CHAPTER II

COMMON FIXED POINT FOR THREE MAPPINGS

IN COMPLETE QUASI GAUGE FUNCTION SPACE

(35 - 51)
COMMON FIXED POINT THEOREM FOR THREE MAPPINGS IN COMPLETE QUASI GAUGE FUNCTION SPACE

(2.1) Quasi-gauge space was first developed by Reilly\(^1\). In his thesis, according to him a quasi-gauge structure for a topological space \((X, J)\) is a family \(p\) of quasi-pseudometrics on \(X\), such that \( J \) has a sub-base i.e. the family

\[
\{ B(x, p, \varepsilon) : x \in X, p \in \mathcal{P}, \varepsilon > 0 \} \quad \text{where} \quad B(x, p, \varepsilon) \text{ is the set } \{ y \in X \mid p(x, y) < \varepsilon \}.
\]

If the topological space \((X, J)\) has a quasi-gauge structure \(p\), then it is called quasi-gauge space and it is denoted by \((X, p)\).

We shall define the quasi-gauge structure in function space.

Let \(X\) be a non-empty set, and \(Y^X\) be a quasi-gauge space, we consider a non-negative real number \(p\) on function space \(Y^X\) having pointwise topology.

On this we define topology called quasi-pseudometric if

\[
(2.1.1) \quad p(f, g)(x) = \sup \{ f(x), g(x) \}
\]

\(^1\) Reilly, I.L. (1)
Satisfying \( p(f,g)(x) = 0 \) for every \( f \) in \( Y^X \) and for all \( p(f,g)(x) = p(f,h)(x) + p(h,g)(x) \) for all \( f,g,h \in Y^X \).

We need some definitions:

**Definition 2.1.1:** A quasi-gauge structure for topological space \((Y^X, p)\) is a family \( P \) of quasi-pseudo metric on \( Y^X \) such that \( P \) has a sub-base.

If a topological space \((Y^X, p)\) has a quasi-gauge structure \( P \), it is called quasigauge function space and is denoted by \((Y^X, P)\). If in addition \((Y^X, P)\) is metrizable, we take \( p \) to consists of \( d \) alone.

**Definition 2.1.2:** If \((Y^X, P)\) is a quasigauge function space, the sequence \( \{f_n\} \) in \( Y^X \) is called left \( p \)-Cauchy where for each \( p \in P \) and each \( \varepsilon > 0 \) there is a \( f \in Y^X \) and an integer \( k \) such that \( p(f,f_m)(x) < \varepsilon \) for all \( m \geq k \).

Similarly the right \( p \)-Cauchy sequence can be defined.

**Definition 2.1.3:** If \((Y^X, P)\) is a quasigauge function space, the sequence \( \{f_n\} \) in \( Y^X \) is called \( p \)-Cauchy where for each \( p \in P \), \( \varepsilon > 0 \) and an integer \( k \) such that

\[ p(f_m, f_n)(x) < \varepsilon \quad \text{for all} \quad m, n \geq k. \]
DEFINITION 2.1.4. An operator $T$ on a quasi-gauge function space $(Y^X, P)$ in to itself is said to be left (right) weak contraction for each $f \in Y^X$ and $p \in P$ there exists $0 \leq \lambda < 1$ such that for each $y$ in $B(f, p, E)$ we have $p[T(f), T(y)](x) \leq \lambda p(f, y)(x)$

$[p(T(y), T(f))(x) \leq \lambda \left[p(g, f)(x)\right].$

2.2. Recently in 1982, we proved the following theorem:

THEOREM 1. Let $T$ be a continuous mapping of a complete metric space X into itself satisfying:

$$(2.2.1) \quad d(Tx, Ty) \leq \alpha \cdot d(v, Ty) \left[1 + d(x, Tx)\right]$$

$$[1 + d(x, y)]$$

$$+ \beta \left[d(x, Tx) + d(y, Ty)\right]$$

$$+ \gamma \left[d(x, Ty) + d(y, Tx)\right] + \delta \cdot d(x, y)$$

for all $x, y \in X$ where $\alpha, \beta, \gamma, \delta \geq 0$ and $\alpha + 2\beta + 2\gamma + \delta < 1$, then $T$ has a unique fixed point in $X$.

