CHAPTER I.

INTRODUCTION

(1-34)


1.1. In 1910, Brower gave, perhaps the first result on fixed point. Brower's fixed point theorem states that if \( S \) denotes the closed unit ball with centre at origin in \( \mathbb{E}^n \) (Euclidean \( n \)-dimensional space) and let \( T \) be a continuous mapping of \( S \) into itself, then \( T \) has at least one fixed point \( x \) in \( X \). We replace \( S \in \mathbb{E}^n \) by any topological space \( X \) homeomorphic with \( S \). This result was later extended by Schauder\(^1\) to compact convex sets in Banach space in 1927. Every continuous mapping on a convex compact subsets \( K \) of Banach space has a fixed point. In 1930 Schauder\(^2\) generalised his own result by relaxing the stringent conditions of compactness of \( K \), i.e. A compact self mapping on a closed bounded convex subset of \( K \) of a Banach space has a fixed point.

Banach\(^3\) proved a very important and now a fundamental result on a fixed point, so called Banach contraction principle in 1922.

If \( T \) is a mapping of a complete metric space \((X,d)\) into itself satisfying

\[ \text{---------------} \]

1) Schauder, J. \hspace{1cm} (1)
2) Schauder, J. \hspace{1cm} (2)
3) Banach, S. \hspace{1cm} (1)

\[ \text{---------------} \]
(1.1.1) \[ d(Tx, Ty) \leq K d(x, y) \]

for all \( x, y \in X \) and for some \( K, 0 \leq K < 1 \), then \( T \) has a unique fixed point.

A mapping \( T \) satisfying (1.1.1) is known as a contraction mapping. It is important to note that any such contraction mapping of \( X \) into itself. However, a continuous mapping is not necessarily a contraction, thus Banach contraction principle has served as the main source containing the important idea of contraction. Fixed point theorems have extensive application in proving the existence and uniqueness theorems of differential equations, integral equations, partial differential equations, random differential and other related areas. It has a very fruitful application in eigen value problems as well as in boundary value problems. For these applications it is enough to refer Smart 4), Kelmogorov and Fomin 5), Szabednely 6), Swaminathan 7) and Czerwik Stenfan 8). Some authors such as Raktoch 9), Boyd and Wong 10) Browder 11) have attempted to

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4) Smart, R.D. \[ (1) \]
5) Kelmogorov, A.N. and Fomin, S.J. \[ (2) \]
6) Szubednely, V. \[ (1) \]
7) Swaminathan, S. \[ (1) \]
8) Czerwik Stenfan \[ (1) \]
9) Raktoch, E. \[ (1) \]
generalize Banach contraction principle by replacing the lipschitz constant \( k \) by some real valued function whose values are less than one.

In 1968 Kannan\(^{12}\) investigated mainly under what condition two mappings, each mapping on a complete metric space \( X \) into itself have a unique fixed point:

\[
(1.1.2) \quad d(Tx,Ty) \leq \alpha \left[ d(x,Tx) + d(y, Ty) \right]
\]
for all \( x, y \in X \) and \( 0 \leq \alpha < \frac{1}{2} \).

Reich\(^{13}\) unified the result of Banach contraction principle and the theorem of Kannan by considering a self mapping of a complete metric space \( X \) satisfying:

\[
(1.1.3) \quad d(Tx,Ty) \leq \alpha d(x,Tx) + \beta d(y, Ty) + \gamma d(x,y)
\]
where \( \alpha, \beta, \gamma \) are non-negative with \( 0 \leq \alpha + \beta + \gamma < 1 \) and \( x, y \in X \), then it has a unique fixed point. Raktoch\(^{14}\) replaced the constant \( \alpha \) with any \( \alpha (x,y) \) the constant depending on \( x, y \) with this view Reich\(^{15}\) has proved fixed point theorem with conditions:

\[\text{References}\]

12) Kannan, R.
13) Reich, S.
14) Raktoch, E.
15) Reich, S.

(1)
(1.1.4) \[ d(Tx, Ty) \leq \alpha (d(x, y)) d(x, Tx) + \beta (d(x, y)) d(y, Ty) \]

\[ + \gamma (d(x, y)) d(x, y) \]

For further work on the same lines we refer Wong \(^{16}\), Byzant \(^{17}\), Fukushima \(^{18}\), Chatterjea \(^{19}\) and others. In 1969, Meir and Keeler \(^{20}\) first defined a mapping known as a weakly uniformly strict contraction which generalizes the result of Boyd and Wong \(^{21}\). Recently Maiti and Pal \(^{22}\) obtained a generalization of Meir and Keeler \(^{23}\) and Boyd and Wong \(^{24}\). Further Maiti, Pal and Achari \(^{25}\) obtained more general results along these lines.

In the year 1973, Hardy and Rogers \(^{26}\), further unified and studied the mapping \(T:X \rightarrow X\) of Reich and Chatterjea (above mentioned) on a complete metric space \(X\) satisfying:

\[ \sum = \text{constant} \]

16) Wong, C.S.  
17) Byzant, V.W.  
18) Fukushima, H.  
19) Chatterjea, S.K.  
20) Meir, A, and Keeler, E.  
22) Boyd, D.W. and Wong, J.S.W.  
22) Maiti, M., and Pal, T.K.  
23) Meir, A, and Keeler, E.  
24) Boyd, D.W. and Wong, J.S.W.  
26) Hardy, G, and Rogers, T.
(1.1.5) and \((T_x, T_y) \leq a_1 d(x, T_x) + a_2 d(y, T_y) + a_3 d(x, T_y)\)
\[
+ a_4 d(y, T_x) + a_5 d(x, y)
\]
for all \(x, y \in X\), with \(a_1 \geq 0, \ 1 = 1, 2, \ldots 5\) and
\[
a_1 + a_2 + a_3 + a_4 + a_5 < 1.
\]

Later on the above work was generalized by Wong\(^\text{20}\), under addition assumption \(a_1 = a_2\) and \(a_3 = a_4\) which includes the work of Kannan\(^\text{28}\), Rakočević\(^\text{29}\), Gupta and Srivastava\(^\text{30}\) and Banach\(^\text{31}\).

