Chapter 6

Clique irreducible and weakly clique irreducible graphs

This chapter deals with two graph classes - the clique irreducible graphs and the weakly clique irreducible graphs. A new graph class called the clique vertex irreducible graphs is also defined. We characterize line graphs and its iterations, Gallai graphs, anti-Gallai graphs and its iterations, cographs and distance hereditary graphs which are clique irreducible, clique vertex irreducible and weakly clique irreducible graphs.

Some results of this chapter are included in the following papers.

(1) Clique irreducibility and clique vertex irreducibility of graphs. (communicated).
(3) On weakly clique irreducible graphs, (communicated).
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6.1 Iterations of the line graph

In this section the line graphs and all its iterations which are clique irreducible and clique vertex irreducible are characterized.

Theorem 6.1.1. Let $G$ be a graph. The line graph $L(G)$ is clique vertex irreducible if and only if $G$ satisfies the following conditions.

(1) Every triangle in $G$ has at least two vertices of degree two.

(2) Every vertex of degree greater than one in $G$ has a pendant vertex attached to it, except for the vertices of degree two lying in a triangle.

Proof. Let $G$ be a graph which satisfies the conditions (1) and (2). The cliques of $L(G)$ are induced by the vertices corresponding to the edges in $G$ which are incident on a vertex of degree at least three, the edges in $G$ which are incident on a vertex of degree two and which do not lie in a triangle and by the edges in $G$ which lie in a triangle. By (2), the cliques in $L(G)$ induced by the vertices corresponding to the edges in $G$ which are incident on a vertex, have a vertex which does not lie in any other clique of $L(G)$. By (1), the cliques in $L(G)$ induced by the vertices which correspond to the edges in $G$ which lie in a triangle, have a vertex which does not lie in any other clique of $L(G)$. Therefore, $G$ is clique vertex irreducible.

Conversely, assume that $L(G)$ is a clique vertex irreducible graph. Let $< u_1, u_2, u_3 >$ be a triangle in $G$. Let $e_1, e_2, e_3$ be the vertices in $L(G)$ which correspond to the edges $u_1u_2, u_2u_3, u_3u_1$ in $G$. $T = < e_1, e_2, e_3 >$ is a clique in $L(G)$. If $d(u_i) > 2$ for two $u_i$, $u_1$ and $u_2$, then there exist $v_1$ and $v_2$ (not necessarily different, but different from $u_3$) such that $u_i$ is adjacent to $v_i$ for $i = 1, 2$. But then, the vertices $e_1$ and $e_3$ will be present in the clique induced by the edges incident
on the vertex \( u_1 \) and the vertices \( e_2 \) and \( e_3 \) will be present in the clique induced by the edges incident on the vertex \( u_2 \). Therefore, every vertex in \( T \) belongs to another clique in \( L(G) \) which is a contradiction to the assumption that \( L(G) \) is clique vertex irreducible. Hence every triangle in \( G \) has at least two vertices of degree two.

Now, let \( u \in V(G) \) and \( N(u) = \{u_1, u_2, ..., u_p\} \), where \( p \geq 2 \) and if \( p = 2 \) then \( u_1 \) is not adjacent to \( u_2 \). Let \( e_i \) be the vertex in \( L(G) \) corresponding to the edge \( uu_i \) in \( G \) for \( i = 1, 2, ..., p \). Let \( C \) be the clique \( < e_1, e_2, ..., e_p > \) in \( L(G) \). If \( u \) has no pendant vertex attached to it then every \( u_i \) has a neighbor \( v_i \neq u \) for \( i = 1, 2, ..., p \). The \( v_i \)'s are not necessarily pairwise different. Moreover, some \( v_i \) can be equal to some \( u_j \) with \( j \neq i \), except in the case \( p = 2 \). Therefore, for each \( i \), every \( e_i \) in \( L(G) \) will be present in another clique, either induced by the edges incident on the vertex \( u_i \) in \( G \) or by the edges in a triangle containing \( u \) and \( u_i \) in \( G \). But this is a contradiction to the assumption that \( L(G) \) is clique vertex irreducible. Hence, every vertex of degree greater than one in \( G \) has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

Fig : 6.1 gives an example of a graph whose line graph is clique vertex irreducible.

\[ \text{Fig : 6.1} \]

**Theorem 6.1.2.** Let \( G \) be a connected graph. The second iterated line graph \( L^2(G) \) is clique vertex irreducible if and only if \( G \) is one of the following graphs.
Proof. By Theorem 6.1.1, $L^2(G)$ is clique vertex irreducible if and only if

(1) Every triangle in $L(G)$ has at least two vertices of degree two.

(2) Every vertex of degree greater than one in $L(G)$ has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

By (2), every non-pendant edge in $G$ must have a pendant edge attached to it on one end vertex and the degree of that end vertex must be two.

Case 1 : $L(G)$ has a triangle.

A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in $G$. Let it correspond to a triangle in $G$. If any of the vertices of this triangle has a neighbor outside the triangle, then two vertices in the corresponding triangle in $L(G)$ have neighbors outside the triangle, which is a contradiction. Therefore, since $G$ is connected, in this case $G$ must be $K_3$.

If the triangle in $L(G)$ corresponds to a $K_{1,3}$ in $G$, then two of the edges of this $K_{1,3}$ cannot have any other edge incident on any of its end vertices. Therefore, $G$ cannot have a vertex of degree greater than three. Moreover, two vertices of $K_{1,3}$ in $G$ must be pendant vertices. Again, by (2) and since $G$ is connected, we conclude that $G$ is either $K_{1,3}$ or the graph (vii).

Case 2 : $L(G)$ has no triangle.
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Since $L(G)$ has no triangle, $G$ cannot have a $K_3$ or a vertex of degree greater than or equal to 3. Therefore, since $G$ is connected, $G$ must be a path or a cycle of length greater than three. Again, by (2), $G$ cannot be a path of length greater than five or a cycle. Therefore $G$ is $K_2$, $P_3$, $P_4$ or $P_5$.

Corollary 6.1.3. Let $G$ be a connected graph. The $n^{th}$ iterated line graph $L^n(G)$ is clique vertex irreducible if and only if $G$ is $K_3$, $K_{1,3}$ or $P_k$ where $n + 1 \leq k \leq n + 3$, for $n \geq 3$.

Theorem 6.1.4. The line graph $L(G)$ is clique irreducible if and only if every triangle in $G$ has a vertex of degree two.

Proof. Let $G$ be a graph such that every triangle in $G$ has a vertex of degree two. Let $C$ be a clique in $L(G)$.

Case 1: The clique $C$ is induced by the vertices corresponding to the edges in $G$ which are incident on a vertex of degree at least three.

An edge of $C$ can be present in another clique of $L(G)$ if and only if the corresponding pair of edges in $G$ lies in a triangle. Thus, if every edge of $C$ lies in another clique of $L(G)$, then $G$ has an induced $K_p$, where $p$ is at least four. But, this contradicts the assumption that every triangle in $G$ has a vertex of degree two.

Case 2: The clique $C$ is induced by the vertices corresponding to the edges in $G$ which are incident on a vertex of degree two and which do not lie in a triangle.

In this case, $C$ is $K_2$ which always has an edge of its own.