In this section we extend our theorem to quasi-gauge function space. In fact we prove:

THEOREM 1. Let $(Y^X, P)$ be a Hausdorff sequentially complete quasi-gauge function space generated by the family $P$ of pseudometric and $T$ is continuous mapping and

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2) Iseki, K., Rajput, S.S. and Sharma, P.L. (1).
(2.2.2) \[ p \left[ T(f), T(g) \right] (x) \leq \alpha \frac{p(g \cdot T(g)) (x) \left[ 1 + p(f \cdot T(f)) (x) \right]}{[1 + p(f \cdot g) (x)]} \]

\[ + \beta \left[ p(f \cdot T(f)) (x) + p(g \cdot T(g)) (x) \right] \]

\[ + \gamma \left[ p(f \cdot T(g)) (x) + p(g \cdot T(f)) (x) \right] \]

\[ + \delta p(f \cdot g) (x) \]

for all \( f, g \in Y^X \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \).

Then \( T \) has a unique fixed point in \((Y^X, p)\).

**Proof:** Let \( f_0 \in Y^X \) and \( \{f_n\} \) be defined by \( f_{2n+1}(x) = T f_{2n}(x) \) for \( n = 1, 2, \ldots \).

Then by hypothesis of the theorem, we have

\[ p \left[ f_{2n+1} \cdot f_{2n+2} \right] (x) = p \left[ T f_{2n} \cdot T f_{2n+1} \right] (x) \]

\[ \leq \alpha \frac{p \left[ f_{2n+1} \cdot T f_{2n+1} \right] (x) \left[ 1 + p(f_{2n} \cdot T f_{2n}) (x) \right]}{[1 + p(f_{2n} \cdot f_{2n+1}) (x)]} \]

\[ + \beta \left[ p(f_{2n} \cdot T f_{2n}) (x) + p(f_{2n+1} \cdot T f_{2n+1}) (x) \right] \]

\[ + \gamma \left[ p(f_{2n} \cdot T f_{2n+1}) (x) + p(f_{2n+1} \cdot T f_{2n}) (x) \right] \]

\[ + \delta p(f_{2n} \cdot f_{2n+1}) \]

\[ \leq \alpha \frac{p \left[ f_{2n+1} \cdot f_{2n+2} \right] (x) \cdot [1 + p(f_{2n} \cdot f_{2n+1}) (x)]}{[1 + p(f_{2n} \cdot f_{2n+1})]} \]
\[ + \beta \left[ p(f_{2n}, f_{2n+1}) (x) + p(f_{2n+1}, f_{2n+2}) (x) \right] \\
+ \gamma \left[ p(f_{2n}, f_{2n+2}) (x) + p(f_{2n+1}, f_{2n+1}) (x) \right] \\
+ \delta p(f_{2n}, f_{2n+1}) (x) \]

Thus
\[ p(f_{2n+1}, f_{2n+2}) (x) \leq \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} p(f_{2n}, f_{2n+1}) (x) \]

Taking \( \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} = h < 1 \), we have
\[ p(f_{2n+1}, f_{2n+2}) (x) \leq h p(f_{2n}, f_{2n+1}) (x) \]

proceeding in this way, we get
\[ p(f_{2n+1}, f_{2n+2}) \leq h p(f_{2n}, f_{2n+1}) (x) \]

\[ \leq h^2 p(f_{2n}, f_{2n+1}) (x) \]
\[ \leq h^{2n} p(f_0, f_1) (x) \]

Now
\[ p(f_n, f_{n+k}) (x) \leq \sum_{i=1}^{n} p(f_{n+i-1}, f_{n+i}) (x) \]
\[ \leq h^n p(f_0, f_1) (x) \rightarrow 0 \text{ as } n \rightarrow \infty \]

Hence the sequence \( \{ f_n \} \) is a Cauchy sequence in \( Y^X \). By the completeness of \( Y^X \), there exists some \( z \) in \( Y^X \) such that
\[ \lim_{n \rightarrow \infty} f_n = z \]
Further the continuity of $T$ implies

$$Tz = T \left( \lim_{n \to \infty} f_n \right) = \lim_{n \to \infty} T f_n = z.$$ Thus

$z$ is a fixed point of $T$.

