CHAPTER IV

CONTRACTION TYPE OF MAPPINGS ON

3-METRIC SPACE

(60 - 75)
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CONTRACTION TYPE OF MAPPING ON 3-METRIC SPACE

4.1 In a paper, Gahler\(^1\) investigated the notion of a 2-metric, a real valued function of a point triples on a set \(X\), whose abstract properties were suggested by the area function for a triangle determined by a triple in euclidean spaces, associated with a given two metric was a natural topology. It is proved that every metric space is a 2-metric space.

For the first time P.L.Sharma and others\(^2\) found contraction type of condition to get fix point in 2-metric space.

Gahler\(^3\) further introduced the notion of \(m\)-metric space.

Our object in this chapter is to introduce 3-metric space using the method of simplex from algebraic topology and fixed contraction type of condition to obtain fixed point theorem.

\[\text{References:}\]
1) Gahler, S \hfill (1)
2) Sharma, P.L, Sharma B.K. and Iseki, K \hfill (1)
3) Gahler, S. \hfill (2)
We need some definitions:

**Definition 1.** A 3-metric space is a space $X$ in which for each quadruple point $x, y, z, w$ there exists a real valued function $P(x, y, z, w)$ such that

(4.1.1.a) To each non-degenerate 2-simplex $< x, y, z >$ in $X$ there is a point $w \in X$ satisfying, $P(x, y, z, w) \neq 0$

(4.1.1 b) $P(x, y, z, w) = 0$ only when the 2-simplex

$< x, y, z, w >$ is degenerate

(4.1.3) $P(x, y, z, w) = P(x, w, z, y) = P(y, z, w, x) = P(z, w, x, y) = \quad$

(4.1.4) $P(x, y, z, w) \leq P(x, y, z, u) + P(x, y, u, w)$

$+ P(x, u, z, w) + P(u, y, z, w)$

Putting $z = w$ here

$0 \leq P(x, y, z, u) + P(x, y, u, z) = 2P(x, y, u, z) \leq 0$

This can further be generalised by means of notion of simplex.

**Definition 2.** A sequence $[x_n]$ is a 3-metric space $X$ is called a convergent sequence if there is an $x \in X$ such that

$\lim P(x_n, x, a, b) = 0$ for all $a, b \in X$. Here $x$ is called the limit of $[x_n]$. 
DEFINITION 3: A sequence $\{x_n\}$ in a 3-metric space $X$ is called a Cauchy sequence, if $\lim P(x_m, x_n, a, b) = 0$ for all $a, b, x \in X$.

DEFINITION 4: A 3-metric space in which every Cauchy sequence converges is called a complete 3-metric space.

DEFINITION 5: A 3-metric space is called bounded, if there exists a constant $K$ such that $P(x, y, z, w) \leq K$ for all $x, y, z, w$.

Now we shall prove.

THEOREM 1: Let $X$ be a complete bounded 3-metric space, and let $f_n(x)$ ($n=1, 2, \ldots$) be a family of mappings of $X$ into itself. Suppose that there exists a sequence of non-negative integers $\{m_n\}$, and non-negative numbers $\alpha, \beta$ such that for all $x, y, a, b \in X$ and every pair $i, j$ with $i \neq j$

\[
(4.1.5) \quad P(f^{m_i}_i(x), f^{m_j}_j(y), a, b) \\
\leq \alpha [P(x, f^{m_i}_i(x), a, b) + P(y f^{m_j}_j(y), a, b)] \\
+ \beta P(x, y, a, b),
\]

where $2\alpha + \beta < 1$. Then the sequence of mappings $\{f_n\}$ has a unique common fixed point.
Proof: Let

\[ g_i = f_i^m \quad (i = 1, 2, \ldots) \], then for \( i \neq j \)

by (4.1.5) we have

\[ P(g_i(x), g_j(y), a, b) \leq \alpha \left[ P(x, g_1(x), a, b) + P(y, g_j(y), a, b) \right] + \beta P(x, y, a, b) \]

we take an element \( x_0 \in X \), then by recursive way, we define a sequence \( \{ x_n \} \) by \( x_n = g_n(x_{n-1}) \) \( (n = 1, 2, \ldots) \) By (4.1.3) we have

\[ P(x_1, x_2, a, b) = P(g_1(x_0), g_2(x_1), a, b) \]

\[ \leq \alpha \left[ P(x_0, x_1, a, b) + P(x_1, x_2, a, b) \right] + \beta (x_0, x_1, a, b) \]

\[ \leq ( \alpha + \beta ) P(x_0, x_1, a, b) + \alpha P(x_1, x_2, a, b) \]

