Chapter 2

Gallai and anti-Gallai graphs

This chapter deals with two graph classes the Gallai graphs and the anti-Gallai graphs. We construct infinitely many pairs of graphs $G$ and $H$ such that $\Gamma(G) = \Gamma(H)$. The existence of a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be $H$-free, for any finite graph $H$ is proved and the forbidden subgraph characterizations of $G$ for which the Gallai graphs and the anti-Gallai graphs are cographs, split graphs and threshold graphs are discussed in detail. If $G$ is a connected cograph without a universal vertex then $\Gamma(G)$ is a cograph if and only if $G = (pK_2)^c$. The relationships between the radius, the diameter and the chromatic number of a graph and its Gallai (anti-Gallai) graph are also studied in detail.

Some results of this chapter are included in the following paper.

2.1 Gallai and anti-Gallai graphs

It is well known [80] that the only pair of non-isomorphic graphs having the same line graph is $K_{1,3}$ and $K_3$. But, we first observe that, in the case of both Gallai and anti-Gallai graphs, there are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs (anti-Gallai graphs).

**Theorem 2.1.1.** There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs.

**Proof.** We prove this theorem by the following two types of constructions.

Type 1 :- Let $G$ be the graph $P_4$ with $n$ independent vertices joined to both its internal vertices and an end vertex attached to $k$ of these $n$ vertices and $H$ be two copies of $K_{1,n+1}$ with $k + 1$ distinct pairs of end vertices made adjacent.

The graph $G$ in type 1 is as follows. Let $v_1, v_2, v_3, v_4$ be an induced $P_4$. Let $v_2$ and $v_3$ be joined to $n$ vertices $u_1, u_2, ..., u_n$. Introduce $k$ end vertices $w_1, w_2, ..., w_k$ such that each $w_i$ is adjacent only to $u_i$ for $i = 1, 2, ..., k$. The edges $v_1v_2, v_2u_1, v_2u_2, ..., v_2u_n$ of $G$, which are vertices of $\Gamma(G)$ will induce a complete graph on $n + 1$ vertices in $\Gamma(G)$. Similarly, $v_3v_4, v_3u_1, v_3u_2, ..., v_3u_n$ will induce another complete graph on $n + 1$ vertices in $\Gamma(G)$. The vertex corresponding to the edge $v_2v_3$ will be adjacent to both the vertices corresponding to $v_1v_2$ and $v_3v_1$. The $k$ vertices corresponding to the edges $u_iv_i$ for $i = 1, 2, ..., k$ will be adjacent to the vertices corresponding to the edges $u_1v_2$ and $u_iv_3$ for $i = 1, 2, ..., k$ respectively.

The graph $H$ in type 1 is as follows. Let $u$ adjacent to $u_1, u_2, ..., u_{n+1}$ and $v$ adjacent to $v_1, v_2, ..., v_{n+1}$ be the two $K_{1,n+1}$ s in $H$. Let $u_1v_1, u_2v_2, ..., u_{k+1}v_{k+1}$ be
Chapter 2: Gallai and anti-Gallai graphs

The $k + 1$ distinct pairs of adjacent vertices in $H$. The vertices corresponding to the edges $uu_1, uu_2, \ldots, uu_{n+1}$ will induce a complete graph on $n + 1$ vertices in $\Gamma(H)$. Similarly, the vertices corresponding to $vv_1, vv_2, \ldots, vv_{n+1}$ will also induce another complete graph on $n + 1$ vertices in $\Gamma(H)$. Again, the vertices corresponding to the edges $u_i v_i$ for $i = 1, 2, \ldots, k + 1$ will be adjacent to the vertices corresponding to the edges $uu_i$ and $vv_i$ for $i = 1, 2, \ldots, k + 1$ respectively.

Therefore, both $\Gamma(G)$ and $\Gamma(H)$ are two copies of $K_{n+1}$ together with $k + 1$ new vertices made adjacent to $k + 1$ distinct vertices of both the copies of $K_{n+1}$.

