Chapter 3

Domination in Graph Classes

In [9], Bacsó and Tuza Z. put forward the following problem.

**Problem**: Let \( P \) be a property of vertex sets in a graph. Characterize all graphs having a dominating set satisfying the property \( P \).

Based on various properties of the vertex set, many domination parameters were introduced and studied. For a detailed study of various domination parameters, the reader may refer to [44].

Inspired by the above problem, in this chapter we define two new domination parameters, cographic domination number \( \gamma_{l_{cd}}(G) \) and global cographic domination number \( \gamma_{g_{cd}}(G) \) based on cographs and some of its properties are discussed.

Some results of this chapter are included in the following paper.

Cographic and global cographic domination number of a graph, Ars Combin., (to appear).
3.1 Cographic domination number

In this section, given any graph $G$, we prove the existence of a cographic dominating set. The relationship between $\gamma$, $\gamma_{cd}$ and $\gamma_i$ of a tree is studied.

**Theorem 3.1.1.** For any graph $G$, there exists a dominating induced subgraph which is a cograph.

**Proof.** The proof is by induction on $n$. For $n \leq 3$, the theorem can be easily verified. Assume that it is true for all graphs with at most $n$ vertices.

Let $G$ be a graph with $n + 1$ vertices. By induction assumption, the graph $G - v$ has a dominating induced subgraph $H$ which is a cograph. If at least one of the vertices in $H$ is adjacent to $v$, then $H$ is a dominating induced subgraph for $G$. If not, $H \cup \{v\}$ is a dominating induced subgraph of $G$ which is also a cograph. Therefore by induction, the theorem is true for all graphs.

**Note:** For any graph $G$, $\gamma(G) \leq \gamma_{cd}(G) \leq \gamma_i(G)$. However, there are graphs with $\gamma(G) < \gamma_{cd}(G) < \gamma_i(G)$. For e.g.:

![Graph](image)

$\gamma(G) = 4$, $\gamma_{cd}(G) = 5$ and $\gamma_i(G) = 6$. 
Lemma 3.1.2. If $T$ is a tree with $\gamma(T) < \gamma_{cd}(T)$, then $T$ must have the graph $G$ in Fig: 3.1 as an induced subgraph.

Proof. Since $\gamma(T) < \gamma_{cd}(T)$, in every dominating set $D$ with cardinality $\gamma(T)$ there exists an induced $P_4 : u_1u_2u_3u_4$. Since $D$ is minimal dominating and $u_i$ for $i = 1,2,3,4$ is adjacent to at least one vertex in the dominating set, there exists at least one $v_i$ in the vertex set of $T$ corresponding to each $u_i$ such that $v_i$ is adjacent only to $u_i$ in $D$ for each $i = 1,2,3,4$. If for one of these $i$, $v_i$ is the only such neighbor of $u_i$ then we can replace $u_i$ by $v_i$ for that $i$ in the dominating set to remove the induced $P_4$ without changing the cardinality. Therefore, there exists at least one induced $P_4$ in $T$ such that each of its vertices is adjacent to a pair of vertices. These twelve vertices together induce the required graph. □

Corollary 3.1.3. For any graph $G$ with less than twelve vertices, $\gamma(G) = \gamma_{cd}(G)$.

Proof. If $G$ has less than twelve vertices, then $G$ cannot have the graph in Fig: 3.1 as an induced subgraph. Hence, $\gamma(G) = \gamma_{cd}(G)$. □

Lemma 3.1.4. If $T$ is a tree with $\gamma_{cd}(T) < \gamma_{i}(T)$, then $T$ has the following graph as an induced subgraph.

![Fig: 3.2](image)

Proof. Since $\gamma_{cd}(T) < \gamma_{i}(T)$, every cographic dominating set $D$ with cardinality $\gamma_{cd}(T)$ will have at least one pair of adjacent vertices, say $u v$. Since $u$ and $v$ are mutually dominating, there exist at least two vertices $u_1$ and $v_1$ in $T$ which are
adjacent only to \( u \) and \( v \), respectively. If these are the only such vertices then we can replace \( u \) by \( u_1 \) or \( v \) by \( v_1 \) in \( T \) to remove the adjacency in \( D \) without affecting the cardinality. Therefore, there exist at least one pair of vertices in \( D \) which has at least two neighbors of their own. These six vertices induce the required graph. \( \square \)

**Corollary 3.1.5.** For any graph \( G \) with less than six vertices, \( \gamma_{cd}(G) = \gamma_i(G) \).

**Proof.** If \( G \) has less than six vertices, then \( G \) cannot have the graph in Fig: 3.2 as an induced subgraph. Hence, \( \gamma_{cd}(G) = \gamma_i(G) \). \( \square \)

