraph of a bipartite graph is bipartite, it follows that $T_k(G)$ is clique perfect.

The $< t >$-property of the various classes of graphs which we have studied in this chapter are summarized in the following table.

<table>
<thead>
<tr>
<th>Class</th>
<th>Satisfy $&lt; t &gt;$-property</th>
<th>Do not satisfy $&lt; t &gt;$-property</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cographs</td>
<td>2, 3</td>
<td>$\geq 4$</td>
</tr>
<tr>
<td>Clique perfect graphs</td>
<td>2, 3</td>
<td>$\geq 4$</td>
</tr>
<tr>
<td>Planar graphs</td>
<td>-</td>
<td>2, 3, 4</td>
</tr>
<tr>
<td>Perfect graphs</td>
<td>-</td>
<td>$\geq 2$</td>
</tr>
</tbody>
</table>

### 4.6 Highly clique imperfect graphs

A graph $G$ is highly clique imperfect if the difference between $\tau_c(G)$ and $\alpha_c(G)$ is arbitrarily large. In [32], a graph $F_t$ satisfying $\tau_c(F_t) - \alpha_c(F_t) = t$, where $t$ is an arbitrary integer is given where the number of vertices in $F_t$ grows exponentially with $t$. However, the following problem is open [73]:

**Problem**: For an arbitrary integer $t$, are there graphs $G$ such that $\tau_c(G) - \alpha_c(G) = t$ where the number of vertices in $G$ is linear in $t$.

In this section, this problem is solved by constructing a family of such graphs.
For each positive integer $t$, define $G_t$ as $K_{1,t+1}$ with 5-cycles attached to $t$ distinct pendant vertices of $K_{1,t+1}$ (Fig: 4.4).

Fig: 4.4

Then $\tau_c(G_t) = 3t + 1$ and $\alpha_c(G_t) = 2t + 1$ so that $\tau_c(G_t) - \alpha_c(G_t) = t$ and the size of $G_t$ is $5t + 2$.

More generally, if $G_{k,t}$ is the graph obtained by replacing the 5-cycles in this example by any odd cycle $C_{2k+1}$, then $\tau_c(G_{k,t}) = (k + 1)t + 1, \alpha_c(G_{k,t}) = kt + 1$ and the number of vertices in $G_{k,t}$ is $(2k + 1)t + 2$ which is also polynomially bounded in $t$. 