Type 2: Let $G$ be the graph $P_1$ with $n$ independent vertices joined to both its internal vertices and an end vertex attached to $k$ of them with $k \geq 1$, together with one end vertex each attached to the two end vertices of $P_1$ and $H$ be two copies of $K_{1,n+1}$ with $k + 1$ distinct pairs of end vertices (one from each star) made adjacent and a single pair made adjacent to another vertex.

The graph $G$ in type 2 can be obtained from the graph $G$ in type 1 by attaching two end vertices $x$ and $y$ to $v_1$ and $v_2$ respectively. In $\Gamma(G)$ the vertices corresponding to the edges $v_1 x$ and $v_3 y$ will be adjacent to the vertices corresponding to the edges $v_1 v_2$ and $v_3 v_4$ respectively. The graph $H$ in type 2 can be obtained from the graph $H$ in type 1 by adding a new vertex $w$ and making it adjacent to both $u_1$ and $v_1$. In $\Gamma(H)$ the vertices corresponding to the edges $wu_1$ and $wv_1$ will be adjacent to the vertices corresponding to the edges $uu_1$ and $vv_1$ respectively.

Therefore, both $\Gamma(G)$ and $\Gamma(H)$ are two copies of $K_{n+1}$ together with $k + 1$ vertices made adjacent to $k + 1$ distinct vertices of both the copies of $K_{n+1}$ and two end vertices made adjacent to one vertex from each of the complete graphs.

The constructions mentioned in type 1 and type 2 are illustrated in Table 2.1.
In both the cases, the graphs $G$ and $H$ have the same Gallai graph. If $n = k$ and $n = k - 1$ in type 1 and type 2 respectively, then the order of $G$ and $H$ is the same.

<table>
<thead>
<tr>
<th>Type 1</th>
<th>Type 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 3$</td>
<td>$n = 3$</td>
</tr>
<tr>
<td>$k = 1$</td>
<td>$k = 1$</td>
</tr>
</tbody>
</table>

Table 2.1

**Theorem 2.1.2.** There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic anti-Gallai graphs.

*Proof.* Let $G$ be a graph with vertex set $\{v_1, v_2, ..., v_n\}$ and an edge $v_i v_j$ such that $G$ is not isomorphic to a graph obtained under permutations of the index set of the vertices which interchange $i$ and $j$ and $\Delta(G)$ is connected. Introduce a vertex
Chapter 2 : Gallai and anti-Gallai graphs

u adjacent to \( v_i \) and \( v_j \). Let \( H_1 \) be the graph obtained by introducing one more vertex \( u_1 \) adjacent to \( u \) and \( v_i \). Let \( H_2 \) be the graph obtained by introducing another vertex \( u_2 \) (\( u_1 \) is absent here) adjacent to \( u \) and \( v_j \). Then by construction \( H_1 \) and \( H_2 \) are non-isomorphic. \( \Delta(H_1) \) is \( \Delta(G) \) together with four more vertices corresponding to \( uv_i, uv_j, uu_1, v_iu_1 \) in which \( uv_i \) and \( uv_j \) are adjacent to each other and to \( v_i, v_j, uu_1 \) and \( v_iu_1 \) are adjacent to each other and to \( uv_i \). \( \Delta(H_2) \) is \( \Delta(G) \) together with four more vertices corresponding to \( uv_i, uv_j, uu_2, v_ju_2 \) in which \( uv_i \) and \( uv_j \) are adjacent to each other and to \( v_i, v_j, uu_2 \) and \( v_ju_2 \) are adjacent to each other and to \( uv_j \). Therefore, \( \Delta(H_1) \) is isomorphic to \( \Delta(H_2) \).

\[ \square \]

2.2 Forbidden subgraph characterizations

Even though the Gallai and the anti-Gallai graphs cannot be characterized using forbidden subgraphs, in this section we prove the existence of a finite forbidden subgraph characterization for the Gallai graph and the anti-Gallai graph to be \( H \)-free and obtain the forbidden subgraph characterizations for the Gallai and the anti-Gallai graphs to be a cograph, a split graph and a threshold graph.

**Notation**: For a connected graph \( H \), let \( \mathcal{G}(H) = \{ G : \Gamma(G) \text{ is } H \text{-free} \} \) and \( \mathcal{G}^*(H) = \{ G : \Delta(G) \text{ is } H \text{-free} \} \).