Hardy and Rogers’ work was further generalized in other direction by Ciric\(^\text{32}\), Sharma\(^\text{33}\), Iseki\(^\text{34}\), Singh and Meada\(^\text{35}\).

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27) Wong, C.S.

28) Kannan, K.

29) Rakočević, B.

30) Gupta, V.K. and Srivastava, P.

31) Banach, S.

32) Ciric, Lj.B.

33) Sharma, A.K.

34) Iseki, K.

35) Singh, S.P. and Meada, B.A.

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Hussain and Sehgal\textsuperscript{36}). There is a multitude of metrical fixed point theorem for mapping satisfying certain contraction type conditions. In all these results one considers sequences of iterates which due to the contraction condition becomes a Cauchy sequence and whose limit is a fixed point of the mappings.

For the first time 'Junge\textsuperscript{37}) took a continuous mapping in place of identity mapping in order to generalize the celebrated Banach contraction principle. He proved

\textbf{Theorem I.} A continuous self mapping $A$ of a complete metric space $(X,d)$ has a fixed point iff there exists $0 \leq \alpha < 1$ and a mapping $S : X \rightarrow X$ which commute with $A$ and satisfy:

\begin{align*}
(1.1.6) & \quad S(X) \subseteq A(X) \\
(1.1.7) & \quad d(Sx, Sy) \leq \alpha d(Ax, Ay) \quad \text{for all } x, y \in X.
\end{align*}

Indeed $S$ and $A$ have a unique fixed point. Recently Mukherjea\textsuperscript{38}) proved

\textbf{Theorem M.} Let $f, g$ be two self mapping on a compact metric space $(X,d)$ such that $fg = gf$, $g(x) \subseteq f(x)$, $f$ is continuous and $f(x) = f(y)$

\begin{align*}
36) & \quad \text{Hussain, S.A. and Sehgal, V.M.} \quad \text{(1.2)} \\
37) & \quad \text{Jungeck, G.} \quad \text{(1)} \\
38) & \quad \text{Mukherjea, R.N.} \quad \text{(1)}
\end{align*}
\[(1.1.8) \quad d(gx, gy) < a_1 d(gx, f(x)) + a_2 d(gy, fy) + a_3 d(gx, fy) + a_4 d(gy, f(x)) + a_5 d(fx, fy) \text{ where } a_1 \geq 0, \sum_{i=1}^{5} a_i = 1\]

then \( f \) and \( g \) have a unique fixed point, further Fisher\(^39\) also proved a theorem related to the above by using the inequality

\[(1.1.9) \quad d(gx, gy) < \max \{ d(fx, fy), d(fx, gx), d(fy, gy) , d(fx, gy), d(fy, gx) \}.\]

Using the technique of Shih, M. and Yeh\(^{40}\), Rao and Rao\(^{41}\), have also proved a result for compact Hausdorff space. We need some definitions:

**Definition 1.1.** Let \( A \) and \( B \) be two self mapping on \( X \) and \( \{x_n\} \) is a sequence in \( X \), then \( \{x_n\} \) is said to be asymptotically \( A \)-regular with respect to \( B \) if \( \lim_{n \to \infty} d(Bx_n, Ax_n) = 0 \) where \( B = \text{identity map} \), the above definition reduces to that of Engle\(^{42}\).

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\(^{39}\) Fisher, B.  
\(^{40}\) Shih, M. and Yeh, C.C.  
\(^{41}\) Rao, I.H.N. and Rao, K.P.R.  
\(^{42}\) Engle, H.M.
**Definition 1.2:** Let \( f \) and \( g \) be two self mapping on \( X \), then \( f, g \) is said to be a weakly commuting pair if

\[
d(fgx, gf(x)) \leq d(gx, f(x)) \quad \forall x \in X.
\]

Clearly, a commuting pair is weakly commuting but the converse is not necessarily true which is clear by the example.

Let \( X = [0,1] \) with the usual metric. Define \( f, g : X \rightarrow X \) by setting \( f(x) = \frac{x}{2} \), \( g(x) = \frac{x}{2+x} \), \( x \in X \). Then for all \( x \in X \), one gets

\[
d(fgx, gf(x)) = \left( \frac{x}{4+x} \right) - \left( \frac{x^2}{4+2x} \right) = \frac{x^2}{(4+2x)(4+x)}
\]

\[
\leq \frac{x^2}{(4+2x)} = \frac{x^2}{2} - \frac{x}{2+x} = d(fx, gx)
\]

so \( f \) and \( g \) commute weakly, but for any non zero \( x \in X \) we have

\[
gf(x) = \frac{x}{(4+x)} > \frac{x}{4+2x} = fg(x)
\]

where \( gf \neq fg \). Thus \( f \) and \( g \) are non-commutative. Recently Khan and Swaleh\(^{43}\) studied Hardy and Roger's condition (cited above) for three self mappings using weakly commuting pairs for unique common fixed points.

It should be remarked here that the constants \( a_i \) need not satisfy \( \sum_{i=1}^{n} a_i < 1 \) as in Hardy and Rogers theorem.

\[43\) Khan, M.S. and Swaleh (1)\]
This theorem gives immediately Mukherjee\textsuperscript{44}) theorem if $S = T$. Further Sastry, Naidu, Rao and Rao\textsuperscript{45}) proved an interesting theorem for three self mappings for arbitrary complete metric space, using (1.1.9) type of mapping. This theorem generalizes the result of Singh and Singh\textsuperscript{46}), Fisher\textsuperscript{47}), and Sharma and Yuel\textsuperscript{48}).