**Theorem 3.1.6.** There is no tree \( T \) which satisfies \( \gamma(T) < \gamma_{cd}(T) = \gamma_i(T) \).

**Proof.** If possible assume that there is a tree \( T \) which satisfies \( \gamma(T) < \gamma_{cd}(T) = \gamma_i(T) \). Let \( D \) be a minimal dominating set of cardinality \( \gamma(T) \). Since \( \gamma(T) < \gamma_{cd}(T) \), by Lemma 3.1.2, \( T \) must contain the graph in Fig: 3.1 as an induced subgraph and the vertices which induce a \( P_4 \) in it must be present in \( D \). Also, none of the vertices of this \( P_4 \) can be replaced without affecting the domination property and without increasing the cardinality of \( D \). To make \( D \) a cographic dominating set, only one vertex is to be replaced, whereas to make \( D \) an independent dominating set, two of the vertices are to be replaced. Since \( D \) is arbitrary, \( \gamma_{cd}(T) < \gamma_i(T) \) which is a contradiction. Hence, the theorem. \( \square \)

### 3.2 Global cographic domination number

We prove the existence of a global cographic dominating set for every graph \( G \) and study the relation between \( \gamma_{cd}(G) \) and \( \gamma_{gcd}(G) \) of various special classes of graphs in this section.
Theorem 3.2.1. Given any graph $G = (V, E)$, there exists a cographic dominating set which dominates $G^c$ also.

Proof. If $D$ is a cographic dominating set in $G$ which dominates $G^c$ also, then there is nothing to prove. Otherwise, there exists at least one vertex, say $v_1$ which is not adjacent to any vertex of $D$ in $G^c$. Adjoin $v_1$ to $D$. If $D \cup \{v_1\}$ does not dominate $G^c$, then there exist a $v_2$ which is not adjacent to any vertex of $D \cup \{v_1\}$ in $G^c$. Adjoin $v_2$ to $D \cup \{v_1\}$. Continue this process until we get a dominating set $D' = D \cup \{v_1, v_2, ..., v_k\}$ which dominates $G^c$. The process will eventually terminate, since $V$ dominates $G^c$. The subgraph induced by $D'$ in $G$ is the join of the subgraph induced by $D$ in $G$ with $K_p$, for some $p$. Therefore, the subgraph induced by $D'$ is also a cograph by the choice of $D$ and since $D \subseteq D'$, $D'$ dominates $G$. Therefore, $D'$ is a cographic dominating set in $G$ which dominates $G^c$ also. $\square$

Note: For any graph $G$, $\gamma_{gcd}(G) \geq \max\{\gamma_{cd}(G), \gamma_{cd}(G^c)\}$.

Lemma 3.2.2. For any graph $G \neq K_1$, $\gamma_{gcd}(G) > 1$.

Proof. If $\gamma_{gcd}(G) = 1$, then $\gamma_{cd}(G) = 1$. Then $G$ has a vertex of full degree and so $G^c$ has an isolated vertex. Therefore, $\gamma_{cd}(G^c) > 1$ and so $\gamma_{gcd}(G) < \gamma_{cd}(G^c)$. This is a contradiction and hence $\gamma_{gcd}(G) > 1$. $\square$

Theorem 3.2.3. If $G$ is a triangle free graph, then $\gamma_{gcd}(G) = \gamma_{cd}(G)$ or $\gamma_{cd}(G) + 1$.

Proof. Let $\gamma_{gcd}(G) \neq \gamma_{cd}(G)$. Let $D$ be a minimum cographic dominating set. Since none of the minimum cographic dominating sets dominate $G^c$, at least one vertex $v$ of $G$ must be adjacent to all the vertices of $D$. Consider $D \cup \{v\}$. Since the graph is triangle free, none of the neighbors of the vertices of $D$ are adjacent
to $v$. Since $D$ is dominating, every vertex of $G$ is either in $D$ or is adjacent to a
vertex of $D$. Therefore, the only neighbors of $v$ are those present in $D$. Hence,
in $G^c$, $v$ dominates all the vertices outside $D$. Also, $D \cup \{v\}$ induces a cograph.
Thus, $D \cup \{v\}$ is a cographic dominating set in $G$ as well as $G^c$, of cardinality
$\gamma_{cd}(G) + 1$. \hfill \Box$

**Remark 3.2.1.** There are graphs for which $\gamma_{gcd}(G) = \gamma_{cd}(G)$ and $\gamma_{gcd}(G) = \gamma_{cd}(G) + 1$. For example, $\gamma_{cd}(C_4) = \gamma_{gcd}(C_4) = 2$, whereas $\gamma_{cd}(C_5) = 2$ and $\gamma_{gcd}(C_5) = 3$. But, the converse need not be true. In the graphs $G_1$ and $G_2$ in Fig: 3.3,
$\gamma_{gcd}(G_1) = \gamma_{cd}(G_1) = 3$ and $\gamma_{gcd}(G_2) = 2$ and $\gamma_{cd}(G_2) = 1$.