Now we prove the uniqueness of $z$.

Let $z$ and $w$ be two fixed point of $T$ such that

$Tz = z$ and $Tw = w$

Then

$$p(z,w)(x) = p(Tz,Tw)(x)$$

$$\leq \alpha p(w,Tw)(x) \frac{[1+p(z,Tz)(x)]}{[1+p(2w)(x)]}$$

$$+ \beta [ p(z,Tz)(x) + p(w,Tw)(x) ]$$

$$+ \gamma [ p(z,Tw)(x) + p(w,Tz)(x) ]$$

$$+ \delta p(z,w)(x)$$

$$\leq (2\gamma + \delta) p(z,w)(x)$$

$$< d(z,w)(x) \text{ as } 2\gamma + \delta < 1.$$ This leads to contradiction. Thus $z = w$. Hence $z$ is unique fixed point of $T$. This completes the proof of the theorem.
Now we extend theorem 1 for two mapping, we prove

**Theorem 2:** Let \( T_1 \) and \( T_2 \) are two continuous mapping on a Hausdorff sequentially complete quasi-gauge function space generated by the family \( P \) of pseudometric and

\[
(2.2.3) \quad p(T_1 f, T_2 g) (x)
\]

\[
\leq \frac{\alpha \cdot p(g, T_2 g) (x) \cdot [1 + p(f, T_1 f) (x)]}{[1 + p(f, g) (x)]}
+ \beta \cdot [p(f, T_1 f) (x) + p(g, T_2 g) (x)]
+ \gamma \cdot [p(f, T_2 g) (x) + p(g, T_1 f) (x)]
+ \delta \cdot p(f, g) (x)
\]

for all \( f, g \in Y^X \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \)

Then \( T_1 \) and \( T_2 \) have a unique common fixed point in \((Y^X, P)\)

**Proof:** Let \( f_0 \in Y^X \) be an arbitrary element and, let \( \{ f_n \} \) be defined as \( f_{2n+1}(x) = T_1 f_{2n}(x) \) and \( f_{2n+2}(x) = T_2 f_{2n+1}(x) \) for \( n = 1, 2, \ldots \)

Then by the hypothesis of the theorem, we have

\[
p(f_{2n+1}, f_{2n+2}) (x) = p(T_1 f_{2n}, T_2 f_{2n+1}) (x)
\]

\[
\leq \frac{\alpha \cdot p(f_{2n+1}, T_2 f_{2n+1}) (x) \cdot [1 + p(f_{2n}, T_1 f_{2n}) (x)]}{[1 + p(f_{2n}, f_{2n+1}) (x)]}
\]
\[ + \beta \left[ p \left( f_{2n}, T_{2} f_{2n+1} \right)(x) + p \left( f_{2n+1}, T_{2} f_{2n+1} \right)(x) \right] \\
+ \gamma \left[ p \left( f_{2n}, T_{2} f_{2n+1} \right)(x) + p \left( f_{2n+1}, T_{1} f_{2n} \right)(x) \right] \\
+ \delta p \left( f_{2n}, f_{2n+1} \right)(x) \]

\[ \leq \delta + p \left( f_{2n+1}, f_{2n+2} \right)(x) \cdot \left[ 1 + p \left( f_{2n}, f_{2n+1} \right)(x) \right] \]

\[ \leq \delta + p \left( f_{2n+1}, f_{2n+2} \right)(x) \cdot \left[ 1 + p \left( f_{2n}, f_{2n+1} \right)(x) \right] \]

\[ + \beta \left[ p \left( f_{2n}, f_{2n+1} \right)(x) + p \left( f_{2n+1}, f_{2n+2} \right)(x) \right] \\
+ \gamma \left[ p \left( f_{2n}, f_{2n+2} \right)(x) + p \left( f_{2n+1}, f_{2n+1} \right)(x) \right] \\
+ \delta p \left( f_{2n}, f_{2n+1} \right)(x) \]