Hence

\[ P(x_1, x_2, a, b) \leq \frac{\alpha + \beta}{1 - \alpha} P(x_0, x_1, a, b) \]

Similarly, we have

\[ P(x_1, x_2, a, b) = P(g_2(x_1), g_3(x_2), a, b) \]

\[ \leq \alpha \left[ P(x_0, x_1, a, b) + P(x_2, x_3, a, b) \right] + \beta P(x_1, x_2, a, b) \]
Hence \[ P(x_2, x_3, \ldots, a, b) \leq \frac{(n+2)}{(1-a)} P(x_1, x_2, a, b) \]
\[ \leq \left( \frac{n+2}{1-a} \right)^2 P(x_0, x_1, a, b) \]

In general, for all \( a, b \in X \), we have

\[ (4.1.7) \quad P(x_0, x_{n+1}, a, b) \leq \left( \frac{n+2}{1-a} \right)^n P(x_0, x_1, a, b) \]

Therefore by (4.1.6) and (4.1.7), for \( n < m \), we have

\[ P(x_n, x_m, a, b) \leq P(x_n, x_m, x_{n+1}) + P(x_n, x_m, x_{n+1}, b) \]
\[ + P(x_n, x_{n+1}, a, b) + P(x_m, x_n, a, b) \]
\[ \leq P(x_n, x_{n+1}, x_m, a) + P(x_n, x_{n+1}, x_m, b) \]
\[ + P(x_n, x_{n+1}, a, b) + P(x_{n+1}, x_m, b) \]
\[ + P(x_m, x_{n+1}, a, b) + P(x_{n+2}, x_{m+1}, x_m, b) \]
\[ + P(x_{n+1}, x_{n+2}, a, b) + P(x_{n+2}, x_m, a, b) \]
\[ + P(x_{n+1}, x_m, a, x_{n+2}) \]
\[ = P(x_n, x_{n+1}, x_m, a) + P(x_{n+1}, x_{n+2}, x_m, a) \]
\[ + P(x_n, x_{n+1}, x_m, b) + P(x_{n+1}, x_{n+2}, x_m, b) \]
\[ + P(x_n, x_m, a, b) + P(x_{n+1}, x_{m+2}, a, b) + P(x_{n+2}, x_m, a, b) \]

\[ \leq \sum_{k=n}^{m} \left( \frac{\theta}{1-\alpha} \right)^k P(x_0, x_1, x_m, a) + \sum_{k=n}^{m} \left( \frac{\theta+\beta}{1-\alpha} \right)^k P(x_0, x_1, x_m, b) \]

\[ + \sum_{k=n}^{m} \left( \frac{\theta+\beta}{1-\alpha} \right)^k P(x_0, x_1, a, b) \]

\[ \leq 3K \sum_{k=n}^{m} \left( \frac{\theta+\beta}{1-\alpha} \right)^k \]

Since \( x \) is bounded, and \( 2\alpha + \beta < 1 \), we have

\[ \lim_{n, m \to \infty} P(x_n, x_m, a, b) = 0 \]

Hence \( \{x_n\} \) is a Cauchy sequence.

Therefore \( \{x_n\} \) has a limit \( u \).