![Fig: 3.3](image)

**Corollary 3.2.4.** If $G$ is a triangle free graph with $\gamma_{gcd}(G) \neq \gamma_{cd}(G)$, then $\gamma_{cd}(G) = \gamma_t(G)$.

**Proof.** Let $D$ be a minimum cographic dominating set of $G$. Since, none of the
minimum cographic dominating sets dominate $G^c$, at least one vertex $v$ of $G$ must
be adjacent to all the vertices of $D$. Since, $G$ is triangle free, no two vertices of
$D$ are adjacent. Therefore, $D$ is an independent dominating set. Hence, $\gamma_{cd}(G) = \gamma_t(G)$. \hfill \Box

**Corollary 3.2.5.** Every tree $T$ has $\gamma_{gcd}(T) = \gamma_{cd}(T)$ or $\gamma_{cd}(T) + 1$. Moreover,
$\gamma_{gcd}(T) = \gamma_{cd}(T) + 1$ only if $T$ is a rooted tree of depth two in which every vertex
(may be except the root) has at least two children.
Proof. The first statement follows from Theorem 3.2.3, since trees are triangle free. Assume that \( \gamma_{gcd}(T) = \gamma_{cd}(T) + 1 \) for a tree \( T \). Then as in the proof of corollary 3.2.4, there exists a minimum cographic dominating set \( D \), which is independent and has a common neighbor \( v \). Since \( D \) is dominating and \( T \) is a tree, \( v \) is not adjacent to any other vertex of \( G \). Now, every vertex of \( D \) has at least two pendant vertices attached to it. Since, otherwise if \( u \in D \) has only one pendant vertex \( w \) attached to it, then \( (D - \{u\}) \cup \{w\} \) is a global dominating set of cardinality \( \gamma_{cd}(T) \), which is a contradiction. Therefore, all the vertices in \( D \) have at least two pendant vertices attached to it and so \( T \) is a rooted tree of depth two with \( v \) as its root in which every vertex has at least two children.

Fig: 3.4 gives examples of trees with \( \gamma_{gcd}(T) = \gamma_{cd}(T) + 1 \).

Lemma 3.2.6. If \( G \) is a disconnected graph, then \( \gamma_{cd}(G^c) \leq 2 \) and \( \gamma_{gcd}(G) = \gamma_{cd}(G) \).

Proof. Since \( G \) is disconnected, \( G^c \) is connected and any two vertices in the two different components of \( G \) dominates \( G^c \). So, \( \gamma_{cd}(G^c) \leq 2 \). Also, in any cographic dominating set of \( G \), there will be at least one vertex from each component. Therefore any cographic dominating set of \( G \) is a cographic dominating set of \( G^c \) also. Hence \( \gamma_{gcd}(G) = \gamma_{cd}(G) \).

Remark 3.2.2. This lemma holds for the domination number and the global domination number of a disconnected graph also.
Theorem 3.2.7. A cograph $G$ without a universal vertex has $\gamma_{gcd}(G) = \gamma_{cd}(G)$ if and only if there exists two vertices $u$ and $v$ such that $N(u)$ and $N(v)$ partitions $V(G)$ or $V(G) - \{u, v\}$.

Proof. If $N(u)$ and $N(v)$ partitions $V(G)$ or $V(G) - \{u, v\}$, the cographic domination number of $G$ is 2. In $G^c$, $\{u, v\}$ itself dominates. Therefore, $\gamma_{gcd}(G) = \gamma_{cd}(G) = 2$.

Conversely, assume that $\gamma_{gcd}(G) = \gamma_{cd}(G)$. Since $\gamma_{gcd}(G) > 1$ and $\gamma_{cd}(G) \leq 2$, we have $\gamma_{gcd}(G) = \gamma_{cd}(G) = 2$. Therefore, there exist two vertices $u$ and $v$ such that $\{u, v\}$ dominates both $G$ and $G^c$. Since, neighbors of $u$ in $G$ will not be adjacent to $u$ in $G^c$, they must be adjacent to $v$ in $G^c$. Hence, no vertex in $N(u)$ is adjacent to $v$ in $G$ and vice versa. Also, since $\{u, v\}$ dominates, $N(u) \cup N(v) = V(G)$ or $V(G) - \{u, v\}$. Therefore, $N(u)$ and $N(v)$ partitions $V(G)$ or $V(G) - \{u, v\}$. \[ \square \]

Fig : 3.5 gives an example of a cograph for which $\gamma_{gcd}(G) = \gamma_{cd}(G)$.