Case 3: The clique $C$ is induced by the vertices corresponding to the edges which lie in a triangle $T$ in $G$. 
Since \( T \) has a vertex \( v \) of degree two, the vertices corresponding to the edges which are incident on \( v \) induce an edge in \( C \) which does not lie in any other clique of \( L(G) \).

Therefore, \( G \) is clique irreducible.

Conversely, assume that \( G \) is a clique irreducible graph. Let \(< u_1, u_2, u_3 >\) be a triangle in \( G \). Let \( e_1, e_2, e_3 \) be the vertices in \( L(G) \) which correspond to the edges \( u_1u_2, u_2u_3, u_3u_1 \) of \( G \). \( T = < e_1, e_2, e_3 > \) is a clique in \( L(G) \). If \( d(u_i) > 2 \) for each \( i \), there exist \( v_1, v_2, v_3 \) such that \( u_i \) is adjacent to \( v_i \) for \( i = 1, 2, 3 \) (\( v_1, v_2 \) and \( v_3 \) are not necessarily different, but they are different from \( u_1, u_2 \) and \( u_3 \)). Then the edges \( e_1e_2, e_2e_3 \) and \( e_3e_1 \) of \( L(G) \) will be present in the cliques induced by edges which are incident on the vertices \( u_1, u_2 \) and \( u_3 \) respectively. Therefore, every edge in \( T \) is in another clique of \( L(G) \), which is a contradiction. \( \Box \)

**Theorem 6.1.5.** The second iterated line graph \( L^2(G) \) is clique irreducible if and only if \( G \) satisfies the following conditions.

1. Every triangle in \( G \) has at least two vertices of degree two.
2. Every vertex of degree three has at least one pendant vertex attached to it.
3. \( G \) has no vertex of degree greater than or equal to four.

**Proof.** Let \( G \) be a graph such that \( L^2(G) \) is clique irreducible. By Theorem 6.1.4, every triangle in \( L(G) \) has a vertex of degree two. Then, we have the following cases.

Case 1: The triangle in \( L(G) \) corresponds to a triangle in \( G \).

Let \(< u_1, u_2, u_3 >\) be a triangle in \( G \). Let \( e_1, e_2, e_3 \) be the vertices in \( L(G) \) which correspond to the edges \( u_1u_2, u_2u_3, u_3u_1 \) of \( G \). At least one of the vertices
of the triangle \(<e_1, e_2, e_3>\) in \(L(G)\) must be of degree two. Let \(e_1\) be a vertex of degree two in \(L(G)\). Since \(e_2\) and \(e_3\) belong to \(N(e_1)\) in \(L(G)\), \(e_1\) has no other neighbors in \(L(G)\). Therefore, the corresponding end vertices, \(u_1\) and \(u_2\) in \(G\) have no other neighbors. Hence (1) holds.

Case 2: The triangle in \(L(G)\) corresponds to a \(K_{1,3}\) (need not be induced) in \(G\).

Let \(e_1, e_2, e_3\) be the vertices in \(L(G)\) corresponding to the edges \(vu_1, uu_2, uu_3\) in \(G\). At least one of the vertices of the triangle \(<e_1, e_2, e_3>\) in \(L(G)\) must be of degree two. Let \(e_1\) be a vertex of degree two in \(L(G)\). Vertices \(e_2\) and \(e_3\) belong to \(N(e_1)\) in \(L(G)\) and hence \(e_1\) has no other neighbors in \(L(G)\). Therefore, the corresponding end vertices, \(u\) and \(u_1\) in \(G\) have no other neighbors. Since \(u\) has no other neighbors (3) holds and since \(u_1\) has no other neighbors (2) holds.

Conversely, assume that \(G\) is a graph which satisfies all the three conditions. A triangle in \(L(G)\) corresponds to a triangle or a \(K_{1,3}\) (need not be induced) in \(G\). A triangle in \(L(G)\) which corresponds to a triangle in \(G\) has at least one vertex of degree two by (1). Again, a triangle in \(L(G)\) which corresponds to a \(K_{1,3}\) in \(G\) has at least one vertex of degree two by (2) and (3). Therefore, every triangle in \(L(G)\) has at least one vertex of degree two and by Theorem 6.1.4, \(L^2(G)\) is clique irreducible.

\[\square\]

**Theorem 6.1.6.** Let \(G\) be a connected graph. If \(G \neq K_3\) then, \(L^3(G)\) is clique irreducible if and only if \(G\) satisfies the following conditions.

(1) \(G\) is triangle free.

(2) \(G\) has no vertex of degree greater than or equal to four.

(3) At least two of the vertices of every \(K_{1,3}\) in \(G\) are pendant vertices.

(4) If \(uv\) is an edge in \(G\), then either \(u\) or \(v\) has degree less than or equal to two.
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Proof. Let $L^3(G)$ be clique irreducible. By Theorem 6.1.5, $L(G)$ satisfies,

(1') Every triangle in $L(G)$ has at least two vertices of degree 2.

(2') Every vertex of degree three in $L(G)$ has at least one pendant vertex attached to it.

(3') $L(G)$ has no vertex of degree greater than or equal to 4.

A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in $G$. Every triangle in $L(G)$ has at least two vertices of degree two implies that every triangle in $G$ has its three vertices of degree two. i.e: $G$ is a triangle, because $G$ is connected. Since $G \neq K_3$, $G$ must be triangle free. Also, every $K_{1,3}$ in $G$ has at least two pendant vertices and the degree of a vertex cannot exceed three. Therefore (1), (2) and (3) hold. Again (3') implies that no edge in $G$ can have more than three edges incident on its end vertices. Therefore, (4) holds.

Conversely, assume that the given conditions hold. Since $G$ is triangle free, a triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in $G$. Therefore, by (2) and (3) every triangle in $L(G)$ has at least two vertices of degree two.

Let $e$ be a vertex of degree three in $L(G)$ and let $uv$ be the corresponding edge in $G$. Since $e$ is of degree three in $L(G)$, the number of edges incident on $u$ in $G$ together with the number of edges incident on $v$ in $G$ is three. If $u$ (or $v$) has three more edges incident on it then $u$ (or $v$) will be of degree at least four which is a contradiction to the condition (2). Therefore, $u$ has two neighbors and $v$ has one neighbor (or vice versa) in $G$. Let $u_1$ and $u_2$ be the neighbors of $u$, and let $v_1$ be the neighbor of $v$ in $G$. Then $<u, v, u_1, u_2> = K_{1,3}$ in $G$ and hence at least two of $v, u_1$ and $u_2$ must be pendant vertices. Since $v$ is not a pendant vertex, $u_1$ and $u_2$ must be pendant vertices. Therefore, $e$ has two pendant vertices attached to it in $L(G)$ corresponding to the edges $uu_1$ and $uu_2$ in $G$. Hence (2') is satisfied.
Again, (2), (3) and (4) together imply (3'). Since the conditions (1'), (2') and (3') are satisfied, by Theorem 6.1.5, $L^3(G)$ is clique irreducible.

**Theorem 6.1.7.** Let $G$ be a connected graph. The fourth iterated line graph $L^4(G)$ is clique irreducible if and only if $G$ is $K_3, K_{1,3}, P_n$ with $n \geq 5$ or $C_n$ with $n \geq 4$.