**Theorem 2.2.1.** The properties of being an element of \( \mathcal{G}(H) \) and \( \mathcal{G}^*(H) \) are vertex hereditary.

**Proof.** Let \( G \in \mathcal{G}(H) \) and \( v \in V(G) \). Consider \( G' = G - \{v\} \). It is required to
prove that \( G' \in \mathcal{G}(H) \). On the contrary assume that \( \Gamma(G') \) has \( H \) as an induced subgraph. Let \( v_1, v_2, \ldots, v_t \) be neighbors of \( v \). Therefore \( \Gamma(G) \) has the vertex set \( V(\Gamma(G')) \cup \{vv_1, vv_2, \ldots, vv_t\} \). In \( \Gamma(G) \), \( vv_i \) is adjacent to \( vv_j \) if \( v_i \) is not adjacent to \( v_j \), and \( vv_i \) will be adjacent to all edges which have \( v_i \) as one end vertex and other end vertex is not \( v_j \) for \( j = 1, 2, \ldots, t \). \( V(\Gamma(G')) \) induce \( \Gamma(G') \) itself. Hence if \( H \) is an induced subgraph of \( \Gamma(G') \) then \( H \) is an induced subgraph of \( \Gamma(G) \) also, which is a contradiction.

The case of \( \mathcal{G}^*(H) \) follows similarly.

Corollary 2.2.2. \( \mathcal{G}(H) \) and \( \mathcal{G}^*(H) \) have vertex minimal forbidden subgraph characterization.

Though many well known classes of graphs admit forbidden subgraph characterizations, the number of such forbidden subgraphs need not be finite. However, for \( \mathcal{G}(H) \) and \( \mathcal{G}^*(H) \) we have

**Theorem 2.2.3.** For every vertex minimal forbidden subgraph of \( \mathcal{G}(H) \) and \( \mathcal{G}^*(H) \), the number of vertices is bounded above by \( n(H) + 1 \).

**Proof.** Let \( \mathcal{F}(H) \) be the collection of all vertex minimal forbidden subgraphs of \( \mathcal{G}(H) \). Let \( L \in \mathcal{F}(H) \). Therefore, \( \Gamma(L) \) has \( H \) as an induced subgraph. The \( n(H) \) vertices of \( H \), which correspond to \( n(H) \) edges of \( L \), say \( e_1, e_2, \ldots, e_{n(H)} \), can cover a maximum of \( n(H) + 1 \) vertices of \( L \), since \( H \) is connected.

We have to prove that \( n(L) \leq n(H) + 1 \). On the contrary assume that \( n(L) > n(H) + 1 \). Then there exists at least one vertex \( v \in V(L) \) which is not an end vertex of any of \( e_1, e_2, \ldots, e_{n(H)} \). Therefore, \( \Gamma(L - v) \) still has \( H \) as an induced subgraph, which contradicts that \( L \) is a vertex minimal forbidden subgraph of \( \mathcal{G}(H) \). Hence,
Chapter 2: Gallai and anti-Gallai graphs

\[ n(L) \leq n(H) + 1. \]

A similar argument holds for \( G^*(H) \) also. \( \square \)

**Corollary 2.2.4.** The number of vertex minimal forbidden subgraphs for \( G(H) \) and \( G^*(H) \) is finite.

**Theorem 2.2.5.** Let \( G \) be a graph. Then, \( \Gamma(G) \) is a cograph if and only if \( G \) does not have the following graphs as induced subgraphs.

\[
\begin{align*}
(i) & \quad P_5 \\
(ii) & \quad C_5 \\
(iii) & \quad K_{2,3} \\
(iv) & \quad \text{Fig. 2.1}
\end{align*}
\]

**Proof.** If \( \Gamma(G) \) is not a cograph then there exists an induced \( P_4 \) in \( \Gamma(G) \), say \( e_1e_2e_3e_4 \). In \( G \), let \( e_1 = u_{11}u_{12}, e_2 = u_{21}u_{22}, e_3 = u_{31}u_{32} \) and \( e_4 = u_{41}u_{42} \).