![Fig : 3.5](image)

Theorem 3.2.8. If $G$ is a planar graph with $\gamma_{cd}(G) \geq 3$, then $\gamma_{gcd}(G) \leq \gamma_{cd}(G) + 2$.

Proof. If possible assume that $\gamma_{gcd}(G) > \gamma_{cd}(G) + 2$. Let $u_1, u_2, u_3$ be three vertices in any $\gamma_{cd}$-set $D$ of $G$. Since $\gamma_{gcd}(G) > \gamma_{cd}(G) + 2$, $D$ cannot dominate $G^c$ and at least three more vertices are to be added to $D$ to make it a global dominating set. Therefore, there exist at least three vertices $v_1, v_2, v_3$ which are adjacent to each other and to every vertex of $D$. Then the subgraph induced by these six vertices
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will be $K_6, K_6 - \{e_1\}, K_6 - \{e_1, e_2\}$ or $K_6 - \{e_1, e_2, e_3\}$ where $e_1, e_2, e_3 \in E(G)$ and are adjacent to each other. Each of the above graph contains $K_{3,3}$ as a subgraph, which is a contradiction to the planarity of $G$. Hence the theorem. \qed

Remark 3.2.3. The converse need not be true. For example, in graphs $G_1, G_2$ and $G_3$ in Fig: 3.6, $\gamma_{cd}(G_1) = \gamma_{gcd}(G_1) = 2$, $\gamma_{cd}(G_2) = 2$, $\gamma_{gcd}(G_2) = 3$, $\gamma_{cd}(G_3) = 2$ and $\gamma_{gcd}(G_3) = 4$.

![Fig: 3.6](image)

Remark 3.2.4. The bound $\gamma_{gcd}(G) \leq \gamma_{cd}(G) + 2$ is strict.

![Fig: 3.7](image)

For example, the plane graph in Fig: 3.7 has $\gamma_{cd} = 3$ and $\gamma_{gcd} = 5$.

3.3 Two constructions

Theorem 3.3.1. Given three positive integers $a$, $b$ and $c$ satisfying $a \leq b \leq c$, there is a graph $G$ such that $\gamma(G) = a$, $\gamma_{cd}(G) = b$, $\gamma_{i}(G) = c$. 
Proof. We shall prove the theorem by constructing the required graph and by analyzing the following cases.

Case 1 : $a = b = c$

Let $G = P_n$ or $C_n$ where $n = 3a$. Then, $\gamma(G) = \gamma_{cd}(G) = \gamma_i(G) = a$.

Case 2 : $a = b < c$

Let $G$ be the graph $P_n$ where $n = 3(a - 1)$ together with $(c - a + 1)$ pendant vertices each attached to an end vertex of $P_n$ and its neighbor. Then, $\gamma(G) = \gamma_{cd}(G) = a$ and $\gamma_i(G) = c$.

Case 3 : $a < b = c$

Let $G$ be $P_n : v_1v_2v_3...v_n$, where $n = 3a - 7$ together with $p = b - a + 2$ vertices $u_1, u_2, \ldots, u_p$, made adjacent to each $v_i$ for $i = 1, 2, 3$ and $4$ and $u_1j$ made adjacent to $u_3j$ for each $j = 1, 2, \ldots, p$.

Then, the vertices $v_1, v_2, v_3$ and $v_4$ dominate all $u_{ij}$ s and $v_5$. To dominate the remaining $(3a - 12)$ vertices of the path, $(a - 4)$ vertices are required. Therefore, $\gamma(G) = a$. At least one vertex among $v_1, v_2, v_3$ and $v_4$ must be replaced to get a cographic dominating set. Remove $v_1$ and include all the $(b - a + 2)$ vertices. But, then $v_3$ is also not required in the dominating set so that $\gamma_{cd}(G) = a - 2 + b - a + 2 = b$. This set is also independent and hence $\gamma_i(G) = b$.