For the point \( u \),

\[ P(u, g_n(u), a, b) \leq P(u, g_n(u), x_{n+1}) + P(u, g_n(u), x_{m+1}, b) + P(u, x_{m+1}, a, b) \]

\[ + P(x_{m+1}, g_n(u), a, b) \]
\[ P(u, x_{m+1}, a, b) + P(g_n(u), g_{m+1}(x_m), a, b) \]
\[ + P(g_n(u), g_{m+1}(x_m), u, b) + P(g_{m+1}(x_m), g_n(u), a, b) \]
\[ \leq P(u, x_{m+1}, a, b) + \alpha [ P(u, g_n(u), a, b) \]
\[ + P(x_m, g_{m+1}(x_m), a, u) \]
\[ + \beta P(u, x_m, u, b) \]
\[ + \alpha [ P(x_m, g_{m+1}(x_m), a, b) + P(u, g_n(u), a, b) ] \]
\[ + \beta P(x_m, u, a, b) \]

Letting \( m \rightarrow \infty \), we have

\[ P(u, g_n(u), a, b) \leq \alpha P(u, g_n(u), a, b) \]
\[ \therefore u = g_n(u) \text{ for } n = 1, 2, \ldots \]

Therefore \( u \) is a common fixed point of \( g_n \). It is easily seen from (4.1.6) that \( u \) is a unique common fixed point of \( g_n \). For the point \( u \),

\[ u = g_n(u) = f_n^m(u) \]
Therefore \( f_n(u) = f_n(f_n^m(u)) = f_n^m(f_n(u)) \).

Hence \( f_n(u) \) is a fixed point of \( g_n \), and
\[
f_n(u) = u \quad (n=1, 2, \ldots)
\]

Therefore \( \{f_n\} \) has a common fixed point \( u \), if \( u \) is another fixed point, then by the hypothesis, we have
\[
P(u, v, a, b) \leq \beta P(u,v,a,b)
\]
which implies \( u = v \). This completes the proof.

2.2 Now we shall extended theorem 1 by proving

**Theorem 2:** Let \( X \) be a complete bounded 3-metric space and let \( f_n(x) \quad (n=1, 2, \ldots) \) be a family of mappings of \( X \) into itself. Suppose that there are non-negative numbers \( a, \beta, \gamma \) such that for all \( x, y, a, b \in X \) and every pair \( i, j \) with \( i \neq j \)

\[
(2.2.1) \quad P(f_i^{m_i}(x), f_j^{m_j}(y), a, b) \leq \alpha [P(x, f_i^{m_i}(x), a, b) + \beta [P(y, f_j^{m_j}(y), a, b)] + \gamma [P(x, y, a, b)]
\]

where \( 2\alpha + 2\beta + \gamma < 1 \). Then the sequence of mappings \( \{f_n\} \) has a unique common fixed point.
Proof: As in theorem 1, let

\[ g_i = f_i^m \quad (i = 1, 2, \ldots) \], then for \( i \neq j \),

by (2.2.1) we have

\[ (2.2.2) \quad P(g_i(x), g_j(y), a, b) \leq \alpha \left[ P(x, g_1(y), a, b) + P(y, g_1(x), a, b) + \beta \right] + \gamma P(x, y, a, b) \]

Take an element \( x_0 \in X \), then we define a sequence

\[ \{x_n\} \quad \text{by} \quad x_n = g_n(x_{n+1}) \quad (n = 1, 2, \ldots) \]

then by (4.1.3), we have

\[ P(x_1, x_2, a, b) = P(g_1(x_0), g_2(x_1), a, b) \]

\[ \leq \alpha \left[ P(x_0, x_1, a, b) + P(x_1, x_2, a, b) \right] + \beta \left[ P(x_0, x_2, a, b) + P(x_1, x_1, a, b) \right] + \gamma P(x_0, x_1, a, b) \]

\[ = (\alpha + \gamma) P(x_0, x_1, a, b) + \alpha P(x_1, x_2, a, b) + \beta P(x_0, x_2, a, b) \]
\[ \sum (\alpha \gamma) \leq (x_0, x_1, a, b) + \alpha P(x_1, x_2, a, b) + \beta [P(x_0, x_1, a, b) + P(x_1, x_2, a, b)] \]

Hence \[ P(x_1, x_2, a, b) \leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta} P(x_0, x_1, a, b) \]