Since \( e_1 \) is adjacent to \( e_2 \), let \( u_{12} = u_{21} \) and let \( u_{11} \) be not adjacent to \( u_{22} \). Since \( e_2 \) is adjacent to \( e_3 \), either \( u_{21} = u_{31} \) or \( u_{22} = u_{31} \).

If \( u_{21} = u_{31} \), then since \( e_1 \) is not adjacent to \( e_3 \), \( u_{11} \) is adjacent to \( u_{32} \). Since \( e_3 \) is adjacent to \( e_4 \), either \( u_{31} = u_{41} \) or \( u_{32} = u_{41} \). If \( u_{31} = u_{41} \), then since \( e_1 \) and \( e_2 \) are not adjacent to \( e_4 \), both \( u_{11} \) and \( u_{21} \) are adjacent to \( u_{42} \). If \( u_{32} = u_{41} \) then \( u_{31} \) is not adjacent to \( u_{42} \).

If \( u_{22} = u_{31} \), then \( u_{21} \) is not adjacent to \( u_{32} \). Again, since \( e_3 \) is adjacent to \( e_4 \), either \( u_{31} = u_{41} \) or \( u_{32} = u_{41} \). If \( u_{31} = u_{41} \), then since \( e_2 \) and \( e_4 \) are not adjacent, \( u_{21} \) is adjacent to \( u_{42} \). If \( u_{32} = u_{41} \) then \( u_{31} \) is not adjacent to \( u_{42} \). The above four resulting graphs are respectively (iv), (vi), (vi) and (i).
Chapter 2: Gallai and anti-Gallai graphs

In (iv), if we add even a single edge the property of $\Gamma(G)$ not being a cograph will be lost. In (vi), $u_{22}$ adjacent to $u_{42}$ gives (vii), $u_{11}$ adjacent to $u_{42}$ gives (ix) and the combination of both gives (v). The addition of these edges will not change the required property either. In (i), $u_{11}$ adjacent to $u_{12}$ gives (ii), $u_{11}$ adjacent to $u_{41}$ gives (viii) and a combination of both gives (iii). Again, the addition of these edges will not change the required property. However, if we add any other edge then the property will be lost.

Conversely, it can be verified that the Gallai graph will not be a cograph if any of the nine graphs listed above is an induced subgraph of $G$. \hfill $\Box$

**Theorem 2.2.6.** Let $G$ be a graph. Then $\Delta(G)$ is a cograph if and only if $G$ does not have the following graphs as induced subgraphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig22.png}
\caption{Fig : 2.2}
\end{figure}

\textit{Proof.} If $\Delta(G)$ is not a cograph then there exists an induced $P_4$ in $\Delta(G)$, say $e_1e_2e_3e_4$. In $G$, let $e_1 = u_{11}u_{12}, e_2 = u_{21}u_{22}, e_3 = u_{31}u_{32}$ and $e_4 = u_{41}u_{42}$.

Since $e_1$ is adjacent to $e_2$, let $u_{12} = u_{21}$ and let $u_{11}$ be adjacent to $u_{22}$. Since $e_2$ is adjacent to $e_3$, either $u_{21} = u_{31}$ or $u_{22} = u_{31}$.

If $u_{21} = u_{31}$ then $u_{22}$ is adjacent to $u_{32}$ and $u_{11}$ is not adjacent to $u_{31}$. Since $e_3$ is adjacent to $e_4$, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then $u_{32}$ is adjacent to $u_{42}$ and $u_{11}$ and $u_{22}$ are not adjacent to $u_{42}$. If $u_{32} = u_{41}$ then $u_{31}$ is adjacent to $u_{42}$.
If \( u_{22} = u_{31} \) then \( u_{12} \) is adjacent to \( u_{32} \). Again, since \( e_3 \) is adjacent to \( e_4 \), either \( u_{31} = u_{41} \) or \( u_{32} = u_{41} \). If \( u_{31} = u_{41} \), then \( u_{32} \) is adjacent to \( u_{42} \) and \( u_{21} \) is not adjacent to \( u_{12} \). If \( u_{32} = u_{42} \) then \( u_{31} \) is adjacent to \( u_{42} \).