Thus,

\[ p \left( f_{2n+1}, f_{2n+2} \right)(x) \leq \frac{p \left( f_{2n}, f_{2n+1} \right)(x)}{1 - \alpha - \beta - \gamma} \] (1 - \alpha - \beta - \gamma)

Taking \( \frac{\beta + \gamma + \delta}{1 - \alpha - \beta - \gamma} \) as \( h \) < 1, we have

\[ p \left( f_{2n+1}, f_{2n+2} \right)(x) \leq h p \left( f_{2n}, f_{2n+1} \right)(x) \]
proceeding in this way, we get

\[ p \left( f_{2n+1}, f_{2n+2} \right) (x) \leq h p \left( f_{2n}, f_{2n+1} \right) (x) \]

\[ \leq h^{2n+1} p \left( f_0, f_1 \right) (x) \]

Finally

\[ p \left( f_n, f_{n+k} \right) (x) \]

\[ \leq \sum_{i=1}^{k} p \left( f_{n+i-1}, f_{n+i} \right) (x) \]

\[ \leq h^n p \left( f_0, f_1 \right) (x) \to 0 \text{ as } n \to \infty \]

\[ \frac{h^n}{(1-h)} \]

Hence the sequence \( \{ f_n \} \) is a Cauchy sequence in \( Y^X \). By completeness of \( Y^X \), there exists some \( z \) in \( Y^X \) such that

\[ \lim_{n \to \infty} f_n = z \]

Further the continuity of \( T_1 \) implies

\[ T_1 z = T_1 \left( \lim_{n \to \infty} f_n \right) = \lim_{n \to \infty} T_1 \left( f_n \right) = z \]
Thus \( z \) is a fixed point of \( T_1 \).

Similarly \( z \) is a fixed point of \( T_2 \).

Hence \( z \) is a common fixed point of \( T_1 \) and \( T_2 \).

Now we shall prove the uniqueness of \( z \).

Let \( z \) and \( w \) be two common fixed points of \( T_1 \) and \( T_2 \) such that

\[
T_1 z = T_2 z = z \quad \text{and} \quad T_1 w = T_2 w = w
\]

Then

\[
p(z, w)(x) = p(T_1 z, T_2 w)(x)
\]

\[
\leq \alpha \cdot p(w, T_2 w)(x) \left[ 1 + p(z, T_1 z)(x) \right]
\]

\[
\leq \left[ 1 + p(z, w)(x) \right]
\]

\[
+ \beta \left[ p(z, T_1 z)(x) + p(w, T_2 w)(x) \right]
\]

\[
+ \gamma \left[ p(z, T_2 w)(x) + p(w, T_1 z)(x) \right]
\]

\[
+ \delta p(z, w)(x)
\]

\[
\leq (z \gamma + \delta)p(z, w)(x)
\]

\[
< p(z, w)(x) \quad \text{as} \quad 2\gamma + \delta < 1.
\]

This leads to contradiction. Thus \( z = w \).

Thus \( z \) is the unique common fixed point of \( T_1 \) and \( T_2 \).

This completes the proof.
Remark. If \( T_1 = T_2 \) we obtain theorem 1.