**Proof.** Let $L^4(G)$ be clique irreducible. Then by Theorem 6.1.6, if $L(G) \neq K_3$ then $L(G)$ must be triangle free. If $L(G) = K_3$ then $G$ is either $K_3$ or $K_{1,3}$. If $L(G)$ is triangle free then $G$ is triangle free and cannot have vertices of degree greater than or equal to three. Therefore, $G$ is either a path or a cycle of length greater than three.

Conversely, if $G$ is $K_3, K_{1,3}, P_n$ or $C_n$ then $L^4(G)$ is either a triangle, a path or a cycle and all of them are clique irreducible.

**Corollary 6.1.8.** For $n \geq 5$, $L^n(G)$ is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible.

### 6.2 Gallai graphs

In this section, we give structural and forbidden subgraph characterizations for the Gallai graph to be clique irreducible, clique vertex irreducible and weakly clique irreducible.

**Theorem 6.2.1.** The Gallai graph $\Gamma(G)$ is clique vertex irreducible if and only if for every $v \in V(G)$, every maximal independent set $I$ in $N(v)$ with $|I| \geq 2$ contains a vertex $u$ such that $N(u) - \{v\} = N(v) - I$. 
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Proof. Let $G$ be a graph such that its Gallai graph $\Gamma(G)$ is clique vertex irreducible. A clique $C$ in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set $I$ of size at least two in $N(v)$. Let $I = \{v_1, v_2, \ldots, v_p\}$, where $p \geq 2$, be a maximal independent set in $N(v)$. Let $e_i$ be the vertex in $\Gamma(G)$ corresponding to the edge $vv_i$ in $G$ for $i = 1, 2, \ldots, p$. Let $C$ be the clique $\langle e_1, e_2, \ldots, e_p \rangle$ in $\Gamma(G)$. Let $e_i$ be the vertex in $C$ which does not belong to any other clique in $G$. Therefore, $e_i$ has no neighbors in $\Gamma(G)$ other than those in $C$. Hence, $N(v_i) - \{v\} = N(v) - I$.

Conversely, assume that for every $v \in V(G)$, every maximal independent set $I = \{v_1, v_2, \ldots, v_p\}$ in $N(v)$ contains a vertex $u$ such that $N(u) - \{v\} = N(v) - I$. If $C$ is a clique of size one, it contains a vertex of its own. Otherwise, let $C$ be defined as above. By our assumption, there exists a vertex $u = v_i$ such that $N(u) - \{v\} = N(v) - I$. Therefore, $e_i$ has no neighbors outside $C$. Hence $C$ has a vertex $e_i$ of its own. \qed

Fig: 6.2 gives an example of a graph whose Gallai graph is clique vertex irreducible.

\begin{center}
\includegraphics[width=0.5\textwidth]{fig.png}
\end{center}

Fig: 6.2

**Theorem 6.2.2.** If $\Gamma(G)$ is clique vertex reducible then $G$ contains one of the graphs in Fig: 6.3 as an induced subgraph.
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Proof. Let $G$ be a graph such that $\Gamma(G)$ is clique vertex reducible and let $C$ be a clique in $\Gamma(G)$ such that each vertex of $C$ belongs to some other clique in $\Gamma(G)$. Consider the order relation $\leq$ among the vertices of $C$ where $e \leq e'$ if $N[e] \leq N[e']$. If $\leq$ is a total ordering, then every vertex adjacent to the minimum vertex $e$ is also adjacent to all the vertices in $C$. Therefore, by maximality of $C$, $e$ cannot have neighbors outside $C$. This is a contradiction to the assumption that $e$ belongs to some other clique of $\Gamma(G)$. So, there exist two vertices $e_1$ and $e_2$ in $C$ which are not comparable. That is, there exist vertices $f_1$ and $f_2$ of $\Gamma(G)$ such that $e_i$ is adjacent to $f_j$ if and only if $i = j$. Let $uv_1$ and $uv_2$ be the edges corresponding to $e_1$ and $e_2$, respectively. Then $v_1$ and $v_2$ are non-adjacent. Let $u_1$ and $u_2$ be the end points of $f_1$ and $f_2$, respectively, which are both different from $v$. $v_1$ and $v_2$.

Case 1: Both $f_1$ and $f_2$ correspond to the edges incident to $v$.

In this case, $u_1$ and $u_2$ are adjacent to $v$. $u_i$ is adjacent to $v_j$ if and only if $i \neq j$ and $u_1$ and $u_2$ can be either adjacent or not. Therefore $< v, v_1, v_2, u_1, u_2 >$ is the graph (i) or (ii) in Fig : 6.3.

Case 2: None of $f_1$ and $f_2$ correspond to the edges incident to $v$.

In this case, $u_1$ and $u_2$ are adjacent to $v_1$ and $v_2$, respectively, and not to $v$. If
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If \( u_1 = u_2 \) then \( G \) contains an induced \( C_4 \). If \( u_1 \neq u_2 \) and \( G \) does not contain an induced \( C_4 \), then \( < v, v_1, v_2, u_1, u_2 > \) is either \( P_5 \) or \( C_5 \).

Case 3: Exactly one of \( f_1 \) and \( f_2 \) correspond to the edges incident to \( v \), say \( f_1 \).

In this case, \( u_1 \) is adjacent to both \( v \) and \( v_2 \) and is not adjacent to \( v_1 \). The vertex \( u_2 \) is adjacent to \( v_2 \) and is not adjacent to \( v \). If \( u_2 \) is adjacent to \( v_1 \) then \( G \) contains an induced \( C_4 \). Otherwise, \( < v, v_1, v_2, u_1, u_2 > \) is the graph (vi) or (vii) in Fig: 6.3.

\[ \square \]

Remark 6.2.1. The converse need not be true. For example consider the graph \( G \) in Fig: 6.4. It contains (iv) in Fig: 6.3 as an induced subgraph. Still \( \Gamma(G) \) is clique vertex irreducible.

![Graphs](image)

**Fig: 6.4**

**Theorem 6.2.3.** The Gallai graph \( \Gamma'(G) \) is clique irreducible if and only if for every \( v \in V(G) \), \( < N(v) >^c \) is clique irreducible.

**Proof.** A clique \( C \) in \( \Gamma(G) \) of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex \( v \in V(G) \) whose other end vertices form a maximal independent set \( I \) of size at least two in \( N(v) \). Therefore, \( C \) has an edge which does not belong to any other clique of \( \Gamma(G) \) if and only if \( I \) has a pair of vertices both of which together does not belong to any other maximal independent set in \( N(v) \). But, this happens if and only if every clique of size at least two in \( < N(v) >^c \) has an edge which does not belong to any other
clique in $<N(v)>^e$, since a maximal independent set in a graph corresponds to a
clique in its complement.

**Theorem 6.2.4.** The second iterated Gallai graph $\Gamma^2(G)$ is clique irreducible if
and only if for every $uv \in E(G)$, either $<N(u) - N(v)>$ and $<N(v) - N(u)>$
are clique vertex irreducible or one among them is a clique and the other is clique
irreducible.

**Proof.** By Theorem 6.2.3, $\Gamma^2(G)$ is clique irreducible if and only if for every $e \in V(\Gamma(G))$, $<N(e)>^e$ is clique irreducible.