Case 4 : $a < b < c$

Let $G$ be $P_n : v_1v_2v_3...v_n$, where $n = 3a - 7$ together with $(b - a)$ vertices made adjacent to $v_4$, $(c - a + 1)$ vertices made adjacent to $v_2$ and $(c - a + 2)$ vertices each made adjacent to $v_1$ and $v_3$ and to each other.
Then, the vertices $v_1, v_2, v_3$ and $v_4$ dominate all pendant vertices attached to them and $v_5$. To dominate the remaining $(3a - 12)$ vertices of the path, $(a - 4)$ vertices are required. Therefore, $\gamma(G) = a$. At least one vertex among $v_1, v_2, v_3$ and $v_4$ must be replaced to get a cographic dominating set. If we remove $v_4$, the $(b - a)$ pendant vertices adjacent to it and $v_5$ are to be adjoined to get a cographic dominating set of cardinality $a - 1 + b - a + 1 = b$. If we remove $v_1$, the $(c - a + 2)$ pendant vertices adjacent to it are to be adjoined. But, then $v_3$ also can be removed from the dominating set to get an independent dominating set of cardinality $(a - 2 + c - a + 2) = c$. Therefore, $\gamma_{cd}(G) = b$ and $\gamma_i(G) = c$.  

Illustration

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$a = b = c = 2$</td>
<td><img src="image" alt="Graph 1" /></td>
</tr>
<tr>
<td>2</td>
<td>$a = b = 3$, $c = 7$</td>
<td><img src="image" alt="Graph 2" /></td>
</tr>
<tr>
<td>3</td>
<td>$a = 5$, $b = c = 7$</td>
<td><img src="image" alt="Graph 3" /></td>
</tr>
<tr>
<td>4</td>
<td>$a = 5$, $b = 7$, $c = 10$</td>
<td><img src="image" alt="Graph 4" /></td>
</tr>
</tbody>
</table>

Table 3.1

**Theorem 3.3.2.** Given two positive integers $a$ and $b$ satisfying $a \leq b$ and $b > 1$, there is a graph $G$ such that $\gamma_{cd}(G) = a, \gamma_{gcd}(G) = b$. 
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Proof. We shall prove the theorem by constructing the required graph and by analyzing the following cases.

Case 1 : \( a = b > 1 \).

G is the graph obtained by taking any graph of order \( a \) and attaching one pendant vertex to each of the vertices.

Case 2 : \( a = 1 \) and \( a < b \).

\( G = K_b \).

Case 3 : \( a = 2 \) and \( a < b \).

\( G \) is \( K_{2b} \) minus a perfect matching.

Case 4 : \( a > 2 \) and \( a < b \).

The graph \( G \) is obtained as per the following constructions based on the Fig : 3.8.

In the Fig : 3.8, the vertices inside each of the circles are independent and the vertices inside both the rectangles form complete graphs. Every vertex \( v_i \) for \( i = 1, 2, ..., a \) is made adjacent to every vertex inside the circle to which an edge is drawn. All the vertices of the smaller rectangle are made adjacent to all the
vertices in the bigger rectangle, all the vertices inside the circle to which an edge is drawn and to $v_a$. Further, $v_{a-1}$ is made adjacent to $v_a$. The graph so obtained is $G$.

Now, if we choose one vertex from each of the circles, we get an independent set of cardinality $a$ which has no common neighbors. Therefore, any dominating set must contain at least $a$ vertices and $\{v_1, v_2, \ldots, v_a\}$ is a cographic dominating set. So $\gamma_{cd}(G) = a$.

The set $\{v_1, v_2, \ldots, v_a\}$ will not dominate $u_i$s in $G^c$. If we remove any one of the $u_i$s from this cographic dominating set, then all the $b - a + 1$ vertices in the corresponding circle must be included to retain the set as a cographic dominating set. Therefore, the cardinality becomes $a - 1 + b - a + 1 = b$.

If we keep all the $u_i$s, then a vertex from any of the circles, except the one corresponding to $v_{a-1}$ cannot be introduced, since otherwise an induced $P_4$ will be present. A vertex from the circle corresponding to $v_{a-1}$ cannot dominate $u_i$s in the complement. Also, a $u_i$ cannot dominate $u_j$ for $i \neq j$. Therefore to get a global cographic dominating set all the $u_i$s must be included. Then the cardinality becomes $a + b - a = b$. In any case, $\gamma_{gcd}(G) = b$. 

Illustration of case 4: $a = 3, b = 5$. 

Fig: 3.9
3.4 Complexity aspects

In this section we discuss the complexity aspects of the newly defined parameters.

**Theorem 3.4.1.** Determining the cographic domination number of a graph is NP-complete.

**Proof.** We prove this by reducing in polynomial time, the dominating set problem in general to the cographic dominating set problem.

Claim: Given a graph $G$, we can construct a graph $H$ in polynomial time such that $G$ has a dominating set of size $k$ if and only if $H$ has a cographic dominating set of size $k + 1$.

Let $G = (V, E)$ where $V = \{v_1, v_2, ..., v_n\}$ be the given graph. Construct $H$ as follows:

Let $V(H) = \{v_1, v_2, ..., v_n\} \cup \{v'_1, v'_2, ..., v'_n\} \cup \{x, y\}$. Make $v_i$ adjacent to $v'_j$ if $v_iv_j \in E(G)$ or $i = j$; $x$ is adjacent to $v'_j$ for every $j$ and $x$ is adjacent to $y$ in $H$.