Similarly, we have

\[ P(x_2, x_3, a, b) = P(g_2(x_1), g_3(x_2), a, b) \leq (\alpha + \gamma) P(x_1, x_2, a, b) + \alpha P(x_2, x_3, a, b) \]

\[ + \beta [P(x_1, x_2, a, b) + P(x_2, x_3, a, b)] \]

Therefore, we have

\[ P(x_2, x_3, a, b) \leq \frac{\alpha + \beta + \gamma}{1 - \alpha - \beta} P(x_1, x_2, a, b) \]

In general, we have

\[(4.2.3) P(x_n, x_{n+1}, a, b) \leq \left(\frac{\alpha + \beta + \gamma}{1 - \alpha - \beta}\right)^n P(x_0, x_1, a, b) \]

Now arguing exactly the same as in theorem 1, since \( X \) is complete and \( 2\alpha + 2\beta + \gamma < 1 \), we have

\[ \lim_{n \to \infty} P(x_n, x_m, a, b) = 0. \]

Hence \( \{x_n\} \) has a limit point \( u \). With the help of theorem 1, the rest of the proof can be written.
4.3 In 1960 topological semifield was defined\(^4\) and was proved that any topological semifield contains a topological field iso-explicit with the real line. These elements are denoted by Greek letter \(\alpha\) and follows the rules operation on real numbers.

We require some definition:

**Definition** : We shall call a commutative associative topological semifield \(i.g\) there is isolated in some set \(K\) satisfying:

1. \(K + K \subseteq K, \ K \cdot K \subseteq K\)
2. \(K - K = E\)
3. The least upper bound and greatest upper bounded exists.
4. For \(a, b, c \in K\), the equation \(a x = b\) has at least one solution in \(K\).
5. The intersection \(K \cap (-E)\) contains only the zero element of the ring.
6. We denote by \(F_\alpha (a \in E)\) the totality of all elements \(x \in E\) satisfying the condition \(a x \in K\). Then the totality of all sets of the form \(\beta + f\alpha (a, \beta, c \in E)\) forms a basis system of closed sets of topological space \(E\).

\(^4\) Antonovskii, M. Ye i Béityanskii, V. G. and Saryanaskov. T. A.
Remark

1. The axiom for a topological semifield are so chosen that its properties recall those of the fields of real numbers.

2. We assume here that multiplication require its partial inveribility. That is why it is called 'Semifield'.

3. An interesting example due to Daktyarer show that the requirement $K^+ K \subseteq K$ in the definition can not be weakened by $K^+ K = K$.

4. It is also known that a commutative topological ring admits at most one semifield structure.

We shall call elements of the set $K$ positive elements of the semifield $E$ and the elements of the set $\bar{K} = K$ will be called boundary elements of the semifield $E$. We agree to write the relation $x \succ y \in K$, $x \succ y \in \bar{K}$ also in the form $x > y$, $x \geq y$ (or in the form $y < x$, $y \leq x$). In particular, the inequality $x \succ 0$ means $x \in K$ and $x \geq 0$ means that $x \in \bar{K}$.

The set $K$ contains elements which are different from zero.

Now we define $\delta$-metric space over topological semifield $E$. 
**DEFINITION:** Let $E$ be a semifield and $K$ is the set of all its positive elements. The set $X$ is called a 3-metric space over the semifield $E$ if there exists a metric $P : X \times X \times X \times X \rightarrow E$ for each quadruple point $x, y, z, w \in X$ such that

(4.3.1a) To each non degenerate 2-simplex $\langle x, y, z \rangle \in X$ there is a point $w \in X$ satisfying $P(x, y, z, w) \neq 0$.

(4.3.1b) $P(x, y, z, w) = 0$ only when the 3-simplex $\langle x, y, z, w \rangle$ is degenerate.

(4.3.2) $P(x, y, z, w) = P(x, w, z, y) = \cdots$.

(4.3.3) $P(x, y, z, w) = P(x, y, z, u) + P(x, y, u, w) + P(x, u, z, w) + P(u, y, z, w)$

**Remark:** A 3-metric space over topological semifield is called bounded if there exists a constant $M$ such that:

$P(x, y, z, b) \leq M$ for all $x, y, a, b \in X$.