All the four resulting graphs are isomorphic to (i) itself. Also, addition of any of the possible edges will leave an induced \( P_4 \) in \( \Delta(G) \) and hence any graph with five vertices which contains (i) as a (not induced) subgraph are also forbidden. Hence all the above graphs are forbidden.

The converse can be easily proved.

<table>
<thead>
<tr>
<th>Split graph</th>
<th>Gallai graph</th>
<th>anti-Gallai graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two vertex disjoint ( P_3 )</td>
<td>( 2 K_3 )</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Threshold graph</th>
<th>Gallai graph</th>
<th>anti-Gallai graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two vertex disjoint ( P_3 )</td>
<td>( 2 K_3 )</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.2

If \( G \) is any graph class that admits a finite forbidden subgraph characterization, then using similar arguments as in Theorem 2.2.5 and Theorem 2.2.6, we can obtain forbidden subgraph characterizations for the Gallai graph and the anti-Gallai graph.
to be in \( \mathcal{G} \). In Table 2.2, we list the forbidden subgraphs for \( \Gamma(G) \) and \( \Delta(G) \) to be a split graph and a threshold graph.

### 2.3 Applications to cographs

In this section we obtain characterizations for the Gallai graph and the anti-Gallai graph of a cograph to be a cograph.

**Theorem 2.3.1.** If \( G \) is a connected cograph without a universal vertex then \( \Gamma(G) \) is a cograph if and only if \( G = (pK_2)^c \).

**Proof.** Let \( G = (pK_2)^c \). Then the number of vertices of \( G \) is \( 2p \) and the number of edges of \( G \) is \( 2p(p-1) \). Let the vertices of \( G \) be \( \{v_{11}, v_{12}, \ldots, v_{1p}, v_{21}, v_{22}, \ldots, v_{2p}\} \) with \( v_{1j} \) and \( v_{2j} \) as the only pair of non-adjacent vertices, for \( j = 1, 2, \ldots, p \). Therefore, the vertices of the Gallai graph are of the form \( v_{ij}v_{i'j'} \) where \( j \neq j' \). By the definition of the Gallai graphs, \( v_{ij}v_{i'j'} \) will be adjacent only to \( v_{ij}v_{1j'} \) or \( v_{ij}v_{2j'} \) and \( v_{1j}v_{i'j'} \) or \( v_{2j}v_{i'j'} \) according to the value of \( i \) and \( i' \). Therefore, \( \Gamma(G) = (pC_2)C_4 \), which is a cograph.

Conversely, assume that \( G \) is a cograph without a universal vertex and \( \Gamma(G) \) is also a cograph. For every \( u \in V(G) \), there exist at least one \( u' \in V(G) \) which is not adjacent to \( u \).

Claim : \( u' \) is the only vertex which is not adjacent to \( u \).

On the contrary assume that there exists another vertex \( u'' \) which is not adjacent to \( u \). Since \( G \) is a connected cograph, \( G = G_1 \vee G_2 \). Let \( u \in V(G_1) \). Since
u is not adjacent to both \( u' \) and \( u'' \), both of them belong to \( V(G_1) \). Since \( G \) has no vertex of full degree, \( G_2 \) must contain at least two non-adjacent vertices \( v_1 \) and \( v_2 \). Then the edges \( u''v_1, v_1u, uv_2, v_2u' \) will induce a \( P_4 \) in \( \Gamma(G) \), which is a contradiction.

Therefore \( G = (pK_2)^c \), where \( 2p = n \). \( \square \)

**Notation**: Consider the class of graphs which are recursively defined as follows:

\[ \mathcal{H}_1 = \{ G : G = (pK_2)^c \cup (K_q), \text{ where } p, q \geq 0 \} \]

\[ \mathcal{H}_i = \{ G : G = (\bigcup H_{i-1}) \cup K_r, \text{ where } H_{i-1} \in \mathcal{H}_{i-1} \text{ and } r \geq 0 \} \text{ for } i > 1 \]

\[ \mathcal{H} = \bigcup \mathcal{H}_i \]

**Theorem 2.3.2.** For a connected cograph \( G \), \( \Gamma(G) \) is a cograph if and only if \( G \in \mathcal{H} \).

**Proof.** Let \( G \) be a cograph other than \( K_q \) with a vertex of full degree. Let \( V_1 \) be the collection of all full degree vertices in \( G \). Define \( G_1 = < V - V_1 > \). \( \Gamma(G_1) \) is an induced subgraph of \( \Gamma(G) \). More precisely, \( \Gamma(G) = \Gamma(G_1) \) together with some isolated vertices. Therefore, \( \Gamma(G) \) is a cograph if and only if \( \Gamma(G_1) \) is a cograph.