Let $\{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ be a minimal dominating set of cardinality $k$ in $G$. Then $\{v'_{i_1}, v'_{i_2}, ..., v'_{i_k}, x\}$ is a minimal dominating set in $H$. Since there is no induced $P_4$ in this set, it is a minimal cographic dominating set in $H$ of cardinality $k + 1$.

Conversely, let $\{u_1, u_2, ..., u_{k+1}\}$ be a cographic dominating in $H$. (By construction of $H$, any minimal dominating set is cographic). One of these $u_i$'s must be $x$ or $y$. Remove that $u_i$. All other $u_i$'s must be either $v_j$ or $v'_k$. Keep each $v_j$ unchanged and replace each $v'_k$ by $v_k$. This new set of cardinality $k$ will be a minimal dominating set of $G$. Since $H$ can be computed from $G$ in time polynomial in size
of $G$, our claim holds.

**Corollary 3.4.2.** The problem of determining the cographic domination number is NP-complete for the class of bipartite graphs.

*Proof.* In the proof above, the graph $H$ constructed from $G$ is bipartite.

**Theorem 3.4.3.** Determining the global cographic domination number of a graph is NP-complete.

*Proof.* Claim: Given a graph $G$, we can construct a graph $H$ in polynomial time such that $G$ has a cographic dominating set of size $k$ if and only if $H$ has a global cographic dominating set of size $k + 1$.

Given a graph $G$ define $H = G \cup K_1$. Clearly, a minimum cographic dominating set in $G$ together with the isolated vertex is a minimal global cographic dominating set in $H$.

Conversely, any minimal global cographic dominating set in $H$ will contain the isolated vertex and the remaining vertices is a minimal cographic dominating set in $G$. Since $H$ can be computed from $G$ in time polynomial in size of $G$, our claim holds.

### 3.5 Domination in NEPS of two graphs

In this section, we study the relation between the domination parameters $\gamma, \gamma_g, \gamma_{cd}, \gamma_{gcd}$ and $\gamma_i$ of $G_1$ and $G_2$ with the NEPS of $G_1$ and $G_2$ for all possible choices of the basis.
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NEPS with basis $B_1$ and $B_2$

The value of \( \gamma(\text{NEPS}(G_1, G_2; B_1)), \gamma_g(\text{NEPS}(G_1, G_2; B_1)), \gamma_{cd}(\text{NEPS}(G_1, G_2; B_1)) \), \( \gamma_{gcd}(\text{NEPS}(G_1, G_2; B_1)), \gamma_{t}(\text{NEPS}(G_1, G_2; B_1)) \) are \( n_1 \cdot \gamma(G_2), n_1 \cdot \gamma_g(G_2), n_1 \cdot \gamma_{cd}(G_2), n_1 \cdot \gamma_{gcd}(G_2) \) and \( n_1 \cdot \gamma_{t}(G_2) \) respectively and the case of \( \text{NEPS}(G_1, G_2; B_2) \) follows similarly.

NEPS with basis $B_3$

In [39] it was conjectured that \( \gamma(G \times H) \geq \gamma(G) \gamma(H) \), where \( \times \) denotes the tensor product of two graphs. But, the conjecture was disproved in [48] by giving a realization of a graph \( G \) such that \( \gamma(G \times G) \leq \gamma(G)^2 - k \) for any non-negative integer \( k \).

Theorem 3.5.1. There exist graphs \( G_1 \) and \( G_2 \) such that \( \sigma(\text{NEPS}(G_1, G_2; B_3)) - \sigma(G_1) \sigma(G_2) = k \) for any positive integer \( k \), where \( \sigma \) denotes any of the domination parameters \( \gamma, \gamma_{cd} \) or \( \gamma_{t} \).

Proof. Let \( G_1 \) be the graph defined as follows. Let \( u_{11}u_{12}u_{13}, u_{21}u_{22}u_{23}, \ldots, u_{k1}u_{k2}u_{k3} \) be \( k \) distinct \( P_3 \) s and let \( u_{j1} \) be adjacent to \( u_{j+1,1} \) for \( j = 1, 2, \ldots, k - 1 \). Then \( \sigma(G_1) = k \). Let \( G_2 \) be \( K_2 \). Then, \( \sigma(G_2) = 1 \). Also, \( \sigma(\text{NEPS}(G_1, G_2; B_3)) = 2k \). Therefore, \( \sigma(\text{NEPS}(G_1, G_2; B_3)) - \sigma(G_1) \sigma(G_2) = k \). \( \square \)

Theorem 3.5.2. The \( \gamma_g \) and \( \gamma_{gcd} \) are neither sub multiplicative nor super multiplicative with respect to the NEPS with basis \( B_3 \). Moreover, given any integer \( k \) there exist graphs \( G_1 \) and \( G_2 \) such that \( \sigma(\text{NEPS}(G_1, G_2; B_3)) - \sigma(G_1) \sigma(G_2) = k \), where \( \sigma \) denotes \( \gamma_g \) or \( \gamma_{gcd} \).

