CHAPTER VIII

FIXED POINTS OF NOT NECESSARILY CONTINUOUS MAPPINGS

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CHAPTER VII

FIXED POINTS OF NOT NECESSARILY CONTINUOUS MAPPINGS

8.1 Banach\(^1\) has proved that a self contraction mapping of a complete metric space is necessarily continuous and has a unique fixed point. However, certain mappings which are not necessarily continuous yet the fixed or common fixed points have been obtained for such type of mappings. For example if \(T\) is a self mapping of a complete metric space \((X, d)\) such that

\[d(Tx, Ty) \leq \alpha \max \{d(x, Ty) + d(y, Tx)\}\]

for all \(x, y \in X\) and \(0 \leq \alpha \leq \frac{1}{2}\), then \(T\) has a unique fixed point. The mapping \(T\) satisfying (8.1.1) is not necessarily continuous, Ciric\(^2\), \(^3\), Pachpatte\(^4\), Kannan\(^5\) and other obtained. Fixed points for different

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1) Banach, S (1)
2) Ciric, L.B. (3)
3) Ciric, L.B. (4)
4) Pachpatte, B.G. (1)
5) Kannan, R. (2)
type of mappings which are not necessarily continuous, Rhoades\(^6\) has given a list of various mappings and most of those mappings are not necessarily continuous.

6.2 Recently in 1975, Gupta and Ranganathan\(^7\) obtained fixed point theorem for a self mapping \(T\) of a complete metric space \(X\), which is as follows.

**THEOREM 6.2.** Let \(T\) be a self mapping of a complete metric space \((X, d)\) such that

\[
(6.2.1) \quad d(T^{p+1}x, T^{p+2}y) \leq \alpha d(T^p x, T^{p+1}x) + \beta d(T^{p+1}x, T^{p+2}y) + \gamma d(T^p x, T^{p+1}y)
\]

for all \(x, y \in X\), \(p\) is a non negative integer and \(\alpha, \beta, \gamma\) are constants such that

\[
\alpha, \beta, \gamma \geq 0 \text{ with } \alpha + \beta + \gamma < 1.
\]

Then, \(T\) has a unique fixed point.

In (6.2.1), \(T\) is not necessarily continuous for \(p = 0\).

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6) Rhoades, B.E. \((1)\)

7) Gupta, V.K. and Ranganathan, S

8) A
More recently in 1976 Achari\textsuperscript{8)} and \textsuperscript{9)} has introduced and studied a new class of mapping $T : X \rightarrow X$ which satisfies

$$\max \left\{ d(x, y) d(x, Ty), 
  d(x, y)d(y Tx), 
  d(y, Tx)d(x, Ty), 
  d(y, Ty) d(y, Tx) \right\}$$

(8.2.2) $d(Tx, Ty) \leq q \max \left\{ d(x, Ty), d(y, Tx) \right\}$

for all $x, y$ in $X$, where $0 \leq q < 1$.

Achari has further shown that $T$ has a unique fixed point in $X$, if $T$ is a self mapping of a complete metric space $(X, d)$ and satisfies the condition (8.2.2).

The object of Chapter VIII is to prove first the:

\textsuperscript{8}) Achari, J \hspace{1cm} (1)
\textsuperscript{9}) Achari, J \hspace{1cm} (2)
**Theorem 1.** Let \((x, d)\) be a complete metric space and let \(T : X \rightarrow X\) satisfying:

\[
\max \{ d(T^p x, T^{p+1} y), d(T^p x, T^{p+1} x), d(T^{p+1} y, T^{p+1} y), d(T^{p+1} y, T^{p+1} x), d(T^p x, T^p y), d(T^p x, T^{p+2} y) \}
\]

(6.3.1) \(d(T^{p+1} x, T^{p+2} y) \leq \alpha \frac{d(T^p x, T^{p+2} y) \max \{ d(T^p x, T^{p+2} y), d(T^{p+1} y, T^{p+1} x) \}}{\max \{ d(T^p x, T^p y), d(T^{p+1} y, T^{p+1} x) \}}\)

for all \(x, y\) in \(X\), where \(0 \leq \alpha < 1\), \(p\) is a non-negative integer. Then \(T\) has a unique fixed point in \(X\), if either \(p = 0\) or \(T\) is continuous.

**Proof:** We prove the theorem for \(p = 0\). The proof in the general case follows on similar lines.

Now, for \(p = 0\), the condition (6.3.1) reduces to
\[
\max \{ d(x, Ty) \leq d(x, T^2 y) \} \bigg) \\
\] \\
\[
d(x, Ty) \leq d(y, Tx) \\
d(Ty, Ty) \leq d(x, T^2 y) \\
d(Ty, T^2 y) \leq d(Ty, Tx) \\
d(x, y) \leq d(x, T^2 y) \\
\]

(8.3.2) \(d(Tx, T^2 y) \leq a \max \{ d(x, T^2 y), d(Ty, Tx) \} \)

We define a sequence \(\{x_n\} \) of elements of \(X\) as follows:

Let \(x_0 \in X\) and let \(x_n = Tx_{n-1} = T^n x_0\),

\(x_{n+1} = Tx_n = T^{n+1} x_0\), for \(n = 1, 2, 3, \ldots\)

then by (8.3.2), we have

\[
d(x_1, x_2) = d(Tx_0, T^2 x_0) \\
\] \\
\[
\max \{ d(x_0, Tx_0) \leq d(x_0, T^2 x_0), \\
d(x_0, Tx_0) \leq d(Tx_0, T^2 x_0) \} \\
\]
\[ d(Tx_0, Tx_0) \leq d(Tx_0, T^2x_0), \]
\[ d(Tx_0, T^2x_0) \leq d(Tx_0, Tx_0), \]
\[ d(x_0, x_0) \leq d(x_0, T^2x_0) \]

\[ \leq a \]

\[ \max \left[ d(x_0, T^2x_0), d(Tx_0, Tx_0) \right] \]
\[ \leq a \]

\[ \max \left[ d(x_0, Tx_0), d(x_0, T^2x_0) \right], o, o, o, o, o \]

\[ \leq a \]

\[ \max \left[ d(x_0, T^2x_0), o \right] \]

\[ \leq a \]
\[ d(x_0, Tx_0) \]
\[ \leq a \]
\[ d(x_0, x_1) \]

Similarly,
\[ d(x_2, x_3) = d(T^2x_0, T^3x_0) = d(Tx_1, T^2x_1) \]
\[ \max \left[ d(x_1, Tx_1), d(x_1, T^2x_1) \right] \]
\[ d(x_1, Tx_1) \leq d(Tx_1, Tx_1), \]
\[ d(x_1, T^2x_1) \leq d(Tx_1, T^2x_1), \]
\[ d(Tx_1, T^2x_1) \leq d(T^2x_1, T^2x_1), \]
\[ d(Tx_1, T^2x_1) \leq d(Tx_1, Tx_1). \]

\[ d(x_1, x_2) \leq (x_1, T^2x_1). \]

\[ \leq a \]

\[ \max \left\{ d(x_1, T^2x_2), d(Tx_1, Tx_2) \right\} \]

\[ \leq a \]

\[ \max \left\{ d(x_1, Tx_2) \leq d(x_1, T^2x_1), 0, 0, 0, 0 \right\} \]

\[ \leq a \]

\[ \max \left\{ d(x_1, T^2x_1), 0 \right\} \]

\[ \leq a \]

\[ d(x_1, Tx_1) \]

\[ \leq a \]

\[ d(x_1, x_2) \]

\[ \leq a^2 d(x_0, x_1) \]

Similarly, we may have

\[ d(x_3, x_4) \leq a d(x_2, x_3) \leq a^3 d(x_0, x_1) \]

and so on.

So, in general, we have

\[ d(x_n, x_{n+1}) \leq a^n d(x_0, x_1). \]
Since $\alpha < 1$, thus $\{x_n\}$ is a Cauchy sequence and from completeness of space, there exists a point $z \in X$, such that

$$\lim_{n \to \infty} x_n = z.$$  \hfill (8.3.3)

Now we shall show that $z$ is the fixed point of $T$.

Consider

$$d(Tz, x_n) = d(Tz, Tx_{n-1}) = d(Tz, T^2x_{n-2})$$

$$\max \left[ d(z, Tz) \cdot d(z, T^2x_{n-2}), \right.$$  

$$d(z, Tx_{n-2}) \cdot d(Tx_{n-2}, Tz),$$

$$d(Tx_{n-2}, Tz) \cdot d(z, T^2x_{n-2}),$$

$$d(Tx_{n-2}, T^2x_{n-2}), d(Tx_{n-2}, Tz)$$

$$\left. d(z, x_{n-2}) \cdot d(z, T^2x_{n-2}) \right]$$

$$\leq \alpha$$

$$\max \left[ d(z, T^2x_{n-2}), d(Tx_{n-2}, Tz) \right]$$
\[
\max \left[ d \left( z, x_n \right), d \left( z, x_{n-1} \right), d \left( x_{n-1}, Tz \right), d \left( x_{n-1}, x_n \right), d \left( x_{n-1}, x_{n-2} \right) \right] \\
\leq \alpha \left( \max \left[ d \left( z, x_n \right), d \left( x_{n-1}, Tz \right) \right] \right)
\]

On letting \( n \to \infty \), and applying (8.3.3), we have

\[
d \left( z, Tz \right) \leq 0
\]

Hence, it follows that \( z = Tz \), i.e. \( z \) is the fixed point of \( T \).

Now for the uniqueness of fixed point, let us suppose that \( w \) and \( z \) are fixed point of \( T \), then \( d \left( z, w \right) > 0 \).

\[
d \left( z, \bar{w} \right) = d \left( Tz, T^2 w \right)
\]
\[
\max \left[ d(z, Tz) \ d(z, T^2w) \right.,
\]
\[
d(z, Tw) \ d(z, Tw),
\]
\[
d(Tw, Tz) \ d(z, T^2w)
\]
\[
d(Tw, T^2w) \ d(Tw, Tz)
\]
\[
d(z, w) \ d(z, T^2w) \right]
\]
\[
\leq a \frac{1}{\max \left[ d(z, T^2w), d(Tw, Tz) \right]}
\]
\[
\leq a \ d(z, w)
\]

Since \( a < 1 \), it follows that \( z = w \).

This completes the proof of our theorem.