If $E$ is a field of real numbers then we arrive at the definition of 3-metric space (defined above) also.
If $X$ consists only two points, we get the definition of metric space over topological semifield.

**Definition**: A sequence $\{x_n\}$ in a $3$-metric space over a topological semifield $X$ is called a Cauchy sequence if $\lim P(x_m, x_n, a, b) \in U$ for all $a, b \in X$, where $u \in E$ is the neighborhood of the origin.

**Definition**: A sequence $\{x_n\}$ in a $3$-metric space over topological semifield $X$ is called a convergent sequence if there is a $x \in X$ such that

$$\lim P(x_n, x, a, b) \in U$$

for all $a, b \in X$.

**Definition**: A $3$-metric space over a topological semifield $X$ in which every Cauchy sequence converges is called a complete $3$-metric space. Now we shall prove.

**Theorem 3**: Let $X$ be a complete $3$-metric space over a topological semifield $E$, $f(x)$ be a mapping on $X$ and for all $a \in X$ such that

$$P(f(x), f(y), a, b) \ll \alpha P(x, y, a, b)$$

where $\alpha$ is a positive number less than $1$, then there is a fixed element $x'$ of the mapping $f$ such that

$$f(x') = x'$$
Proof: Take an element $x_0 \in X$, then by recursive we define a sequence \( \{x_n\} \) by

\[ x_{n+1} = f(x_n) \quad (n = 0, 1, 2, \ldots) \] then we have

\[ P(x_0, x_1, a, b) = P(f(x_0) f(x_1), a, b) \]

\[ \ll a P(x_0, x_1, a, b) \]

Again

\[ P(x_0, x_1, x_2, a, b) = P(f(x_1) f(x_2), a, b) \]

\[ \ll a P(x_1, x_2, a, b) \]

\[ \ll a^2 P(x_0, x_1, a, b) \]

In general

\[ P(x_0, x_n+1, a, b) \ll a^n P(x_0, x_1, a, b) \]

For $n < m$, we have

\[ P(x_n, x_m, a, b) \ll P(x_n, x_m, x_{n+1}) + P(x_n, x_m x_{n+1}, b) \]

\[ + P(x_n, x_{n+1}, a, b) + P(x_{n+1}, x_m, a, b) \]

\[ \ll P(x_n, x_{n+1}, x_m, a) + P(x_n, x_{n+1}, x_m, b) \]

\[ + P(x_n, x_{n+1}, a, b) + P(x_{n+1}, x_m, x_{n+2}, b) \]

\[ + P(x_{n+1}, x_{n+2}, a, b) + P(x_{n+2}, x_m, a, b) \]

\[ + P(x_{n+1}, x_m, a, x_{n+2}) \]
= P (x_n, x_1, x_m, a) + P (x_{n+1}, x_{n+2}, x_m, a)
+ P (x_n, x_{n+1}, x_m, b) + P (x_{n+1}, x_{n+2}, x_m, b)
+ P (x_n, x_m, a, b) + P (x_{n+1}, x_{n+2}, a, b)
+ P (x_{n+2}, x_m, a, b)

\leq \sum_{k=n}^{m} a^k P (x_0, x_1, x_m, a) + \sum_{k=n}^{m} b^k P (x_0, x_1, x_m, b)
+ \sum_{k=n}^{m} a^k P (x_0, x_1, a, b)

\leq a k a^n

(1-a)

Hence \{x_n\} is a Cauchy seq^n. Since X is complete,
So \{x_n\} converges to a point u.
For the point u
P (u, f(u), a, b) \leq P (u, f(u), a, x_{m+1})
+ P (u, f(u), x_{m+1}, b) + P (u, x_{m+1}, a, b)
+ P (x_{m+1}, f(u), a, b)

\in U+U+a+u

Hence P (u, f(u), a, b) \in U, Thus u = f (u)

Now suppose \nu is another fixed point. then
P (u, \nu, a, b) = P (f(u), f(\nu), a, b)

\leq a P (u, y, a, b)

which implies that u = \nu

This complete the proof.