The well known result of Jungck has been generalises by Converse\textsuperscript{49}), Cheh-Chih Yeh\textsuperscript{50}), Fisher\textsuperscript{51}), Khan\textsuperscript{52}), Khan and Imdad\textsuperscript{53}), Park\textsuperscript{54}), Park and Rhoades\textsuperscript{55}), Singh\textsuperscript{56}) for complete metric space, Khan\textsuperscript{57}) in uniform space and Cheh-Chih Yeh\textsuperscript{58}) in L-spaces.

\begin{align*}
\text{44)} & \quad \text{Mukherjee, R.N.} \\
\text{45)} & \quad \text{Sastry, K.P., Naidu, S.V., Rao, I.H.N. and Rao, K.P.R.} \\
\text{46)} & \quad \text{Singh, S.L. and Singh, S.P.} \\
\text{47)} & \quad \text{Fisher, B.} \\
\text{48)} & \quad \text{Sharma, P.L. and Yuel, A.L.} \\
\text{49)} & \quad \text{Converse, V.} \\
\text{50)} & \quad \text{Cheh-Chih, Yeh} \\
\text{51)} & \quad \text{Fisher, B.} \\
\text{52)} & \quad \text{Khan, M.S.} \\
\text{53)} & \quad \text{Khan, M.S. and Imdad, M.} \\
\text{54)} & \quad \text{Park, S.} \\
\text{55)} & \quad \text{Park, S. and Rhoades, B.E.} \\
\text{56)} & \quad \text{Singh, S.L.} \\
\text{57)} & \quad \text{Khan, M.S.} \\
\text{58)} & \quad \text{Cheh-Chih, Yeh}
\end{align*}
1.2. FIXED POINT THEOREM ON COMPLETE QUASI GAUGE SPACE

Quasi-gauge space was first developed by Reilly in his thesis, According to him a quasi-gauge structure for topological spaces \((X, \mathcal{J})\) is a family of quasi-pseudo metrics on \(X\). Such that \(\mathcal{J}\) has a sub-base i.e. the family \(\{B(x, p, G) : x \in X, p \in P, G > 0\}\) where \(B(x, p, G)\) is the set \(\{y \in X | p(x, y) < G\}\). If the space has such structure, it is known as quasi-gauge space and is denoted by \((X, P)\).

We shall first define the quasi gauge structure in function space.

Let \(X\) be a nonempty set and \(Y^X\) is a quasi-gauge space. We consider a non-negative real number \(p\) on function space \(Y^X\) having pointwise topology. We define topology called quasi-pseudometric if \(p(f, g)(x) = \sup (f(x), g(x))\) satisfying \(p(f, g)(x) = 0\) for every \(f \in Y^X\) and for all

\[
p(f, g)(x) = p(f, h)(x) + p(h, g)(x)
\]

for all elements \(f, g, h \in Y^X\). Thus we define

**Definition 1.** A quasi-gauge structure for topological space \((Y^X, P)\) is a family \(P\) of quasi-pseudometric on \(Y^X\), such that \(P\) has a sub-base. If a topological space \((Y^X, P)\) has a quasi-gauge structure \(P\). It is called quasi-gauge function space \((Y^X, P)\).

59) Reilly, I.L.
If in addition \((Y^X, p)\) is metrizable, we take \(P\) to consist of \(d\) alone.

**DEFINITION 2:** The sequence \(\{f_n\} \subseteq Y^X\) is called left \(p\)-cauchy where for each \(p \in P\) and \(\varepsilon > 0\) there is an \(f \in Y^X\) and integer \(K\) such that \(p(f, f_m)(x) < \varepsilon\) for all \(m \geq K\) similarly right \(p\)-cauchy sequence can be defined.

**DEFINITION 3:** The sequence \(\{f_n\} \subseteq Y^X\) is called \(p\)-cauchy if for \(p \in P, \varepsilon > 0\) and there exists an integer \(K\) such that

\[ p(f_m, f_n)(x) < \varepsilon \quad \text{for all} \quad m, n \geq K. \]

Now we define contraction mapping:

**DEFINITION 4:** An operator \(T\) on a quasi-gauge function space \((Y^X, p)\) into itself is said to be weak contraction for each \(f \in Y^X\), \(p \in P\) if there exists \(0 \leq \alpha < 1\) such that

\[ p(Tf, Tg)(x) \leq \alpha \cdot p(f, g)(x). \]

Similarly right contraction can be defined. Recently in 1982 we\(^\text{60}\) proved the following theorem:

**THEOREM A:** Let \(T\) be a continuous mapping of a complete metric space \(X\) into itself satisfying

\[ \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (1) \]

\(\text{60)}\) Iseki, K., Rajput, S.S. and Sharma, P.L.
\[(1.2.1) \quad d(Tx, Ty) \leq \alpha \cdot d(y, Ty) \left[ 1 + d(x, Tx) \right] \]

\[\left[ 1 + d(x, y) \right] \]

\[+ \beta \cdot \left[ d(x, Tx) + d(y, Ty) \right] + \gamma \cdot \left[ d(x, Ty) + d(y, Tx) \right] + \delta \cdot d(x, y) \]

for all \( x, y \in X \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \)

then \( T \) has a unique fixed point in \( X \). The object of Chapter II is to extend theorem A to quasi-gauge function space. In fact we prove

**Theorem 1:** Let \( (Y^X, P) \) be a Hausdorff sequentially complete quasi-gauge function space generated by the family \( P \) of pseudo-metric and \( T \) is continuous mapping and

\[(1.2.2) \quad p(Tf, Tg)(x) \leq \alpha \cdot \left[ p(g, Tg)(x) \left[ 1 + p(Tf, Tg)(x) \right] \right] \]

\[\left[ 1 + p(f, g)(x) \right] \]

\[+ \beta \cdot \left[ p(f, Tf)(x) + p(g, Tg)(x) \right] \]

\[+ \gamma \cdot \left[ p(f, Tg)(x) + p(g, Tf)(x) \right] + \delta \cdot p(f, g)(x) \]

for all \( f, g \in Y^X \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \).