Proof. Case 1: \( k \leq 0 \) is even
Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; B_3)) = 2$. Therefore, the required difference is $2 - 2n$ which can be zero or any negative even integer.

Case 2: $k < 0$ is odd or $k = 1$

Let $G_3 = P_3$ and $G_1$ be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; B_3)) = 3$. Therefore, the required difference is $3 - 2n$ which can be one or any negative odd integer.

Case 3: $k > 1$

Let $G_3$ be as in Case 2. Let $G_4$ be the graph defined as follows. Let $u_{11}u_{12}u_{13}$, $u_{21}u_{22}u_{23}$, ..., $u_{k1}u_{k2}u_{k3}$ be $k$ distinct $P_3$ s and let $u_{j1}$ be adjacent to $u_{j+1,1}$ for $j = 1, 2, ..., k - 1$. Then $\sigma(G_4) = k$. Also, $\sigma(\text{NEPS}(G_4, G_3; B_3)) = 3k$. Therefore, the required difference is $k$. □

**NEPS with basis $B_4$**

**Vizing’s conjecture [75]:** The domination number is super multiplicative with respect to the cartesian product i.e; $\gamma(G \Box H) \geq \gamma(G) \gamma(H)$.

**Remark 3.5.1.** There are infinitely many pairs of graphs for which equality holds in the Vizing’s conjecture [62].

**Remark 3.5.2.** Vizing’s type inequality does not hold for cographic, global cographic and independent domination numbers. For example, let $G$ be the graph obtained by attaching $k$ pendant vertices to each vertex of a path on four vertices. Then, $\gamma_{cd}(G) = \gamma_{gcd}(G) = k + 3$ and $\gamma_{cd}(G \Box G) = \gamma_{gcd}(G \Box G) = 16k + 8$. For $k \geq 10$, $\gamma_{cd}(G \Box G) < \gamma_{cd}(G)^2$.

**Theorem 3.5.3.** There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_4))$
\( \sigma(G_1)\sigma(G_2) = k \) for any positive integer \( k \), where \( \sigma \) denotes any of the domination parameters \( \gamma, \gamma_{cd} \) or \( \gamma_i \).

**Proof.** Let \( G_1 = P_n \) and \( G_2 = K_2 \). Then, \( \sigma(G_1) = \lceil \frac{n+2}{3} \rceil \) [44] and \( \sigma(G_2) = 1 \). Also, \( \sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) = \lceil \frac{n+2}{2} \rceil \) [47]. Therefore, for any positive integer \( k \), if we choose \( n = 6k - 2 \) the claim follows. \( \square \)

**Theorem 3.5.4.** The \( \gamma_g \) and \( \gamma_{gcd} \) are neither sub multiplicative nor super multiplicative with respect to the NEPS with basis \( \mathcal{B}_4 \). Moreover, given any integer \( k \) there exist graphs \( G_1 \) and \( G_2 \) such that \( \sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) - \sigma(G_1)\sigma(G_2) = k \), where \( \sigma \) denotes \( \gamma_g \) or \( \gamma_{gcd} \).

**Proof.** Case 1: \( k \leq 0 \) is even.

Let \( G_1 = K_n \) and \( G_2 = K_2 \). Then, \( \sigma(G_1) = n \) and \( \sigma(G_2) = 2 \). But, \( \sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) = 2 \). Therefore, the required difference is \( 2 - 2n \) which can be any positive even integer.

Case 2: \( k < 0 \) is odd.

Let \( G_3 = P_3 \) and \( G_1 \) be as in Case 1. Then \( \sigma(G_3) = 2 \). Also, \( \sigma(\text{NEPS}(G_1, G_3; \mathcal{B}_4)) = 3 \). Therefore, the required difference is \( 3 - 2n \) which can be any negative odd integer.

Case 3: \( k \geq 1 \).