Let $e = uv \in E(G)$, $N(u) - N(v) = \{u_1, u_2, ..., u_p\}$ and $N(v) - N(u) = \{v_1, v_2, ..., v_l\}$. Also let $e_i = uu_i$ for $i = 1, 2, ..., p$ and $f_j = vv_j$ for $j = 1, 2, ..., l$. $N_{\Gamma(G)}(e) = \{e_1, e_2, ..., e_p, f_1, f_2, ..., f_l\}$. $<N(e)>^e$ is clique irreducible if and only if every maximal independent set $I$ in $<N(e)>$ has a pair of vertices of its own. $e_i$
is not adjacent to $e_j$ if and only if $u_i$ is adjacent to $u_j$. Similarly, $f_i$ is not adjacent
to $f_j$ if and only if $v_i$ is adjacent to $v_j$. So, $I = \{e_{i_1}, e_{i_2}, ..., e_{i_k}, f_{j_1}, f_{j_2}, ..., f_{j_l}\}$ if
and only if $\{u_{i_1}, u_{i_2}, ..., u_{i_k}\}$ is a clique in $<N(u) - N(v)>$ and $\{v_{j_1}, v_{j_2}, ..., v_{j_l}\}$ is
a clique in $N(v) - N(u)$. Therefore, every maximal independent set $I$ in $N_{\Gamma(G)}(e)$
has a pair of vertices of its own if and only if either both $<N(u) - N(v)>$ and
$<N(v) - N(u)>$ are clique vertex irreducible or one among them is a clique and
the other is clique irreducible.

**Theorem 6.2.5.** If $\Gamma(G)$ is clique reducible then $G$ contains one of the following
graphs as an induced subgraph.
Proof. Let \( \Gamma(G) \) be a clique reducible graph. By Lemma 1.1.9 and Lemma 1.1.12, \( \Gamma(G) \) contains at least one of the Hajo's graph as an induced subgraph. A Hajo's graph is an induced subgraph of \( \Gamma(G) \) if and only if \( G \) contains one of the graphs in Fig : 6.5 as an induced subgraph. Hence the theorem.

Remark 6.2.2. The converse need not be true. Let \( G \) be the graph in Fig : 6.6.

\[
V(G) = \{v, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8\}. \]

Let \( < v, v_1, v_2, v_3, u_1, u_2, u_3 \) be the graph (i) in Fig : 6.5 and let \( w_i \)s for \( i = 1, 2, \ldots, 8 \) induce a complete graph.
Also, let $w_1$ be adjacent to $\{v_1, v_2, v_3\}$, $w_2$ be adjacent to $\{v_1, v_2, u_3\}$, $w_3$ be adjacent to $\{v_1, u_2, v_3\}$, $w_4$ be adjacent to $\{v_1, u_2, u_3\}$, $w_5$ be adjacent to $\{u_1, v_2, v_3\}$, $w_6$ be adjacent to $\{u_1, v_2, u_3\}$, $w_7$ be adjacent to $\{u_1, u_2, v_3\}$, $w_8$ be adjacent to $\{u_1, u_2, u_3\}$, and $v$ adjacent to $w_i$ for $i = 1, 2, \ldots, 8$.

In $\Gamma(G)$ the vertices corresponding to the edges with one end vertex $v$ induces $K_6$ minus a perfect matching in which the vertices of each of the eight triangles are adjacent to another vertex each. The remaining vertices induce the graph $H = 4K_{1,8}$. Therefore, $\Gamma(G)$ is clique irreducible.

**Theorem 6.2.6.** The Gallai graph of a graph $G$, $\Gamma(G)$, is weakly clique irreducible if and only if for every vertex $u \in V(G)$, $< N(u) >^c$ is weakly clique irreducible.

**Proof.** Let $G$ be a graph such that $\Gamma(G)$ is weakly clique irreducible. Let $u_1u_2$ be an edge in $< N(u) >^c$ and let $e_i$ be the vertex in $\Gamma(G)$ corresponding to the edge $uu_i$ in $G$ for $i = 1, 2$. Since $\Gamma(G)$ is weakly clique irreducible and $e_1e_2$ is an edge in $\Gamma(G)$, let $C = < e_1, e_2, \ldots, e_k >$ be the essential clique in $\Gamma(G)$ which contains the edge $e_1e_2$. For $i = 3, 4, \ldots, k$, let $u_1u_i$ be the edge in $G$ corresponding to the vertex $e_i$ in $\Gamma(G)$. Let $e_i e_j$ be the essential edge in $C$. Therefore, there exist no independent set in $N(u)$ which contains both the vertices $u_i$ and $u_j$. Hence, there is no clique in $< N(u) >^c$ which contains the edge $u_iu_j$, other than the clique $S = < u_1, u_2, \ldots, u_k >$. Therefore, $S$ is an essential clique in $< N(u) >^c$ which contains the edge $u_1u_2$. Since the edge $u_1u_2$ was arbitrary, $< N(u) >^c$ is weakly clique irreducible.

The converse can be proved similarly. \qed
6.3 Iterations of the anti-Gallai graph

In this section the anti-Gallai graph and all its iterations which are clique irreducible, clique vertex irreducible and weakly clique irreducible are characterized.

**Theorem 6.3.1.** The anti-Gallai graph $\Delta(G)$ is clique vertex irreducible if and only if $G$ does neither contain $K_4$ nor one of the Hajo’s graphs as an induced subgraph.

**Proof.** Let $G$ be a graph which does neither contain $K_4$ nor one of the Hajo’s graphs as an induced subgraph. The cliques of $\Delta(G)$ are induced by the edges corresponding to the edges of $G$ incident on a vertex of degree at least 3 whose other end vertices induce a complete graph and by the edges which lie in a triangle. In the first case $G$ contains an induced $K_4$, which is a contradiction. Therefore, the cliques of $\Delta(G)$ are induced by the edges which lie in a triangle. Let $< u_1, u_2, u_3 >$ be a triangle in $G$. Let $e_1, e_2, e_3$ be the vertices in $\Delta(G)$ corresponding to the edges $u_1u_2, u_2u_3, u_3u_1$ in $G$. Then $< e_1, e_2, e_3 >$ is a clique in $\Delta(G)$. If a vertex $e_i$ for $i = 1, 2, 3$ lies in another clique of $\Delta(G)$, then the edge corresponding to $e_i$ lies in another triangle. Therefore, the end vertices of the edge corresponding to $e_i$ in $G$ has a neighbor $v_i$ for $i = 1, 2, 3$. $v_i \neq v_j$ if $i \neq j$ and $v_1, v_2, v_3$ are not adjacent to $u_3, u_1, u_2$, respectively, since otherwise $G$ contains a $K_4$, which is a contradiction. Then, $< u_1, u_2, u_3, v_1, v_2, v_3 >$ is one of the Hajo’s graphs, a contradiction. Hence, $G$ is clique vertex irreducible.

Conversely, assume that $G$ is clique vertex irreducible. If $G$ contains $K_4$ or one of the Hajo’s graphs as an induced subgraph, then there exists a clique in $\Delta(G)$, corresponding to a triangle in $G$, which shares each of its vertices with some other
Lemma 6.3.2. If $G$ is $K_4$-free then $\Delta(G)$ is diamond free.

Proof. Let $G$ be a graph which does not contain $K_4$ as an induced subgraph. Therefore, a triangle in $\Delta(G)$ can only be induced by a triangle in $G$. If two vertices of the triangle in $\Delta(G)$ have a common neighbor, then it forces $G$ to have a $K_4$, a contradiction. Therefore, $\Delta(G)$ is diamond free.