Then \( T \) has a unique fixed point in \( (Y^X, P) \).

Now we extend theorem 1 for two mapping and prove:

**Theorem 2:** Let \( T_1 \) and \( T_2 \) are two continuous mapping on a Hausdorff sequentially complete quasi-gauge function space generated by the family \( P \) of pseudo-metric and
(1.2.3) \[ p(T_1 f, T_2 g)(x) \leq \alpha p(g, T_2 g)(x) \left[ 1 + p(f, T_1 f)(x) \right] \]
\[ + \beta \left[ p(f, T_1 f)(x) + p(g, T_2 g)(x) \right] \]
\[ + \gamma \left[ p(f, T_2 g)(x) + p(g, T_1 f)(x) \right] + \delta p(f, g)(x) \]

for all \( f, g \in Y^K \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \).

Then \( T_1 \) and \( T_2 \) have a unique common fixed point in \( (Y^K, p) \).

Now we shall further prove:

**Theorem 3**: Let \( E, F \) and \( T \) are three continuous mappings onto a Hausdorff sequentially complete quasi-gauge function space generated by the family \( p \) of pseudometrics and

(1.2.4) \[ ET = TE, FT = TF \] and \( E(Y^K) \subset T(Y^K), F(Y^K) \subset T(Y^K) \)

(1.2.5) \[ p(Ef, Fg)(x) = \alpha p(Tg, Tg)(x) \left[ 1 + p(TF, EF)(x) \right] \]
\[ + \beta \left[ p(TF, EF)(x) + p(Tg, Tg)(x) \right] \]
\[ + \gamma \left[ p(TF, Fg)(x) + p(Tg, EF)(x) \right] + \delta p(TF, Tg)(x) \]

for all \( f, g \in Y^K \) where \( \alpha, \beta, \gamma, \delta \geq 0 \) and \( \alpha + 2\beta + 2\gamma + \delta < 1 \).

Then \( E, F \) and \( T \) have a unique fixed point in \( Y^K \).
1.3. There are many generalizations of the classical contraction mapping theorem of S. Banach.

Suppose $X$ denote a Banach Space with the norm $||.||$, and let $C$ be a closed subset of $X$. The transformation $T : C \rightarrow C$ is called contraction if there exists a constant $0 \leq \alpha < 1$ such that for arbitrary $x, y \in C$, the inequality

$$||Tx - Ty|| \leq \alpha \cdot ||x - y||,$$

is true. It is called nonexpansive if the same condition with $\alpha = 1$ holds. By Banach contraction principle each contraction of $C$ has exactly one fixed point. The same is true if we assume that only some powers of $T$ are contraction, but is not true for non-expansive mappings. However, Browder\(^{61}\) has proved that every non-expansive mapping of a closed bounded convex subset of a uniformly convex Banach space has at least one fixed point. Kirk\(^{62}\) proved similar theorem in the space with normal structure. Goebel\(^{63}\) has given a simple proof of the above result of Browder and Kirk. Recently Kannan\(^{64}\) has prove a theorem for the mapping which satisfy:

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61) Browder, F.E. 
62) Kirk, W.A. 
63) Goebel, K. 
64) Kannan, R.
\[(1.3.2) \quad \|Tx - Ty\| \leq \frac{1}{2} \left\{ \|x-Tx\| + \|y-Ty\| \right\} \quad x, y \in C \]

where C is closed bounded convex subset of reflexive Banach space. It is to be noted that the reflexivity of the space and the normed structure of C being consequence of uniform convexity. It is also to be remarked that the mapping (1.3.2) is neither weaker nor stronger than the non-expansive mapping (1.3.1), yet it appears that most of the fixed point theorem for non-expansive mapping also holds for mapping which are continuous and satisfy (1.3.2). In a paper Goebel, Kirk and Shimii\(^{65}\) proved similar theorem for the mapping:

\[(1.3.3) \quad \|Tx-Ty\| \leq a_1 \|x-y\| + a_2 \|x-Tx\| + a_3 \|y-Ty\| + a_4 \|x-Ty\| + a_5 \|y-Tx\| \]

where \( \frac{5}{i=1} a_i = 1 \)

Recently Diviccaro, Fisher and Sessa\(^{66}\) established

**Theorem**: Let T and I be two weakly commuting mapping of C into itself satisfying:

\[(1.3.4) \quad \|Tx-Ty\|^p \leq a \|x-y\|^p - (1-a) \max \{\|Tx-Ix\|^p, \|Tx-I\|^p \} \]

\(^{65}\) Goebel, K. Kirk, W.A. and Shimii, T.N. \hspace{1cm} (1)

\(^{66}\) Diviccaro, M.L., Fisher, B. and Sessa, S. \hspace{1cm} (1)
If \( p = 1 \), we obtain the result of Fisher and Sessa \(^{67}\). First the result of Gregus \(^{68}\) was generalized by Delbosco, Ferrero and Rossati \(^{69}\). The notion of asymptotic regularity of a mapping was first introduced by Browder and Petryshyn \(^{70}\) for Banach space \( X \). This notion is found to be useful in proving the existence of fixed point.

