CHAPTER VI

SOME MORE PROBLEMS IN BANACH SPACE

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6.1 There are many generalizations of the classical contraction mapping theorem of S. Banach.

Suppose $X$ denote a Banach space with the norm $||.||$ and let $C$ we the closed sub set of $X$. The transformation $F: C \rightarrow C$ is called contraction if there exists a constant $0 < \alpha < 1$ such that for arbitrary $x, y \in C$, the inequality

$$(6.1.1) \quad ||F(x) - F(y)|| \leq \alpha \cdot ||x - y||,$$

is true. It is called non expansive. If the same condition with $\alpha = 1$ holds. By Banach contraction principle each contraction of $C$ has exactly one fixed point. The same is true, if we assume that only some power of $F$ are contraction, but it is not true for non expansive mappings. However, Browder$^1$ has proved that every non expansive mapping of a closed bounded convex sub set of a uniformly convex Banach space has at least one fixed point. Kirk$^2$ proved similar theorems in the space with normal structure. Goebel$^3$ has given a simple proof

1) Browder, F.E.
2) Kirk, W.A.
3) Goebel, K

(2)
(1)
of the above result of Browder and Kirk. Then Kannan\(^4\) has proved a theorem for the mapping which satisfy.

\[
(6.1.2) \quad ||T(x)-T(y)|| \leq \frac{1}{2} \left( ||x-T(x)|| + ||y-T(y)|| \right)
\]

\[x, y \in C;\]

where \(C\) is closed bounded convex sub set of reflexive Banach space. It is to be noted that the reflexivity of the space and the normal structure of \(C\) being consequences of uniform convexity of the space. It is also to be remarked that the mapping \((6.1.2)\) is neither weaker nor stranger than the non expansive mapping \((6.1.1)\), yet most of the theorem for non expansive mapping also holds, for mapping \((6.1.2)\). Further Goebel, Kirk and Shimmi\(^5\) proved similar theorem satisfying:

\[
(6.1.3) \quad ||T(x)-T(y)|| \leq a_1 ||x-y|| + a_2 ||x-T(x)|| + a_3 ||y-T(y)|| + a_4 ||x-T(y)|| + a_5 ||y-T(x)||
\]

where \(a_1 + a_2 + a_3 + a_4 + a_5 < 1\).

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4) Kannan, R

5) Goebel, K, Kirk, W.A. and Shimmi, T.N. (1)
It is known that (6.1.3) is much wider class of mapping. Many fixed point theorems for non expansive mappings have been devised in recent years for examples, Browder and Petryshyn\(^6\), Belluce and Kirk Setteiner\(^7\), Diaz and Metcalf\(^8\), Kirk\(^9\),

6.2. Our object of this chapter is to prove some fixed point theorems using a new symmetric rational fraction.

We shall be first concerned with the mapping, which satisfies the following contractive (new type) of conditions;

Let \( T \) be a mapping of \( X \) into it self, such that

\[
(6.2.1) \quad \|T(x)-T(y)\| \leq \frac{\|x-Tx\| + \|y-Ty\|}{\|x-Ty\| + \|y-Tx\| + \|x-Tx\| + \|y-Ty\| + \|x-Ty\| + \|y-Tx\|}
\]

for all \( x, y \in X, \quad x \neq y \) and

\[
\|x-Tx\| + \|y-Ty\| + \|x-Ty\| + \|y-Tx\| = 0.
\]

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6) Browder, F.E. and Petryshyn (1)

7) Belluce, L.P and Kirk, W.A. and Steiner, F.E. (1)

8) Diaz, J.B and Metcalf, F.T. (1)

9) Kirk, W.A. (2)
**Definition**: For a bounded set $K \subset X$, the diameter of $K$ denoted by $D(K)$ is defined as

$$D(K) = \text{Sup} \left\{ \| x - y \| : x, y \in K \right\}$$

A bounded convex subset $K$ of a Banach space is said to have a normal structure, if for each convex subset $H$ of $K$ with more than one point $x \in H$ such that

$$\text{Sup} \left\{ \| x - y \| : y \in H \right\} < D(H)$$

The convex null of $K$ is denoted by $C_0(K)$ and the closed convex null of $K$ is denoted by $\overline{C_0(K)}$.