Theorem 6.3.3. The second iterated anti-Gallai graph $\Delta^2(G)$ is clique vertex irreducible if and only if $G$ does not contain $K_4$ as an induced subgraph.

Proof. By Theorem 6.3.1, $\Delta^2(G)$ is clique vertex irreducible if and only if $\Delta(G)$ does neither contain $K_4$ nor one of the Hajo's graphs as an induced subgraph.

Let $G$ be a graph which does not contain $K_4$ as an induced subgraph. Therefore, $G$ does not contain $K_5$ as an induced subgraph and hence $\Delta(G)$ does not contain $K_4$ as an induced subgraph. Again, by Lemma 6.3.2, $\Delta(G)$ cannot have diamond as an induced subgraph and hence it does not contain any of the Hajo's graph as an induced subgraph. Hence, $\Delta^2(G)$ is clique vertex irreducible.

Conversely, assume that $\Delta^2(G)$ is clique vertex irreducible. If $G$ contains $K_4$ as an induced subgraph then in $\Delta(G)$ the vertices corresponding to the edges of this $K_4$ induce $K_6$ minus a perfect matching which is the fourth Hajo's graph, a contradiction. Therefore, $G$ does not contain $K_4$ as an induced subgraph.

Theorem 6.3.4. The $n^{th}$ iterated anti-Gallai graph $\Delta^n(G)$ is clique vertex irreducible if and only if $G$ does not contain $K_{n+2}$ as an induced subgraph.

Proof. By Theorem 6.3.3, $\Delta^n(G)$ is clique vertex irreducible if and only if $\Delta^{n-2}(G)$
does not contain $K_4$ as an induced subgraph. $\Delta^{n-2}(G)$ does not contain $K_4$ as an induced subgraph if and only if $\Delta^{n-3}(G)$ does not contain $K_5$ as an induced subgraph. Proceeding like this, we get that $\Delta(G)$ does not contain $K_{n+1}$ as an induced subgraph if and only if $G$ does not contain $K_{n+2}$ as an induced subgraph. Therefore, $\Delta^n(G)$ is clique vertex irreducible if and only if $G$ does not contain $K_{n+2}$ as an induced subgraph.

**Theorem 6.3.5.** The anti-Gallai graph $\Delta(G)$ is clique irreducible if and only if $G$ does not contain $K_4$ as an induced subgraph.

**Proof.** Let $G$ be a graph which does not contain $K_4$ as an induced subgraph. By Lemma 6.3.2 and Lemma 1.1.10, $\Delta(G)$ is clique irreducible.

Conversely, if $G$ contains a $K_1 = \langle u_1, u_2, u_3, u_4 \rangle$, then it follows that the clique in $\Delta(G)$, corresponding to the triangle $\langle u_1, u_2, u_3 \rangle$ in $G$, shares each of its edges with some other clique. Therefore, if $\Delta(G)$ is clique irreducible, then $G$ cannot have $K_4$ as an induced subgraph. 

**Theorem 6.3.6.** The $n^{th}$ iterated anti-Gallai graph $\Delta^n(G)$ is clique irreducible if and only if $G$ does not contain an induced $K_{n+3}$.

**Proof.** By Theorem 6.3.5, $\Delta^n(G)$ is clique irreducible if and only if $\Delta^{n-1}(G)$ does not contain an induced $K_4$. $\Delta^{n-1}(G)$ does not contain an induced $K_4$ if and only if $\Delta^{n-2}(G)$ does not contain an induced $K_5$. Proceeding like this, we get, $\Delta(G)$ does not contain an induced $K_{n+2}$ if and only if $G$ does not contain an induced $K_{n+3}$. Therefore, $\Delta^n(G)$ is clique irreducible if and only if $G$ does not contain an induced $K_{n+3}$.

**Theorem 6.3.7.** The anti-Gallai graph of a graph $G$, $\Delta(G)$ is weakly clique irreducible if and only if $G$ is $K_4$-free.
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Proof. Let $<u_1, u_2, ..., u_k>$ be a clique of size greater than or equal to four in $G$. Let $e_{ij}$ be the vertex corresponding to the edge $u_iu_j$ in $G$ for $i, j \in \{1, 2, ..., k\}$ and $i \neq j$. (Note that $e_{ij} = e_{ji}$). Consider the edge $e_{12}e_{13}$ in $\Delta(G)$. The cliques in $\Delta(G)$ obtained from the clique $<u_1, u_2, ..., u_k>$ in $G$, which contains the edge $e_{12}e_{13}$ are $<e_{12}, e_{13}, ..., e_{1k}>$ and $<e_{12}, e_{23}, e_{31}>$. Both these cliques are not essential, since all of their edges are present in at least one of the cliques $<e_{21}, e_{23}, ..., e_{2k}>$, $<e_{31}, e_{32}, ..., e_{3k}>$ or $<e_{1i}, e_{ij}, e_{j1}>$ for $i, j \in \{3, 4, ..., k\}$ and $i \neq j$. Similarly, if there is any other clique which contains the vertices $u_1, u_2$ and $u_3$ in $G$, then the corresponding cliques in $\Delta(G)$ will not be essential. Therefore, $\Delta(G)$ is not weakly clique irreducible.

Conversely, assume that $G$ is $K_4$-free. Then by Theorem 6.3.5, $\Delta(G)$ is clique irreducible and hence is weakly clique irreducible.

Corollary 6.3.8. The anti-Gallai graph of a graph $G$, $\Delta(G)$ is weakly clique irreducible if and only if it is clique irreducible.

Corollary 6.3.9. The $n^{th}$ iterated anti-Gallai graph $\Delta^n(G)$ is weakly clique irreducible if and only if it is $K_{n+3}$-free.

6.4 Cographs

In this section the cographs which are clique irreducible, clique vertex irreducible and weakly clique irreducible are characterized.

Lemma 6.4.1. If $G^c$ has at least three non-trivial components then $G$ is clique reducible.
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Proof. Let $G$ be a graph such that $G^c$ has at least three non trivial components. Let $H_1, H_2, ..., H_p$ be the components of $G^c$. Let $G_i = H_i^c$ for $i = 1, 2, ..., p$. Then, $G = G_1 \lor G_2 \lor ... \lor G_p$. Also, any clique of $G$ is the join of the cliques of $G_i$ s for $i = 1, 2, ..., p$. At least three of the $H_i$ s are non-trivial and hence at least three of the $G_i$ s have more than one clique. Let $C_{ij}$ for $j = 1, 2$ be any two of the cliques of $G_i$ for $i = 1, 2, 3$. Let $S_i$ be a clique of $G_i$ for $i = 4, 5, ..., p$. Consider the clique $C_{11} \lor C_{21} \lor C_{31} \lor S_4 \lor ... \lor S_p$. Every edge of this clique is present in at least one of the cliques $C_{11} \lor C_{21} \lor C_{31} \lor S_4 \lor ... \lor S_p$, $C_{12} \lor C_{21} \lor C_{31} \lor S_4 \lor ... \lor S_p$. Therefore, $G$ is clique reducible.

Lemma 6.4.2. If $G^c$ has at least two non-trivial components then $G$ is clique vertex reducible.