We need some definitions:

**Definition 1.3.1.1.** Let \( T \) and \( E \) be two self-mapping of Banach space \( X \) and \( \{x_n\} \) be a sequence in \( X \). Then \( \{x_n\} \) is said to be asymptotically \( E \)-regular with respect to \( T \) if

\[
(1.3.5) \quad ||T x_n - E x_n|| \to 0 \text{ as } n \to \infty.
\]

If \( T \) is the identity map of \( X \), we get the definition of Browder and Petryshyn referred above. Sessa \(^{71}\) generalizing a result of Das and Naik \(^{72}\) defined two mappings \( E \) and \( T \) of a metric space \((X, d)\) into itself to be weakly commuting if

\[
(1.3.6) \quad d(E T x, T E x) \leq d(T x, E x)
\]

for all \( x, y \in X \). Two commuting mapping is also weakly commuting but the converse is not true.

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67) Fisher, B. and Sessa, S.
68) Gregus, JX.
69) Delbosco, D., Ferrero, O. and Rossati, F.
70) Browder, F. E. and Petryshyn, W. V.
71) Sessa, S.
72) Das, K. M. and Naik, K. V.
Drawing inspiration from condition (1.3.3) and a well known result of Jungck\textsuperscript{73}, in Chapter III, first we present a theorem:

**Theorem 1.** Let $T, E, F$ be three selfmaps of closed convex subset $C$ of reflexive Banach space $X$ satisfying:

$$(1.3.7) \quad \|Ex-Fy\| \leq a_1 \|Ex-Tx\| + a_2 \|Fx-Tx\| + a_3 \|Ey-Ty\| + a_4 \|Fy-Ty\| + a_5 \|Ex-Ty\|$$

$$+ a_6 \|Fx-Ty\| + a_7 \|Ey-Tx\| + a_8 \|Fy-Tx\| + a_9 \|Tx-Ty\|$$

for all $x, y \in C$ where $a_1 = a_2 = a_3 = + \|x-y\|, i=1,2, \ldots, 9$ are nonnegative function such that

$$(1.3.8) \quad \max \left\{ \sup_{x, y \in C} \left( a_1 + a_2 + a_3 + a_6 \right), \sup_{x, y \in C} \left( a_3 + a_4 + a_7 + a_8 \right), \sup_{x, y \in C} \left( a_5 + a_6 + a_7 + a_8 + a_9 \right) \right\} < 1.$$ 

(1.3.9) If $T$ is continuous

(1.3.10) $T$ weakly commute with $E$ and $F$ and

(1.3.11) there exists a sequence which is asymptotically

\textsuperscript{73) Jungck, G.}
E-regular and F-regular w.r. to T, then T, E and F have unique common fixed point.

CONTRACTION TYPE OF MAP:ING ON 3-METRIC SPACE :

1.4. In a paper S. Gahler\textsuperscript{74}) investigated the notion of a 2-metric, a real valued function of a point triples on a set X, whose abstract properties were suggested by the area function for a triangle determined by a triple in Euclidean spaces, associated with a given two metric was a natural topology. It is proved that every metric space is a 2-metric space.

For the first time P.L. Sharma and others\textsuperscript{75}) found contraction type of condition to get fixed point in 2-metric space, Gahler\textsuperscript{76}) further introduced the notion of m-metric space. Our object in this Chapter is to introduce 3-metric space using the method of simplex from algebraic topology and fixed contraction type of condition to obtain fixed point theorem.

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74) Gahler, S. \hfill (1)
75) Sharma, P.L., Sharma, B.K. and Iseki, K. \hfill (1)
76) Gahler, S. \hfill (2)
We need some definitions:

**DEFINITION 1.** A 3-metric space is a space $X$ in which for each quadruple point $x, y, z, w$ there exists a real valued function $f(x, y, z, w)$ such that

1. for each non-degenerate 2-simplex $<x, y, z>$ in $X$, then is a point $w \in X$ satisfying, $f(x, y, z, w) \neq 0$

2. $f(x, y, z, w) = 0$ only when the 3-simplex $<x, y, z, w>$ is degenerate

3. $f(x, y, z, w) = f(x, w, z, y) = f(y, z, w, x) = f(z, w, x, y) = \ldots \ldots$

4. $f(x, y, z, w) \leq f(x, y, z, u) + f(x, y, u, w) + f(x, u, z, w) + f(u, y, z, w) + f(x, y, z, u) + f(x, y, u, z) \leq 0$

This can further be generalized by means of notion of simplex.

**DEFINITION 2.** A sequence $\{x_n\}$ is a 3-metric space $X$ is called a convergent sequence if there is an $x \in X$ such that

$$\lim f(x_n, x, a, b) = 0$$

for all $a, b \in X$.

Here $x$ is called the limit of $\{x_n\}$.

**DEFINITION 3.** A sequence $\{x_n\}$ in a 3-metric space $X$ is called a Cauchy sequence, if $\lim f(x_n, x_m, a, b) = 0$ for all $a, b \in X$. 


**DEFINITION 1:** A 3-metric space in which every Cauchy sequence converges is called a **complete 3-metric space**.

**DEFINITION 2:** A 3-metric space \( X \) is called bounded, if there exists a constant \( k \) such that \( \varphi(x, y, z, w) \leq k \), for all \( x, y, z, w \in X \).

The object of chapter IV is proved following theorem.

**THEOREM 1.** Let \( x \) be a complete bounded 3-metric space, and let \( f_n(x) \) (\( n=1, 2, \ldots, \)) be a family of mappings of \( X \) into itself. Suppose that there exists a sequence of non-negative integers \( \{m_n\} \), and non-negative numbers \( \alpha, \beta \) such that for all \( x, y, a, b \in X \), and every pair \( i, j \) with \( i \neq j \).

\[
\varphi(f_{1}^{m_{i}}(x), f_{1}^{m_{j}}(y), a, b) \\
\leq \alpha \left[ \varphi(x, f_{1}^{m_{i}}(x), a, b) + \varphi(y, f_{1}^{m_{j}}(y), a, b) \right] \\
+ \beta \varphi(x, y, z, b). 
\]

where \( 2 \alpha + \beta < 1 \). Then the sequence of mappings \( \{f_n\} \) has a unique common fixed point.