We first prove the following lemma:

**Lemma**: Let $K$ be a subset of a Banach space $X$, and let $T$ be a mapping of $K$ into itself, such that for $x \in K$, $T$ satisfies (6.2.1). Then

$$\| T^n x - T x \| \leq \frac{n}{n+1} \| T^{n-1} x \| - \| T^n x \|$$

and for positive $m$ and $n$,

$$\| T^n x - T^m x \| \leq \| x - T x \| .$$

**Proof**:

$$\frac{\left\{ \| T^{n-1} x - T^n x \| + \| T^n x - T^{n+1} x \| \right\} \cdot \left\{ \| T^{n-1} x - T^{n+1} x \| \right\}}{\| T^n - T x \|}$$

$$\leq \frac{\left\{ \| T^{n-1} x - T^n x \| + \| T^n x - T^{n+1} x \| + \| T^{n-1} x - T^{n+1} x \| \right\}}{\| T^n - T x \|}$$
\[ \leq \frac{1}{2} \left\{ \left\| \mathbf{T}^{-1} \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| + \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^{n+1} \mathbf{X} \right\| \right\} \cdot \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^{n+1} \mathbf{X} \right\| \]

Thus
\[ \frac{1}{2} \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^{n+1} \mathbf{X} \right\| \leq \frac{1}{2} \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| \]

:\[ \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^{n+1} \mathbf{X} \right\| \leq \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| \]

\[ \frac{1}{2} \leq \left\| \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| \]

Now
\[ \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| \leq \frac{1}{2} \left\{ \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| + \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| \right\} \]

\[ \left\{ \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| + \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| + \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| \right\} \]

\[ + \left\{ \left\| \mathbf{T}^{n+1} \mathbf{X} - \mathbf{T}^n \mathbf{X} \right\| + \left\| \mathbf{T}^{n-1} \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| + \left\| \mathbf{T}^{n+1} \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| \right\} \]

\[ \leq \frac{1}{2} \left\{ \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| + \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| \right\} \]

\[ \leq \frac{1}{2} \left\{ \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| + \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| \right\} \]

\[ \leq \frac{1}{2} \left\{ \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| + \left\| \mathbf{T}^n \mathbf{X} - \mathbf{T}^m \mathbf{X} \right\| \right\} \]
Thus

\[
\left\| T^n x - T^m x \right\| \leq \frac{1}{2} \left( \left\| T^n x - T^m x \right\| + \left\| T^{n-1} x - T^m x \right\| \right)
\]

\[
+ \left. \left\| T^{n-1} x - T^m x \right\| \right\| T^n x - T^m x \right\| + \left\| T^{n-1} x - T^m x \right\| \right\| T^n x - T^m x \right\| \right)
\]

\[
\leq \frac{1}{2} \left( \left\| T^n x - T^m x \right\| \right)^2 + \left\| T^{n-1} x - T^m x \right\|^2 \right)
\]

\[
\iff \left\| T^n x - T^m x \right\| \leq \frac{1}{2} \left( \left\| T^n x - T^m x \right\| \right)^2 + \left\| T^{n-1} x - T^m x \right\|^2 \right)
\]

\[
\leq \left\| T^n x - T^m x \right\| \leq \left\| x - T x \right\| \right)
\]

Now we state our theorem

**Theorem 1.** Let \( K \) be non-empty, bounded, closed, convex sub-set of a reflexive Banach space \( X \) and let \( K \) have normal structure. If \( T: K \to K \) is continuous and satisfies (6.2.1) then \( T \) has a unique fixed point in \( K \).

**Proof.** Since \( K \) is reflexive Banach space, every descending chain of non-empty closed convex sub-sets of \( X \) has non-empty intersection [Kirk\(^{10}\)].

Hence we may use Zorn’s lemma to obtain a sub-set \( K_1 \) of \( K \) minimal with respect to being closed, convex and invariant under \( T \).

10) Kirk, W.A.
If $D(k_1) = 0$, there is nothing to prove.

Suppose $D(k_1) > 0$.

Since $K$ has a normal structure, there is any point $y \in K$ s. t.

$$\sup \{ \| x-y \|, x \in K \} \leq \gamma < D(k_1)$$

Thus $\| y - Ty \| \leq \gamma$ and by Lemma $D(o(y)) \leq \gamma$.

Let $H = \{ x \in K \mid D(o(x)) \leq \gamma \}$ and let

$G = \overline{CO}(T(H))$. Then $G$ is closed, convex and non empty.

Let $g \in G$.

Then

Case I. $g = Th$ for some $h \in H$

Then $\| g - Tg \| = \| Th - Tg \| \leq \| h - Th \| \leq \gamma$

Hence $g \in H$ and $Tg \in G$.