Proof. Let $G$ be a graph whose complement has at least two non trivial components. Let $H_i, G_i, C_{ij}$ for $i = 1, 2, ..., p$ and $j = 1, 2$ and $S_i$ for $i = 3, 4, ... p$ be defined as in the proof of Lemma 6.4.1 and consider the clique $C_{11} \lor C_{21} \lor C_{31} \lor S_4 \lor ... \lor S_p$. Every vertex of this clique is present in at least one of the cliques $C_{11} \lor C_{22} \lor C_{31} \lor S_4 \lor ... \lor S_p$, $C_{12} \lor C_{21} \lor C_{31} \lor S_4 \lor ... \lor S_p$. Therefore, $G$ is clique vertex reducible.

Remark 6.4.1. If $G$ is clique irreducible then $G^c$ is either connected or has exactly two non trivial components and if $G$ is clique vertex irreducible then $G^c$ is either connected or has exactly one non-trivial component.

Lemma 6.4.3. The clique vertex reducible graphs and the clique reducible graphs are closed for the operations of union and join.

Theorem 6.4.4. A cograph $G$ is clique vertex irreducible if and only if it can be reduced to a trivial graph by recursively deleting universal vertices in each of the components.
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**Proof.** The proof is by induction on $|V| = n$. For $n = 1$ the theorem is trivially true. Assume that the theorem is true for any cograph with less than $n$ vertices. A disconnected graph is clique vertex irreducible if and only if each of its components is clique vertex irreducible. Therefore, we may assume that, $G$ is a connected cograph with $n$ vertices. Then $G = G_1 \vee G_2$. If both $G_i$s are not complete, then $G^c$ will have at least two non trivial components which by Lemma 6.4.2 is a contradiction. Therefore, let $G_1$ be complete. Every vertex of $G_1$ is a universal vertex of $G$. Deleting these vertices we get a cograph $G_2$ with less than $n$ vertices.

Any clique $C$ of $G_2$ corresponds to a clique $G_1 \vee C$ of $G$ and hence has a vertex which does not lie in any other clique of $G_2$. Therefore, $G_2$ is a clique irreducible cograph with less than $n$ vertices and hence by the induction hypothesis $G_2$ can be reduced to trivial graph by deleting universal vertices. Hence, the theorem.

**Theorem 6.4.5.** A connected cograph $G$ is clique irreducible if and only if $G = G_1 \vee G_2 \vee K_p$ where $G_1$ and $G_2$ are clique vertex irreducible cographs such that $G_i^c$ is connected for $i = 1, 2$ and $p \geq 0$.

**Proof.** Let $G = G_1 \vee G_2 \vee K_p$ where $G_1$ and $G_2$ are connected clique vertex irreducible cographs and $p \geq 0$. Any clique of $G$ is of the form $H = H_1 \vee H_2 \vee K_p$, where $H_1$ and $H_2$ are cliques of $G_1$ and $G_2$ respectively. Since, $G_1$ and $G_2$ are clique vertex irreducible, there exist vertices $v_1 \in H_1$ and $v_2 \in H_2$ such that they do not lie in any other clique of $G$. Therefore, the edge $v_1v_2$ of $H$ does not lie in any other clique of $G$ and hence $G$ is clique irreducible.

Conversely, assume that $G$ is clique irreducible. Since $G$ is a cograph $G^c$ must be disconnected. Therefore by Lemma 6.4.1, $G^c$ has exactly two non trivial components. So, $G = G_1 \vee G_2 \vee K_p$, where $G_i^c$ and $G_j^c$ are both connected. Let $H_{11}$ and $H_{12}$ be any two cliques of $G_1$ and $H_{21}$ and $H_{22}$ be any two cliques of $G_2$. 
$H = H_{11} \lor H_{21} \lor K_p$ is a clique of $G$. Every edge in $H_{11}$, every edge which joins $H_{11}$ to a vertex of $K_p$ and every edge in $K_p$ will be present in the clique $H_{11} \lor H_{22} \lor K_p$. Again, every edge in $H_{21}$, every edge which joins $H_{21}$ to a vertex of $K_p$ and every edge in $K_p$ will be present in the clique $H_{12} \lor H_{21} \lor K_p$. But, $H$ has an edge which does not lie in any other clique of $G$. Therefore, that edge must be an edge which joins a vertex of $H_{11}$ to a vertex of $H_{21}$. Let that edge be $u_1u_2$. But, then $u_1$ and $u_2$ cannot be present in any other clique of $G_1$ and $G_2$ respectively. Therefore, $G_1$ and $G_2$ are clique vertex irreducible.

**Theorem 6.4.6.** The weakly clique irreducible cographs can be recursively characterized as follows.

1. $K_1$ is a weakly clique irreducible cograph.
2. If $G_1$ and $G_2$ are weakly clique irreducible cographs, then so is their union $G_1 \cup G_2$.
3. If $G_1$ is a weakly clique irreducible cograph, then so is $G_1 \lor K_p$.
4. If $G_1$ and $G_2$ are non-complete weakly clique irreducible cographs, then $G_1 \lor G_2$ is a weakly clique irreducible cograph if and only if every edge in $G_i$ belongs to at least one vertex essential clique, for $i = 1, 2$.

**Proof.** The graph $K_1$ is weakly clique irreducible and union of any two weakly clique irreducible graphs is weakly clique irreducible. The cliques of $G_1 \lor K_p$ are of the form $H_1 \lor K_p$, where $H_1$ is a clique in $G_1$. If $H_1$ is essential in $G_1$ then so is $H_1 \lor K_p$ in $G_1 \lor K_p$. If $H_1$ is an isolated vertex $u$, then again $H_1 \lor K_p$ is an essential clique in $G_1 \lor K_p$ with all edges with one end vertex $u$ as essential edges. Therefore, $G_1 \lor K_p$ is weakly clique irreducible if $G_1$ is weakly clique irreducible.
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Let $G_1$ and $G_2$ be non-complete weakly clique irreducible cographs such that every edge in $G_i$ belongs to at least one vertex essential clique, for $i = 1, 2$. If $H_i$ is a vertex essential clique in $G_i$ where $v_i \in V(H_i)$ is the vertex which does not belong to any other clique in $G_i$ for $i = 1, 2$ then $H_1 \lor H_2$ is an essential clique in $G_1 \lor G_2$ where $v_1v_2$ is an essential edge. Therefore, every edge in $E(G_i)$ belongs to an essential clique in $G_1 \lor G_2$, since every edge in $G_i$ belongs to at least one vertex essential clique, for $i = 1, 2$. Let $u \in V(G_1)$ and $v \in V(G_2)$. Consider the edge $uv \in E(G_1 \lor G_2)$.

Case 1: $u$ and $v$ are isolated vertices in $G_1$ and $G_2$ respectively.

In this case, $uv$ is a clique and is essential.

Case 2: $u$ is an isolated vertex in $G_1$, but $v$ is not an isolated vertex in $G_2$.

Let $v' \in N(v)$. There exist a vertex essential clique $C$ in $G_2$ which contains the edge $vv'$. Let $w$ be the essential vertex in $C$. Therefore, $uw$ is an essential edge in the clique $\{u\} \lor C$. Hence the edge $uv$ belongs to the essential clique $\{u\} \lor C$ in $G_1 \lor G_2$.

The case where, $u$ is not an isolated vertex in $G_1$, but $v$ is an isolated vertex in $G_2$ can be proved similarly.