If \( G_1 \) is a connected cograph then \( G_1 \) has no vertex of full degree and hence \( \Gamma(G_1) \) is a cograph if and only if \( G_1 = (pK_2)^c \). Therefore, \( \Gamma(G) \) is a cograph if and only if \( G = (pK_2)^c \cup (K_q) \in \mathcal{H}_1 \).

If \( G_1 \) is disconnected, then consider each of the connected components of \( G_1 \). If the removal of all full degree vertices from each of the components of \( G_1 \) preserves connectedness then as above each of these components must be of the form \( (pK_2)^c \cup (K_q) \). Therefore, \( G = (F_1 \cup F_2 \cup \ldots \cup F_p) \cup K_q \) where each \( F_i \in \mathcal{H}_1 \) and \( q \geq 0 \). Consequently, \( G \in \mathcal{H}_2 \).
Chapter 2: Gallai and anti-Gallai graphs

If any of the components of $G_1$, say $G_2$, is disconnected then repeat the above process to get $G_1 \in \mathcal{H}_2$ and hence $G = (H_1 \cup H_2 \cup \ldots \cup H_r) \lor K_s$ where each $H_i \in \mathcal{H}_2$ and $r \geq 0$. Consequently, $G \in \mathcal{H}_3$.

This process must terminate since the number of vertices of $G$ is finite. Therefore for a connected cograph $G$, $\Gamma(G)$ is a cograph if and only if $G \in \mathcal{H}$.

\[ \text{Theorem 2.3.3.} \quad \text{For a connected cograph } G, \Delta(G) \text{ is a cograph if and only if} \]

(i) $G = G_1 \lor G_2$, where $G_1$ is edgeless and $G_2$ does not contain $P_4$ as a subgraph (which need not be induced) or

(ii) $G$ is $C_4$.

\[ \text{Proof.} \quad \text{Let } G \text{ be a connected cograph whose } \Delta(G) \text{ is also a cograph. Since } G \text{ is a connected cograph, } G = G_1 \lor G_2. \text{ Let } G_1 \text{ be an edgeless graph and } u \in V(G_1). \text{ If } G_2 \text{ contains a } P_4, \text{ say } v_1v_2v_3v_4, \text{ then the edges } v_1v_2, v_2u, uv_3, v_3v_4 \text{ of } G \text{ induce a } P_4 \text{ in } \Delta(G), \text{ which is a contradiction. Therefore, if } G_1 \text{ is edgeless then } G_2 \text{ does not contain } P_4 \text{ as a subgraph.} \]

Let $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. If $G_1$ contains one more vertex, say $v$, not adjacent to $u_1$ and $v_1$, then the edges $u_1v_1, v_1u_2, u_2v_2, u_2u$ of $G$ induce a $P_4$ in $\Delta(G)$, which is a contradiction. If $v$ is adjacent to at least one of the vertices, say $v_1$, then the edges $u_1u_2, u_2v_1, v_1v_2, v_2v$ of $G$ induce a $P_4$ in $\Delta(G)$, which is a contradiction. A similar argument holds also for the vertex set of $G_2$. Therefore both $G_1$ and $G_2$ are $K_2$-s and $G = C_4$.

Conversely, assume that $G$ is a cograph of type (i) or (ii). Then $G$ does not contain any of the graphs in Fig: 2.2 as an induced subgraph and hence $\Delta(G)$ is a cograph by Theorem 2.2.6. \qed
2.4 Chromatic number

In this section we study the relation between the chromatic numbers of $G$, $\Gamma(G)$ and $\Delta(G)$.

**Theorem 2.4.1.** Given two positive integers $a, b$, where $a > 1$, there exists a graph $G$ such that $\chi(G) = a$ and $\chi(\Gamma(G)) = b$.