2.4 Now we shall generalize the above result by taking three mappings. We prove:

**Theorem 3.** Let \( E, F \) and \( T \) are three continuous mapping onto a Hausdorff sequentially complete quasi-gauge function space generated by the family \( \mathcal{P} \) of pseudometric and

\[
E(Y^X) \subseteq T(Y^X), \quad F(Y^X) \subseteq T(Y^X)
\]

(2.2.4) \( ET = TE, \quad FT = TF \) and

\[
p(E_{f^*}F_g)(x) \leq \alpha p(T_{g^*}F_g)(x) + \beta p(T_{f^*}E_g)(x) + \gamma [ p(T_{g^*}E_g)(x) + p(T_{f^*}F_g)(x) ] + \delta p(T_{g^*}T_g)(x)
\]

for all \( f, g \in Y^X \) where \( \alpha, \beta, \gamma, \delta \geq 0 \)

and \( \alpha + 2\beta + 2\gamma + \delta < 1 \).
Then E, F, and T have a unique fixed point in Y^n.

Proof: Let f_0 \in Y^n and as

E (Y^n) \subset T (Y^n) and F (Y^n) \subset T (Y^n),

We define a sequence \{T f_n\} as follows:

(2.2.6) T f_{2n+1} (x) = E f_{2n} (x)

T f_{2n+2} (x) = F f_{2n+1} (x) for n = 1, 2, ...

Now by the hypothesis of the theorem, we have

\text{p} (T f_{2n+1}, T f_{2n+2}) (x) = p (E f_{2n}, F f_{2n+1}) (x)

\leq \alpha \frac{p (T f_{2n+1} + F f_{2n+1}) (x) \left[ 1 + p (T f_{2n}, E f_{2n}) (x) \right] + \beta \left[ p (T f_{2n}, E f_{2n}) (x) + p (T f_{2n+1}, F f_{2n+1}) (x) \right] + \gamma \left[ p (T f_{2n}, F f_{2n+1}) (x) + p (T f_{2n+1}, E f_{2n}) (x) \right] + \delta p (T f_{2n}, T f_{2n+1})}
\[ p(T_{2n+1}, T_{2n+2})(x) \leq \frac{p(T_{2n}, T_{2n+1})(x) + p(T_{2n+1}, T_{2n+2})(x)}{1 + p(T_{2n}, T_{2n+1})(x)} \]

\[ + \beta \left[ p(T_{2n}, T_{2n+1})(x) + p(T_{2n+1}, T_{2n+2})(x) \right] \]

\[ + \gamma \left[ p(T_{2n}, T_{2n+2})(x) + p(T_{2n+1}, T_{2n+1})(x) \right] \]

\[ + \delta p(T_{2n}, T_{2n+1}) \]

Thus

\[ p(T_{2n+1}, T_{2n+2})(x) \leq \frac{\beta + \gamma + \delta}{1 - \beta - \gamma} p(T_{2n}, T_{2n+1})(x) \]

Taking \[ \frac{\beta + \gamma + \delta}{1 - \beta - \gamma} = h < 1 \], we have

\[ p(T_{2n+1}, T_{2n+2})(x) \leq h p(T_{2n}, T_{2n+1})(x) \]

Proceeding in this way, we have

\[ p(T_{2n+1}, T_{2n+2})(x) \leq h p(T_{2n}, T_{2n+1})(x) \]

\[ \leq \cdots \]

Finally we have

\[ p(T_{n}, T_{n+k}) \leq \frac{h^n}{(1-h)} p(T_{0}, T_{1}) \rightarrow 0 \text{ as } n \rightarrow \infty \]
Hence \( \{ T_n \} \) is a Cauchy sequence.

By the completeness of \( Y^X \), \( \{ T_n \} \) converges to a point \( z \) in \( Y^X \).