Case 3: $u$ and $v$ are not isolated vertices in $G_1$ and $G_2$ respectively.

Let $u' \in N(u)$ and $v' \in N(v)$. Let $H_1$ and $H_2$ be the vertex essential cliques in $G_1$ and $G_2$ respectively, which contains the edges $uu'$ and $vv'$ respectively. Let $w_i$ be the essential vertex in $H_i$ for $i = 1, 2$. Therefore, $w_1w_2$ is an essential edge in the clique $H_1 \lor H_2$. Hence the edge $uv$ belongs to the essential clique $H_1 \lor H_2$ in $G_1 \lor G_2$. 
Therefore, every edge in $G_1 \vee G_2$ belongs to an essential clique and hence it is weakly clique irreducible.

Conversely, assume that $G$ is a weakly clique irreducible cograph. If $G$ is disconnected then it is the union of weakly clique irreducible cographs. If $G$ has universal vertices then it is the join of a weakly clique irreducible graph with $K_p$, where $p$ is the number of universal vertices.

Therefore, let $G$ be a connected cograph without universal vertices. Hence, $G = G_1 \vee G_2$ where both $G_1$ and $G_2$ are not complete. None of the edges in $E(G_1) \cup E(G_2)$ are essential, since both $G_1$ and $G_2$ contains more than one clique. Therefore an essential edge in $G_1 \vee G_2$, if it exist, must be of the form $uv$, where $u \in V(G_1)$ and $v \in V(G_2)$. Then, $u$ and $v$ are essential vertices of $G_1$ and $G_2$ respectively. Hence, for $i = 1, 2$, the edges of $G_i$ can be covered by essential cliques if and only if every edge in $G_i$ belongs to at least one vertex essential clique. Therefore, if $G_1$ and $G_2$ are non-complete weakly clique irreducible cographs, then $G_1 \vee G_2$ is a weakly clique irreducible cograph if and only if every edge in $G_i$ belongs to at least one vertex essential clique, for $i = 1, 2$.

Hence, the theorem. □

\section{Distance hereditary graphs}

In this section the distance hereditary graphs which are clique irreducible, clique vertex irreducible and weakly clique irreducible are characterized.

\textbf{Lemma 6.5.1.} The clique vertex reducible (clique reducible) graphs are closed.
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under the operations of attaching a pendant vertex, a true twin and a false twin.

Proof. Let $G$ be a clique vertex reducible (clique reducible) graph and $C$ be a clique in $G$, all of whose vertices (edges) are present in some other clique in $G$.

The cliques of the graph obtained by attaching a pendant vertex $u$ to a vertex $v$ of $G$ are the cliques of $G$ together with the clique $uv$. Therefore $C$ is a clique in this new graph and all of its vertices (edges) are present in some other clique.

The cliques of the graph obtained by attaching a true twin $u$ to the vertex $v$ of $G$ are the cliques of $G$ which does not contain the vertex $v$ and the cliques of $G$ which contains $v$ together with the vertex $u$. If $v \notin C$, then $C$ is a clique in the new graph and all its vertices (edges) are present in some other clique. If $v \in C$, then all the vertices (edges) in $C$ other than $u$ (the edges with one end vertex $u$) are already present in some other clique. Since $v$ is (the edges with one end vertex $v$ are) present in some other clique, $u$ (the edges with one end vertex $u$) also must be present in some other clique.

The cliques of the graph obtained by attaching a false twin $u$ to the vertex $v$ of $G$ are the cliques of $G$ and the cliques of the form $(S \cup \{u\}) - \{v\}$, where $S$ is a clique in $G$ which contains the vertex $v$. Therefore, $C$ is a clique in this new graph and all of its vertices (edges) are present in some other clique.

Theorem 6.5.2. The clique vertex irreducible distance hereditary graphs can be recursively characterized as follows.

(1) $K_1$ is a clique vertex irreducible distance hereditary graph.

(2) If $G$ is a clique vertex irreducible distance hereditary graph, then so is the graph obtained by attaching a pendant vertex to a vertex $v \in V(G)$, where $v$ satisfies either $N(v)$ is not complete or there exists $w \in N(v)$ such that $N(w) = N(v)$. 

$\Box$
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(3) If $G$ is a clique vertex irreducible distance hereditary graph, then so is the graph obtained by attaching a true twin.

(4) If $G$ is a clique vertex irreducible distance hereditary graph, then so is the graph obtained by attaching a false twin to a vertex $v \in V(G)$, where $v$ satisfies $\langle N(v) \rangle$ is complete.

Proof. The graph $K_1$ is clique vertex irreducible. Let $G$ be a clique vertex irreducible graph. Let $G'$ be a graph obtained by attaching a pendant vertex $u$ to a vertex $v$ where $v$ satisfies the conditions in theorem. The cliques of $G'$ are precisely, the cliques of $G$ and the edge $uv$. The clique $uv$ contains the vertex $u$ which does not belong to any other clique of $G'$. Every clique of $G'$ which does not contain $v$ also has a vertex which does not lie in any other clique of $G'$, since $G$ is clique vertex irreducible. Let $C$ be a clique of $G$ which contains the vertex $v$. If $N(v)$ is not complete then $C$ contains a vertex $v' \neq v$ which is not present in any other clique of $G$ and hence of $G'$. If $N(v)$ is complete, then $C$ contains a vertex which does not belong to any other clique of $G'$ if and only if there exist a vertex $w \in V(C)$ which does not belong to any other clique of $G$. i.e; if and only if $N(w) = N(v)$.

Let $G$ be a clique vertex irreducible graph. Let $G'$ be the graph obtained by attaching a true twin $u$ to a vertex $v$ of $G$. The cliques of $G'$ are precisely, the cliques of $G$ which does not contain $v$ and the cliques of $G$ which contains $v$ together with the vertex $u$. Each such clique contains a vertex which does not lie in any other clique of $G'$, since $G$ is clique vertex irreducible and hence $G'$ is also clique vertex irreducible.

Let $G'$ be the graph obtained by attaching a false twin $u$ to a vertex $v$ of $G$. The cliques of $G'$ are the cliques of $G$ together with the cliques of the form
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(C \cup \{u\}) - \{v\} \text{ where } C \text{ is a clique of } G \text{ which contains } v. \text{ The cliques of } G' \text{ which does not contain } v \text{ will continue to have a vertex which does not lie in any other clique. Let } C \text{ be a clique of } G \text{ which contains the vertex } v. \text{ Every vertex of the clique } C \text{ other than } v \text{ will be present in the clique } (C \cup \{u\}) - \{v\} \text{ also. Therefore, } C \text{ contains a vertex which does not lie in any other clique of } G' \text{ if and only if } v \text{ does not belong to any other clique of } G, \text{ which happens if and only if } <\mathcal{N}(v)> \text{ is complete.}

Also, any distance hereditary graph } G \text{ can be obtained from } K_1 \text{ by the operations of attaching pendant vertices, introducing true twins and introducing false twins (Lemma 1.1.3) and by Lemma 6.5.1, the theorem follows.

\begin{theorem}
The weakly clique irreducible distance hereditary graphs can be recursively characterized as follows.