*Proof.* If $a = 1$ then $G$ must be a graph without edges, which makes $\Gamma(G)$ empty. So we can assume that $a > 1$.

Let $G$ be the graph $K_a$ together with $b - 1$ end vertices attached to any one of the vertices. Then $\Gamma(G)$ is $a - 1$ copies of $K_b$ sharing $b - 1$ vertices in common together with some isolated vertices. Clearly, $\chi(G) = a$ and $\chi(\Gamma(G)) = b$. \qed

**Lemma 2.4.2.** The anti-Gallai graph of any graph $G$ cannot be bipartite except for the $K_3$-free graphs.

*Proof.* If $u_1$ is adjacent to $u_2$ in $\Delta(G)$ then the corresponding edges, say $e_1$ and $e_2$, lie in a $K_3$, say $e_1e_2e_3$. Then the vertex $u_3$ in $\Delta(G)$ which corresponds to $e_3$ will be adjacent to both $u_1$ and $u_3$. Therefore, $u_1u_2u_3$ induces a cycle of odd length in $\Delta(G)$ and hence $\Delta(G)$ cannot be bipartite. \qed

**Theorem 2.4.3.** Given two positive integers $a, b$, where $b < a, b \neq 2$, there exists a graph $G$ such that $\chi(G) = a$ and $\chi(\Delta(G)) = b$. Further, for any odd integer $a$, there exists a graph $G$ such that $\chi(G) = \chi(\Delta(G)) = a$.

*Proof.* Since the anti-Gallai graph of a graph $G$ cannot be bipartite except for the triangle free graphs (Lemma 2.4.2), $b = \chi(\Delta(G)) \neq 2$ for any $G$. 

By Myceili’s construction [11] there exists a triangle-free graph $H$ with chromatic number $a$. If we choose $G = H$, then $\Delta(G)$ is a trivial graph and hence $b = 1$. For $2 < b < a$, there exists an induced subgraph $H'$ of $H$ whose chromatic number is $b$. Let $v_1, v_2, \ldots, v_n$ be the vertices of $H'$. Let $G$ be the graph obtained from $H$ by joining all vertices of $H'$ to a new vertex $u$. Since $b < a$, $\chi(G) = a$ itself. If $v_i$ and $v_j$ are adjacent (or non-adjacent) in $H'$ then the vertices corresponding to $uv_i$ and $uv_j$ are adjacent (or non-adjacent) in $\Delta(G)$. Therefore, the vertices corresponding to the edges $uv_1, uv_2, \ldots, uv_n$ induce an $H'$ in $\Delta(G)$. Again for any pair of adjacent vertices, say $v_i$ and $v_j$ in $H'$, the vertices corresponding to the edges $uv_i$ and $uv_j$ are adjacent to the vertex corresponding to $v_1v_2$. Therefore $\Delta(G)$ is $H'$ together with one vertex each adjacent to both the end vertices of each edge in $H'$. For $b > 2$, $\chi(\Delta(G)) = \chi(H') = b$.

If $a$ is an odd integer then $\chi(K_a) = a$ and $\chi(\Delta(G)) = \chi(L(G)) = \chi'(K_a) = a$, where $\chi'$ is the edge chromatic number. □

The triangle free graph $H$ having chromatic number $a = 4$ obtained using Myceili’s construction, the graph $G$ in the above theorem having $\chi(G) = a = 4$ and its anti-Gallai graph having $\chi(\Delta(G)) = b = 3$ are illustrated in Fig : 2.3.
2.5 Radius and diameter

The relation between the radius and the diameter of $G$ with its Gallai and anti-Gallai graphs are studied in this section.

**Theorem 2.5.1.** Let $G$ be a graph such that $\Gamma(G)$ is connected. Then $r(\Gamma(G)) \geq r(G) - 1$ and $d(\Gamma(G)) \geq d(G) - 1$.

**Proof.** Let $r(\Gamma(G)) = r$. Then there exists an edge $uv$ in $G$ such that the vertex corresponding $uv$ in $\Gamma(G)$ is at a distance less than or equal to $r$ from every other vertex in $\Gamma(G)$. Hence, any vertex of $G$ is at a distance less than or equal to $r + 1$ from both $u$ and $v$. We have $r(G) \leq r + 1$, which implies $r(\Gamma(G)) \geq r(G) - 1$.