Thus \( \{ E_n \} \) and \( \{ F_{2n+1} \} \) also converge to \( z \). As \( G \), \( F \) and \( T \) are continuous we have

\[
E(T_{2n}) \rightarrow Ez, \quad F(T_{2n+1}) \rightarrow Fz
\]

As \( T \) commutes with \( E \) and \( F \), we have

\[
E(T_{2n})(x) = T(EF_{2n})(x)
\]

and

\[
F(T_{2n+1})(x) = T(FF_{2n+1})(x) \quad n = 0, 1, 2, \ldots
\]

Thus taking \( n \rightarrow \infty \), we have

\[
Ez(x) = Tz(x) = Fz(x)
\]

Hence

\[
T(Tz) = T(Ez) = ET(z) = E(Ez)
\]

\[
= E(Fz) = T(Fz) = F(Tz)
\]

\[
= F(Ez) = F(Fz)
\]

we have

\[
p(Ez, F(Ez))(x)
\]
$$\alpha \cdot p(T(Ez), F(Ez))(x) \left[ 1 + p(Tz, Ez)(x) \right]$$

$$\left[ 1 + p(Tz, T(Ez))(x) \right]$$

$$+ \beta \left[ p(Tz, Ez)(x) + p(T(Ez), F(Ez))(x) \right]$$

$$+ \gamma \left[ p(Tz, F(Ez))(x) + p(T(Ez), Ez)(x) \right]$$

$$+ \delta \ p(Tz, T(Ez))(x)$$

$$\lesssim (2\gamma + \delta) \ p(Ez, F(Ez))(x)$$

$$< p(Ez, F(Ez))(x), \text{ as } 2\gamma + \delta < 1.$$ 

This leads to a contradiction. Hence

$$Ez = F(Ez)$$

Also $$Eu = F(Eu) = T(Eu) = E(Su)$$

Thus, Eu is a common fixed point of E, F and T.

Let z and w be two different common fixed point of E, F, and T such that

$$Ez = Fz = Tz = z \text{ and}$$

$$Ew = Fw = Tw = w$$

Then

$$p(z, w)(x) = p(Ez, Fw)(x)$$
\[
\mathcal{A} = \alpha \left( \mathcal{I}_{Ez} \mathcal{F} \right)(x) \left[ 1 + p \left( \mathcal{I}_{Ez} \mathcal{F} \right)(x) \right] \\
+ \beta \left[ p \left( \mathcal{I}_{Ez} \mathcal{F} \right)(x) + p \left( \mathcal{I}_{Tw} \mathcal{F} \right)(x) \right] \\
+ \gamma \left[ p \left( \mathcal{I}_{Ez} \mathcal{F} \right)(x) + p \left( \mathcal{I}_{Tw} \mathcal{E} \right)(x) \right] \\
+ \delta \left( \mathcal{I}_{Ez} \mathcal{F} \right)(x)
\]

\[
\leq \left( \alpha \gamma + \delta \right) p \left( \mathcal{Ez} \mathcal{F} \right)(x) < p \left( \mathcal{Ez} \mathcal{F} \right)(x) \quad \text{so} \quad 2\gamma + \delta < 1.
\]

This leads to contradiction, hence \( z = w \).

Thus \( E, F, T \) have a unique common fixed point this complete the proof.

Remark

1) Suppose \( E = F \) and \( T = I \) in theorem 3, we obtain theorem 1 which generalised our theorem A.

2) Taking \( E = F, T = I, \alpha = \beta = \gamma = 0 \) in theorem 3, we obtain the generalization of Banach contradiction theorem for complete quasi-ganage function space.

3) Banach, S \hspace{1cm} (1)
3) Taking $E = F$, $T = I$ and $\alpha = \gamma = \delta = 0$ in the theorem 3, we obtain the extension of result due to Kannan 4) for quasi gauge function space.

4) Taking $E = F$, $T = I$ and $\alpha = \beta = \delta = 0$ in theorem 3, we obtain the extension of result due to Fisher 5) for quasi-gauge function space.

5) Taking $E = F$, $T = I$, and $\alpha = \gamma = 0$ in theorem 3, we obtain the extension of result due to Ciric 6) for quasi gauge function space.

6) Taking $T = I$ in theorem 3, we obtain theorem 2.

7) Taking $E = F = T$, we get theorem 1.

\[ \text{\small 4) Kannan, R (1)} \]
\[ \text{\small 5) Fisher (5)} \]
\[ \text{\small 6) Ciric (2)} \]