1. } K_2 \text{ is a clique irreducible distance hereditary graph.}

2. If } G \text{ is a clique irreducible distance hereditary graph then so is the graph obtained by attaching a pendant vertex.}

3. If } G \text{ is a clique irreducible distance hereditary graph then so is the graph obtained by attaching a true twin.}

4. If } G \text{ is a clique irreducible distance hereditary graph then so is the graph obtained by attaching a false twin to a vertex } V \text{ if } <\mathcal{N}(v)> \text{ is clique vertex irreducible.}

\end{theorem}

\textbf{Proof.} The graph } K_2 \text{ is clique irreducible. Let } G \text{ be a clique irreducible graph. Let } G' \text{ be the graph obtained by attaching a pendant vertex } u \text{ to a vertex } v \text{ of } G. \text{ The cliques of } G' \text{ are precisely, the cliques of } G \text{ and the edge } uv. \text{ Every clique}
contains an edge which does not lie in any other clique of \( G' \) and hence \( G' \) is clique irreducible.

Let \( G \) be a clique irreducible graph. Let \( G' \) be the graph obtained by attaching a true twin \( u \) to a vertex \( v \) of \( G \). The cliques of \( G' \) are precisely, the cliques of \( G \) which does not contain \( v \) and the cliques of \( G \) which contains \( v \) together with the vertex \( u \). Every such clique contains an edge which does not lie in any other clique, since \( G \) is clique irreducible and hence \( G' \) is also clique irreducible.

Let \( G' \) be the graph obtained by attaching a false twin \( u \) to a vertex \( v \) of \( G \). The cliques of \( G' \) are the cliques of \( G \) together with the cliques of the form \( (C \cup \{u\}) - \{v\} \) where \( C \) is a clique of \( G \) which contains \( v \). The cliques of \( G' \) which does not contain \( v \) will continue to have an edge which does not lie in any other clique. Let \( C \) be a clique of \( G \) which contains the vertex \( v \). Every edge of \( C \) which does not contain \( v \) will be present in the clique \( (C \cup \{u\}) - \{v\} \) also. Therefore, \( C \) contains an edge which does not lie in any other clique of \( G' \) if and only if there exists an edge \( vv' \) which does not lie in any other clique of \( G \). Therefore, the vertex \( v' \) is not present in any clique of \( N(v) \) other than \( C - \{v\} \). So, \( N(v) \) is clique vertex irreducible.

The converse follows by Lemma 1.1.3 and by Lemma 6.5.1. \( \square \)

**Lemma 6.5.4.** The class of weakly clique reducible graphs is closed under the operations of attaching pendant vertices, true twins and false twins.

**Proof.** Let \( G \) be a weakly clique reducible graph and let \( e \) be the edge which is not covered by any of the essential cliques in \( G \).

Let \( G' \) be the graph obtained from \( G \) by attaching a pendant vertex. The essen-
tial cliques of $G'$ are the essential cliques of $G$ together with the newly introduced edge. But, these essential cliques will not cover the edge $e$.

Let $G'$ be the graph obtained from $G$ by attaching a true twin $v$ to a vertex $u$. The essential cliques of $G'$ are the essential cliques of $G$ which does not contain the vertex $u$ and the cliques of the form $C \cup \{v\}$, where $C$ is an essential clique in $G$ which contains the vertex $u$. Still, the edge $e$ is not covered by essential cliques.

Let $G'$ be the graph obtained from $G$ by attaching a false twin $v$ to a vertex $u$. The essential cliques of $G'$ are the essential cliques of $G$ which does not contain the vertex $u$, the cliques of the form $(C - \{u\}) \cup \{v\}$ and $C$, where $C$ is an essential clique in $G$ which contains the vertex $u$ and which has an essential edge with one end vertex $u$. Again, the edge $e$ is not covered by the essential cliques.

Hence the lemma. \(\square\)

**Theorem 6.5.5.** A distance hereditary graph $G$ is weakly clique irreducible if and only if all its induced subgraphs are weakly clique irreducible.

**Theorem 6.5.6.** A distance hereditary graph $G$ is weakly clique irreducible if and only if $G$ does not contain $F_{19}$ in Fig. 1.9 as an induced subgraph.

**Proof.** By Theorem 6.5.5, $G$ is weakly clique irreducible if and only if all its induced subgraphs are weakly clique irreducible. But, a graph $G$ is hereditary weakly clique irreducible if and only if $G$ does not contain any of the graphs in Fig. 1.9 as an induced subgraph (Lemma 1.1.11). But, $G$ cannot have any of the graphs $F_1, F_2, \ldots, F_{18}$ as an induced subgraph, since they contain gem as an induced subgraph (Lemma 1.1.4). Hence, the theorem. \(\square\)

**Corollary 6.5.7.** A cograph $G$ is weakly maximal clique irreducible if and only if
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$G$ does not contain $F_{19}$ in 1.1.9 as an induced subgraph.

Proof. Since, cographs are a subclass of distance hereditary graphs (Lemma 1.1.5) and $F_{19}$ in Fig : 1.9 is a cograph, the corollary follows. \qed

Theorem 6.5.8. The weakly clique irreducible distance hereditary graphs can be recursively characterized as follows.

1. $K_2$ is a weakly clique irreducible distance hereditary graph.

2. If $G$ is a weakly clique irreducible distance hereditary graph then so is the graph obtained by attaching pendent vertices to the vertices of $G$.

3. If $G$ is a weakly clique irreducible distance hereditary graph then so is the graph obtained by attaching true twins to the vertices of $G$.

4. If $G$ is weakly clique irreducible distance hereditary graph then so is the graph obtained by attaching false twins to a vertex $u$ where $N(u)$ is $C_4$-free is also weakly clique irreducible.

Proof. The graph $K_2$ is weakly clique irreducible. Let $G$ be a weakly clique irreducible distance hereditary graph. If $G$ does not have $F_{19}$ as an induced subgraph then a graph obtained by any of the above operations also cannot have $F_{19}$ as an induced subgraph. Therefore, they are all weakly clique irreducible.

Conversely, by the recursive definition of distance hereditary graphs (Lemma 1.1.3), it is enough if we could prove that, attaching a false twin $v$ to a vertex $u$ which contains a $C_4 = u_1, u_2, u_3, u_4$ in $N(u)$, gives a weakly clique reducible graph. Clearly, $u, v, u_1, u_2, u_3, u_4$ is $F_{19}$.

Hence the theorem. \qed
List of some open problems

1. Characterize non-isomorphic graphs of the same order having isomorphic
   Gallai graphs (anti-Gallai graphs).

2. Characterize graphs $G$ for which the Gallai and the anti-Gallai operators
   commute.

3. Characterize graphs $G$ for which $\Gamma(G) = \Delta(G)$.

4. Characterize all connected graphs which satisfy $\gamma(G) = \gamma_{cd}(G)$.

5. Characterize all connected graphs which satisfy $\gamma_{cd}(G) = \gamma_{gcd}(G)$.

6. Identify the domination parameters which satisfy Vizing's type relation under
   any of the graph products.

7. Characterize the clique perfect graphs [73].

8. Identify special classes of clique perfect graphs.

9. Estimate sharp upper bounds for the clique transversal number for special
   classes of graphs and characterize the graphs which attains this upper bound.

10. Does there exist graph classes which satisfy the $< t >$-property for every $t$?

11. Characterize the clique irreducible graphs, the clique vertex irreducible graphs
    and the weakly clique irreducible graphs.