Now we shall further prove,
THEOREM 3: Let $X$ be a complete bounded $2$-metric space and let $f_n(x)$ ($n = 1, 2, \ldots$) be a family of mappings of $X$ into itself. Suppose that there are nonnegative numbers $\alpha, \beta, \gamma$ such that for all $x, y, z, b \in X$ and every pair $i, j$ with $i \neq j$.

\begin{align*}
(1.4.6) & \quad \psi \left( f_i^{m_i}(x), f_j^{m_j}(y), a, b \right) \\
& \leq \alpha \left[ \psi(x, f_i^{m_i}(x), a, b) + \psi(y, f_j^{m_j}(y), a, b) \right] \\
& + \beta \left[ \psi(x, f_j^{m_j}(y), a, b) + \psi(y, f_i^{m_i}(x), a, b) \right] \\
& + \gamma \psi(x, y, a, b)
\end{align*}

where $2\alpha + 2\beta + \gamma < 1$. Then the sequence of mappings $\{f_n\}$ has a unique common fixed point.

(1.4.7) In 1960 topological semifield was defined\textsuperscript{77} and was proved that any topological semifield contains a topological field isomorphic with the real line. The elements are denoted by Greek letter $\alpha$ and follows the rules operation on real numbers.

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\textsuperscript{77} Antonovskii, M. Ya.; Boitanskii, V. G. and Sarymsakov, T. A. (1), (2),
We require:

**Definition:** Let $E$ be a semifield and $K$ is the set of all its positive elements. The set $X$ is called a 3-metric space over the semifield $E$ if there exists a metric,

$$\mathcal{P}: X \times X \times X \times X \rightarrow K$$

for each quadruple point $x, y, z, w \in X$ such that

(1.4.8) To each nondegenerate 2-simplex $< x, y, z >$ in $X$ there is a point $w \in X$, satisfying $\mathcal{P}(x, y, z, w) \neq 0$.

(1.4.9) $\mathcal{P}(x, y, z, w) = 0$ only when the 3-simplex $< x, y, z, w >$ is degenerate.

(1.4.10) $\mathcal{P}(x, y, z, w) = \mathcal{P}(x, w, z, y) = \ldots$

(1.4.11) $\mathcal{P}(x, y, z, w) \ll \mathcal{P}(x, y, z, u) + \mathcal{P}(x, y, u, w) + \mathcal{P}(x, u, z, w) + \mathcal{P}(u, y, z, w)$

Now we shall prove theorem:

**Theorem 3:** Let $x$ be a complete 3-metric space over a topological semifield $E$, $f(x)$ be a mapping on $X$ and for all $x \in X$ such that $\mathcal{P}(f(x), f(y), a, b) \ll x \mathcal{P}(x, y, a, b)$
Where $a$ is a positive number less than 1, then there is a fixed element $x^1$ of the mapping $f$ such that

$$f(x^1) = x^1$$

**Fixed point theorems in 2-Banach spaces**

1.5 In a paper Gahler\textsuperscript{78}) defines a linear 2-normed space to be pair $(L, ||.||)$ where $L$ is a linear space and $||.||$ is a nonnegative real valued function defined on $L$ where $a$, $b$, $c \in L$ and

$$||a, b|| = ||b, a||$$

(1.5.1)

$$||a, \beta b|| = |\beta| \cdot ||a, b||,$$  \text{where } \beta \text{ is real,} \tag{1.5.2}$$

(1.5.3)

$$||a, b + c|| \leq ||a, b|| + ||a, c||.$$  

Here $||.||$ is called a 2-norm.

Suppose $B$ denotes Banach space with norm $||.||$ and let $C$ be a closed sub set of $B$. Generalizing the result of Browder\textsuperscript{79}), Goebel, Kirk and Shimmi\textsuperscript{80}) proved an

\textbf{References:}

78) Gahler, S  \hspace{2cm} (1)

79) Browder, E  \hspace{2cm} (2)

80) Goebel, K; Kirk, W.A. and Shimmi, T.N.  \hspace{2cm} (1)
interesting result. Further Goebel and Zlotkiewicz\textsuperscript{81)} have proved a theorem for transformation with non expansive iteration. In fact they proved:

**Theorem GZ.** If $C$ is a closed and convex sub set of Banach space and if $F: C \rightarrow C$ satisfies

1) $F^2 = I$ (I is the identify mapping)

2) $||F(x) - F(y)|| \leq k ||x - y||$, where $0 \leq k < 1$.

Then $F$ has at least one fixed point.

Further Khan\textsuperscript{82)} generalised the above result for two mappings and proved:

**Theorem K.** Let $K$ be a closed and convex subset of a Banach space $X$, let $F: K \rightarrow K$, $G: K \rightarrow K$ satisfying

1) $F$ and $G$ commute

2) $F^2 = I$, $G^2 = I$, $I$ is the identify mapping

3) $||F(x) - F(y)|| \leq \alpha ||G(x) - G(y)||$

\textsuperscript{81)} Goebel, K and Zlotkiewicz, E \hfill (1)

\textsuperscript{82)} Khan, M. S. \hfill (3)
for every $x, y \in K$ and $0 \leq \alpha < 2$, then there exists at least one fixed point $x_0 \in K$, s.t.

$$F(x_0) = G(x_0).$$

Further if $0 \leq \alpha < 1$, then $x_0$ is the unique and $x_0 = F(x_0) = G(x_0)$.

We proved the following theorem

**Theorem 1.** Let $F$ be a mapping of a Banach space $X$ into itself and $F$ satisfies

1. $F^2 = I$ where $I$ is the identity mapping.