Case II. Let $g = \sum_{i=1}^{n} \lambda_i Th_i, h_i \in H_i$

$\lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1$

Then

$$\| Tg - g \| = \| Tg - \sum_{i=1}^{n} \lambda_i Th_i \|$$

$$\leq \sum_{i=1}^{n} \lambda_i \| T^2 h_i - Th_i \| \leq \gamma.$$  

So $\| g - Tg \| \leq \gamma$, $g \in H$ and $Tg \in G$. 


Case 3. \( g \) is the limit of terms of the form

\[
\sum_{i=1}^{n} \lambda_i \text{Th}_i \quad \text{where } \lambda_i \geq 0, \ h_i \in H, \ \sum_{i=1}^{n} \lambda_i = 1
\]

Then for any such term we have

\[
||Tg-g|| \leq \left( \sum_{i=1}^{n} \lambda_i h_i \right) ||Tg-g|| + \left( \sum_{i=1}^{n} \lambda_i \text{Th}_i \right) ||Tg-g||
\]

\[
\leq y
\]

Thus \( g \in H \), and \( Tg \in G \).

Since \( K_1 \) is minimal, \( K_1 = G \).

But \( D(G) = D(\overline{C_0(T(H))}) = D(T(H)) \)

\[
= \text{Sup} \left\{ \left( ||Tx-Ty|| : x, y \in H \right) \right\}
\]

\[
\leq y < D(K_1)
\]

Thus \( D(K_1) > 0 \) leads to the contradiction.

Hence \( D(K_1) = 0 \) and \( I \) has a fixed point.

Say \( p \) in \( K_1 \).

Thus \( p \) is unique by hypothesis.
6.3 Now we shall generalise our theorem 1 for three mapping and prove:

**Theorem 2.** Let $K$ be non-empty, bounded, closed, convex subset of a reflexive Banach space $X$, and $K$ have normal structure. $E$, $F$ and $T$ be three continuous self mappings of $K$.

Satisfying

\[(6.3.1) \quad ET = TE, \quad FT = TF, \quad E(X) \subseteq T(X) \quad \text{and} \quad F(X) \subseteq T(X),\]

\[(6.3.2) \quad \|Ex_{Fy}\| \leq \left\{ \frac{\|Tx-Ex\| + \|Ty-Fy\|}{\|Tx-Ex\| + \|Ty-Fy\| + \|Tx-Fy\| + \|Ty-Ex\|} \right\} \]

for all $x, y$ in $X$ and $\|Tx-Ex\| + \|Ty-Fy\| + \|Tx-Fy\| + \|Ty-Ex\| \neq 0$.

Then $E$, $F$ and $T$ have a unique common fixed point.

**Proof.** Let $x_0$ be a point in $X$. Let $x_1 \in X$ be such that

$Tx_1 = Ex_0$ and let $x_2 \in X$ be such that $Tx_2 = Fx_1$.

In general we can choose $x_{2n+1}$ and $x_{2n+2}$ so that

\[(6.3.3) \quad Tx_{2n+1} = Ex_{2n}, \quad Tx_{2n+2} = Fx_{2n+1}\]

for $n = 0, 1, 2, \ldots \ldots$. We can do this, since (6.3.1) holds. It follows from (6.3.2) that

\[
\|Tx_{2n+1} - Tx_{2n+2}\| = \|Ex_{2n} - Fx_{2n+1}\| \leq \frac{\|Tx_{2n} - Ex_{2n}\| + \|Tx_{2n+1} - Fx_{2n+1}\| + \|Tx_{2n+2} - Ex_{2n}\| + \|Tx_{2n+1} - Fx_{2n+1}\|}{\|Tx_{2n} - Ex_{2n}\| + \|Tx_{2n+1} - Fx_{2n+1}\| + \|Tx_{2n+2} - Ex_{2n}\| + \|Tx_{2n+1} - Fx_{2n+1}\|}
\]
\[
\begin{align*}
\left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n+1}} - T_{x_{2n+2}} \right| \right\} \left\{ \left| T_{x_{2n}} - T_{x_{2n+2}} \right| + \left| T_{x_{2n+1}} - T_{x_{2n+2}} \right| \right\} \\
\leq \left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n+1}} - T_{x_{2n+2}} \right| \right\} \\
\leq \frac{1}{2} \left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n+1}} - T_{x_{2n+2}} \right| \right\}
\end{align*}
\]