Let $d(G) = d$. There exist two vertices $u$ and $v$ such that $d(u, v) = d$. Let $uu_1u_2...u_{d-1}v$ be a shortest path connecting $u$ and $v$ in $G$.

Claim:- $d_{\Gamma(G)}(uu_1, u_{d-1}v) = d - 1$.

$uu_1, u_1u_2,..., u_{d-1}v$ is a path of length $d - 1$ connecting $uu_1$ and $u_{d-1}v$ in $\Gamma(G)$. Therefore, $d_{\Gamma(G)}(uu_1, u_{d-1}v) \leq d - 1$.

It is required to prove that $d_{\Gamma(G)}(uu_1, u_{d-1}v) = d - 1$. On the contrary assume that there exists an induced path $uu_1, v_1v'_1, v_2v'_2, ... v_{k-1}v'_{k-1}, u_{d-1}v$ of length $k$ in $\Gamma(G)$ connecting $uu_1$ and $u_{d-1}v$, where $k < d - 1$. Then there exists a path of length less than or equal to $d - 1$ connecting $u$ and $v$ in $G$, which contradicts $d(u, v) = d$. Hence, $d_{\Gamma(G)}(uu_1, u_{d-1}v) = d - 1$.

Since there exist two vertices of $\Gamma(G)$ which are at a distance $d - 1$, $d(\Gamma(G))$ must be greater than or equal to $d - 1$. Hence, $d(\Gamma(G)) \geq d(G) - 1$. \qed
**Remark 2.5.1.** If $a$ and $b$ are two positive integers such that $a > 1$ and $b \geq a - 1$ then there exist graphs $G'$ and $G''$ whose Gallai graphs are connected and $r(G') = a$, $r(\Gamma(G')) = b$, $d(G'') = a$ and $d(\Gamma(G'')) = b$.

**Theorem 2.5.2.** If $G$ is a graph such that $\Delta(G)$ is connected and $r(G) > 1$, $r(\Delta(G)) \geq 2(r(G) - 1)$ and $d(\Delta(G)) \geq 2(d(G) - 1)$.

**Proof.** Let $r(\Delta(G)) = r > 1$. There exists an edge $uv$ in $G$ such that the vertex corresponding to $uv$ in $\Delta(G)$ is at a distance less than or equal to $r$ from every other vertex in $\Delta(G)$. Let $w \in V(G)$. Since $G$ is connected there exists at least one edge with $w$ as an end vertex, say $ww'$. There exists a path of length less than or equal to $r$ from $ww'$ to $uv$ in $\Delta(G)$. Any two adjacent edges in $\Delta(G)$ belong to a triangle and hence this path induces a path of length less than or equal to $\frac{r}{2}$ from either $u$ or $v$ to $w$ or $w'$. Therefore, any vertex is at a distance less than or equal to $\frac{r}{2} + 1$ from both $u$ and $v$. Hence $r(G) \leq \frac{r}{2} + 1$, which implies that $r(\Delta(G)) \geq 2(r(G) - 1)$.

Let $d(G) = d$. There exist two vertices $u$ and $v$ such that $d(u, v) = d$. Let $uu_1u_2...u_{d-1}v$ be a shortest path connecting $u$ and $v$. Consider $d(uu_1, u_{d-1}v)$ in $\Delta(G)$. If it is $k$, then there exists a path of length less than or equal to $\frac{k}{2} + 1$ in $G$ connecting $u$ and $v$. Therefore, $\frac{k}{2} + 1 \geq d$, which implies $k \geq 2(d - 1)$. However, $d(\Delta(G)) \geq k$. Hence, $d(\Delta(G)) \geq 2(d(G) - 1)$. 

**Remark 2.5.2.** If $a$ and $b$ are two positive integers such that $a > 1$ and $b \geq 2(a - 1)$ then there exist graphs $G'$ and $G''$ whose anti-Gallai graphs are connected with $r(G') = a$, $r(\Delta(G')) = b$, $d(G''H) = a$ and $d(\Delta(G'')) = b$. 