2. $$\| F(x) - F(y) \| \leq a \frac{\| x - F(x) \| \| y - F(y) \| \| x - y \|}{\| x - y \|} + b \left( \| x - F(x) \| + \| y - F(y) \| \right) + c \| x - y \|$$

for every $x, y \in X$ and $x \neq y$ where $4a + 4b + c < 2$. Then $F$ has a unique fixed point.

The object of Chapter V is to generalise our result and extend it to 2-Banach space.

In fact we prove:

**Theorem 2.** Let $F$ be a mapping of a 2-Banach space $X$ into itself. If $F$ satisfies the conditions

$$83)\quad \text{Rajput, S.S. and Sharma, P.L.} \quad (1)$$
(1.5.4) \[ F^2 = I, \] where \( I \) is the identity mapping.

(1.5.5) \[ ||F(x) - F(y)\| a \| \leq \alpha \frac{||x - F(x)\| a || \cdot ||y - F(y)\| a ||}{||x - y\| a ||} + \beta \{ ||x - F(x)\| a || + ||y - F(y)\| a || \} \]

\[ + \gamma \{ ||x - F(x)\| a || + ||y - F(x)\| a || \} + \delta ||x - y\| a || \]

for every \( x, y \in X \), \( a \in X \) where \( \alpha \geq \alpha \), \( \beta \geq \delta \), \( 2 + \delta < 1 \)

and \( 4\alpha + 4\beta + 4\gamma + \delta < 2 \), then \( F \) has at least one fixed point.

**Theorem 3.** Let \( F, G, H \) are three mappings of a 2-Banach space \( X \) into itself such that

(1.5.6) \[ FG = GF, \quad GH = HG \quad \text{and} \quad FH = HF \]

(1.5.7) \[ F^2 = I, \quad G^2 = I \quad \text{and} \quad H^2 = I, \] where \( I \) is the identity mapping.

(1.5.8) \[ \| F(x) - F(y), a \| \leq \alpha \| GH(x) - GH(y), a \| \]

\[ + \beta \left( \| GH(x) - F(x), a \| + \| GH(y) - F(y), a \| \right) \]

\[ + \gamma \| GH(x) - GH(y), a \| \]
for every \( x, y \in X \) and \( 0 \leq \alpha, \beta \) such that \( \alpha + 4\beta < 2 \).

Then there exists at least one fixed point \( x_0 \in X \) such that \( F(x_0) = GH(x_0) \) and \( FG(x_0) = H(x_0) \). Further if \( 0 \leq \alpha < 1 \), then \( x_0 \) is the unique common fixed point of \( F, G \) and \( H \).

**SOME MORE PROBLEMS IN BANACH SPACE**

1.6 In this chapter VI deals with some new type of mapping in Banach space Browder\(^{84}\), Kirk\(^{85}\), Goebel, K\(^{86}\), Goebel, Kirk and Shimmi\(^{87}\), Browder and Petryshyn\(^{88}\), Belluce and Kirk etc\(^{89}\), Diaz and Metcalf\(^{90}\), Kirk\(^{91}\) have proved many interesting result in Banach space.

We shall prove ;

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84) Browder, E
85) Kirk, W.A.
86) Goebel, K
87) Goebel, K; Kirk, W.A. and Shimmi, T.N.
88) Browder and Petryshyn W.V
89) Belluce, L.P and Kirk, W.A. and Steinier, F.E.
90) Diaz, J.B and Metcalf, F.T.
91) Kirk, W.A.
Theorem 1. Let \( K \) be non-empty, bounded, closed convex sub set of a reflexive Banach space \( X \) and let \( K \) have normal structure. If \( T : K \rightarrow k \) is continuous and satisfies (6.2.1), then \( T \) has a unique fixed point in \( K \).

Theorem 2. Let \( K \) be nonempty, bounded, closed, convex sub set of a reflexive Banach space \( X \), and \( K \) have normal structure. \( E, F \) and \( T \) be three continuous self mappings of \( K \), satisfying (6.3.1), (6.3.2). Then \( E, F \) and \( T \) have a unique common fixed point.

Theorem 3. Let \( K \) be a nonempty, bounded, closed convex subset of a reflexive Banach space \( X \) and let \( K \) have normal structure. \( E, F \) and \( T \) be three continuous self mappings of \( K \), satisfying (6.4.1), (6.4.2). Then \( E, F \) and \( T \) have a unique common fixed point.
SOME FIXED POINT THEOREM FOR EXPANSION MAPPINGS

1.7.1 B.E. Rhoades\textsuperscript{92}) Summarised contractive mappings of some types and discussed on the fixed point. He considered 250 types of contractive mappings and analyzed the relationships among them, and obtained some general theorems on fixed point. These 250 types are based on 25 types that $d(f(x), f(y))$ is governed by $d(x, f(x))$, $d(y, f(y))$, $d(x, f(x))$, $d(y, f(x))$ and $d(x, y)$. In this chapter VII, we shall define expansive mappings which correspond to some contractive mapping and even some more which are not given in Rhoades work and consider the existence of fixed point.

Let $X$ denote the complete metric space with a metric $d$ and $f$ is a mapping of $X$ into itself.

We shall prove:

**THEOREM 1.** If there exists non-negative reals $a_1$, $a_2$, $a_3$, $a_4$, $a_5$, $a_6$, $a_7$, $a_8$, $a_9$ with $a_1 + a_2 + a_4 + a_6 + a_9 > 1$ and $a_3 + a_7 = \frac{1}{2}$,

such that

$$d(f(x), f(y)) \leq a_1 d(x, f(x)) + a_2 d(y, f(y)) + a_3 d(x, f(y))$$

$$+ a_4 d(y, f(x)) + a_5 d(y, f^2(x)) + a_6 d(x, f^2(x))$$

$$+ a_7 d(f(x), f^2(x)) + a_8 d(f(y), f^2(x)) + a_9 d(x, y)$$

\textsuperscript{92}) Rhoades, B.E. (1)
for each \( x, y \in X \) with \( x \neq y \) and \( f \) is onto, then \( f \) has a fixed point.