Let \( G_4 = P_n \) and \( G_5 = P_1 \). Then, \( \sigma(G_4) = \lceil \frac{n+2}{3} \rceil \) and \( \sigma(G_5) = 2 \). For any positive integer \( k \), if we choose \( n = 3k + 4 \), then \( \sigma(\text{NEPS}(G_4, G_5; \mathcal{B}_4)) = n \). (Note that the value is \( n + 1 \) only when \( n = 1, 2, 3, 5, 6, 9 \) [47]). Therefore the required difference is \( k \).
Chapter 3: Domination in Graph Classes

NEPS with basis $B_5$ and $B_6$

**Theorem 3.5.5.** There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_5)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer $k$, where $\sigma$ denotes any of the domination parameters $\gamma, \gamma_{cd}$ or $\gamma_i$.

**Proof.** Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor \frac{n+2}{3} \rfloor$ and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_5)) = \lfloor \frac{n+2}{2} \rfloor$. For a positive integer $k$, if we choose $n = 6k - 2$ then the difference is $k$. Hence, the theorem. \qed

**Theorem 3.5.6.** There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_5)) - \sigma(G_1)\sigma(G_2) = k$ for any negative integer $k$, where $\sigma$ denotes $\gamma_g$ or $\gamma_{gcd}$.

**Proof.** Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor \frac{n+2}{3} \rfloor$ and $\sigma(G_2) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_5)) = \lfloor \frac{n+2}{2} \rfloor$. Therefore, if we choose $n = 6k - 2$, the required difference is $-k$. \qed

NEPS with basis $B_7$

**Theorem 3.5.7.** The $\gamma, \gamma_i$ and $\gamma_g$ are sub multiplicative with respect to the NEPS with basis $B_7$.

**Proof.** Let $D_1 = \{u_1, u_2, ..., u_s\}$ be a dominating set of $G_1$ and $D_2 = \{v_1, v_2, ..., v_t\}$ be a dominating set of $G_2$. Consider the set $D = \{(u_1, v_1), (u_1, v_2), ..., (u_1, v_t), ..., (u_s, v_1), (u_s, v_2), ..., (u_s, v_t)\}$. Let $(u, v)$ be any vertex in $\text{NEPS}(G_1, G_2; B_7)$. Since $D_1$ is a $\gamma$-set in $G_1$, there exists at least one $u_i \in D_1$ such that $u = u_i$ or $u$ is adjacent to $u_i$. Similarly, there exists at least one $v_j \in D_2$ such that $v = v_j$ or $v$ is
adjacent to $v_j$. Therefore, $(u_i, v_j)$ dominates $(u, v)$ in $\text{NEPS}(G_1, G_2; B_7)$. Hence,

$$\gamma(\text{NEPS}(G_1, G_2; B_7)) \leq \gamma(G_1) \gamma(G_2).$$

Similar arguments hold for the independent domination and global domination numbers also.

\[ \square \]

Remark 3.5.3. The difference between $\gamma(G_1) \gamma(G_2)$ and $\gamma(\text{NEPS}(G_1, G_2; B_7))$ can be arbitrarily large. Similar is the case for $\gamma_i$ and $\gamma_g$. For, let $G_1$ be the graph, $n$ copies of $C_4$ s with exactly one common vertex. Then, $\gamma(G_1) = \gamma_i(G_1) = n + 1$. Also, $\gamma(\text{NEPS}(G_1, G_1; B_7)) \leq n^2 + 3$ and $\gamma_i(\text{NEPS}(G_1, G_1; B_7)) \leq n^2 + 5$. Also, $\gamma_g(K_n) = n, \gamma_g(P_3) = 2$ and $\gamma_g(\text{NEPS}(G_2, G_3; B_7)) = n + 2$, if $n > 1$.

Theorem 3.5.8. The $\gamma_{cd}$ and $\gamma_{gcd}$ are neither sub multiplicative nor super multiplicative with respect to the NEPS with basis $B_7$. Moreover, for any integer $k$ there exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_7)) - \sigma(G_1) \sigma(G_2) = k$, where $\sigma$ denotes $\gamma_{cd}$ or $\gamma_{gcd}$.

Proof. Case 1: $k \leq 0$

Let $G_1$ be the graph $P_3$ with $k$ pendant vertices each attached to all the three vertices of the $P_3$. Let $G_2$ be the graph $P_4$ with $k$ pendant vertices each attached to all the four vertices of the $P_4$. So, $\sigma(G_1) = 3$ and $\sigma(G_2) = k + 3$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_7)) = 2k + 10$. Therefore, the required difference is $1 - k$.

Case 2: $k \geq 0$

Let $G_1$ be as in Case 1 and $G_3$ be the graph $P_6$ with $k$ pendant vertices each attached to all the six vertices of the $P_6$. So, $\sigma(G_3) = k + 5$. Also, $\sigma(\text{NEPS}(G_1, G_3; B_7)) = 4k + 14$. Therefore, the required difference is $k - 1$. \[ \square \]