1.7.2 Now we shall prove the following theorem

\textbf{Theorem 2.} If

\[
d(f(x), f(y)) \geq \frac{\alpha \, d(x, f(x)), d(y, f(y))}{d(x, y)} + \beta \, d(x, y)
\]

where \( \alpha + \beta > 1 \), then for each \( x, y \in X \) with \( x \neq y \) and \( f \) is onto then \( f \) has a fixed point.

1.7.3 Now we shall prove

\textbf{Theorem 3.} If there exists non-negative real \( \alpha > 1 \) such that

\[
(1.7.3.1) \quad d(f(x), f(y)) \geq \alpha \left( d(x, f(x)), d(y, f(y)) \right)^{\frac{1}{2}}
\]

for each \( x, y \in X \) with \( x \neq y \) and \( f \) is onto then \( f \) has a fixed point.

1.7.4 Now we shall prove a different type of result.

\textbf{Theorem 4.} If there exists \( \alpha > 1 \) such that
\[ (1.7.4.1) \quad d(f(x), f(y)) \geq \frac{2a \cdot d(x f(x)) \cdot d(y f(y))}{d(x, f(x)) + d(y, f(y))} \]

then \( f \) has a fixed point.

1.7.6 Now we shall prove:

**Theorem 5.** If there exists \( a > 1 \) such that

\[ (1.7.5.1) \quad d(f(x), f(y)) \geq \frac{3a \cdot d(x, f(x)) \cdot d(y, f(y)) \cdot d(x, y)}{d(x, f(x)) \cdot d(y, f(y)) + d(x f(x)) \cdot d(y f(y)) \cdot d(x, y)} \]

then \( f \) has a fixed point.

Our theorems here extend and generalize the result of Williams and others\(^{93}\), Wang and others\(^{94}\) and Rhoades\(^{95}\).

**Fixed points of not necessarily continuous mappings**

1.8 Banach\(^{96}\) has proved that a self contraction mapping of complete metric space is necessarily continuous and it has a unique fixed point. However, certain mappings

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93) Williams, B.B. and Gillespie, A.A.
94) Wong, S.-K., Li By, Gao 2. M and 9 Seki, K
95) Rhoades, B.E.
96) Banach, S
which are not necessarily continuous yet the fixed or common fixed points have been obtained for such type of mappings. For example of $T$ is a self-mapping of a complete metric space $(X, d)$ such that

\[(1.8.1) \quad d(Tx, Ty) \leq \alpha [d(x, Ty) + d(y Tx)]\]

for all $x, y \in X$ and $0 \leq \alpha \leq 1$, then $T$ has a unique fixed point. The mapping $T$ satisfying

\[(1.8.1) \quad \text{is not necessarily continuous. Ciric}^{97), \ 98), \ Pachpatte^{99), \ Kanann}^{100)} \text{ and others obtained fixed points for different type of mappings which are not necessarily continuous.} \]

\hspace{1cm} \text{Rhaodes}^{101) \text{ has given a comprehensive list of various mappings and most of these mappings are not necessarily continuous.} \]

\begin{align*}
97) \text{ Ciric, L. B.} & \quad (3) \\
98) \text{ Ciric, L. B.} & \quad (4) \\
99) \text{ Pachpatte, B.G.} & \quad (1) \\
100) \text{ Kannan, R} & \quad (2) \\
101) \text{Rhaades, B.E.} & \quad (1)
\end{align*}
Recently in 1975, Gupta and Ranganathan obtained fixed point theorem for a self-mapping \( T \) of a complete metric space \( X \), which is as follows.

**Theorem G.R.** Let \( T \) be a self-mapping of a complete metric space \( (X, d) \) such that

\[
(1.8.3) \quad d(T^{p+1}x, T^{p+2}y) \leq \alpha d(T^p x, T^{p+1} y)
\]

\[
+ \beta d(T^{p+1} y, T^{p+2} y) + \gamma d(T^p x, T^p y)
\]

for all \( x, y \) in \( X \), \( p \) is a non-negative integer and \( \alpha, \beta, \gamma \) are constants such that \( \alpha + \beta + \gamma < 1 \). Then, \( T \) has a unique fixed point. In (1.8.3), \( T \) is not necessarily continuous for \( p = 0 \). More recently in 1976 Achari has introduced and studied a new class of mapping \( T : X \rightarrow X \) which satisfies a condition of the type

\[
(1.8.4) d(Tx, Ty) \leq q \frac{\max \{d(x, Tx), d(x, Ty), d(y, Tx), d(y, Ty)\}}{\max \{d(x, Tx), d(y, Tx)\}}
\]

102) Gupta, V.K. and Ranganathan, S.
103) Achari, J.
104) Achari, J.
for all \(x, y \in X\), where \(0 \leq q < 1\).

Now we present a theorem in which we obtain a fixed point.

**Theorem 1.** Let \((X, d)\) be a complete metric space and let
\[ T: X \rightarrow X \]
satisfying:

\[
\max \left\{ d(T^n x, T^n y), d(T^n y, T^n x) \right\} < q^n \max \left\{ d(T^{n+1} x, T^{n+1} y), d(T^{n+1} y, T^{n+1} x) \right\}
\]

for all \(n \geq 0\) and for all \(x, y \in X\), where \(0 \leq q < 1\), \(p\) is a nonnegative integer. Then \(T\) has a unique fixed point in \(X\), if either \(p = 0\) or \(T\) is continuous.