Thus

\[
\frac{1}{2} \left| T_{x_{2n+1}} - T_{x_{2n+2}} \right| \leq \frac{1}{2} \left| T_{x_{2n}} - T_{x_{2n+1}} \right|
\]

\[
\therefore \left| T_{x_{2n+1}} - T_{x_{2n+2}} \right| \leq \left| T_{x_{2n}} - T_{x_{2n+1}} \right|
\]

\[
\vdots \leq \left| T_{x_{2n-1}} - T_{x_{2n}} \right|
\]

\[
\vdots \leq \left| x_0 - T_{x_0} \right|
\]

Further

\[
\left| T_{x_{2n+1}} - T_{x_{2m}} \right| = \left| E_{x_{2n}} - F_{x_{2m-1}} \right|
\]

\[
\left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n-1}} - T_{2m} \right| \right\} \left\{ \left| T_{x_{2n}} - T_{x_{2m}} \right| + \left| T_{x_{2n-1}} - T_{x_{2m+1}} \right| \right\} \\
\leq \left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2m-1}} - T_{x_{2m}} \right| + \left| T_{x_{2n}} - T_{x_{2m}} \right| + \left| T_{x_{2n-1}} - T_{x_{2m+1}} \right| \right\}
\]
\[
\left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n}} - T_{x_{2m}} \right| + \left| T_{x_{2m}} - T_{x_{2m+1}} \right| + \cdots + \left| T_{x_{2m-1}} - T_{x_{2m}} \right| \right\} \\
\left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n}} - T_{x_{2m}} \right| + \left| T_{x_{2m}} - T_{x_{2m+1}} \right| + \cdots + \left| T_{x_{2m-1}} - T_{x_{2m}} \right| \right\} \\
\leq 2 \left| T_{x_{2n+1}} - T_{x_{2m}} \right|
\]

Thus

\[
\left\{ \left| T_{x_{2n+1}} - T_{x_{2m}} \right| \right\}^2 \leq \frac{1}{2} \left\{ \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n}} - T_{x_{2m}} \right| + \left| T_{x_{2n}} - T_{x_{2n+1}} \right| + \left| T_{x_{2n+1}} - T_{x_{2m}} \right| + \left| T_{x_{2m-1}} - T_{x_{2m}} \right| \right\}
\]
\[
\leq \frac{1}{2} \left[ \left\| \| \mathbf{t}_{2n-1} - \mathbf{t}_{2n+1} \| \right\|^2 + \left\| \| \mathbf{t}_{2m-1} - \mathbf{t}_{2m} \| \right\|^2 \right] \\
\leq \| \mathbf{x}_0 - \mathbf{t}_0 \| \\
\cdot \cdot \cdot \quad \| \mathbf{t}_{2n+1} - \mathbf{t}_{2m} \| \leq \| \mathbf{x}_0 - \mathbf{t}_0 \|
\]

The rest of the theorem follows from theorem 1.

Thus \( \{ \mathbf{t}_n \} \) converges to \( \mathbf{x} \in \mathbf{X} \). It follows from the hypothesis that \( \{ \mathbf{e}_{2n} \} \) and \( \{ \mathbf{e}_{2n+1} \} \) also converges to \( \mathbf{x} \). Since \( E, F \) and \( T \) are continuous we have,

\[
E(\mathbf{t}_{2n}) \rightarrow E \mathbf{x}, \quad F(\mathbf{t}_{2n+1}) \rightarrow F \mathbf{x}
\]

From (6.3.1) we have

(6.3.4) \quad E(\mathbf{t}_{2n}) = T(\mathbf{e}_{2n}), \quad F(\mathbf{t}_{2n+1}) = T(\mathbf{e}_{2n+1})

for all \( n = 0, 1, 2, \ldots \). Taking \( n \rightarrow \infty \), we have

(6.3.5) \quad \mathbf{e}_x = \mathbf{t} = F \mathbf{x}.

(6.3.6) \quad T(\mathbf{t}) = T(\mathbf{e}) = E(\mathbf{t}) = E(\mathbf{e}) = T(\mathbf{e}) = F(\mathbf{t}) = E(F \mathbf{e})

also

\[
\| \mathbf{e}_x - F(\mathbf{e}) \| \leq \frac{\| \| \mathbf{t} - \mathbf{e} \| + \| \mathbf{T}(\mathbf{e}) - F(\mathbf{e}) \| + \| \mathbf{T}(\mathbf{e}) - \mathbf{e} \|}{\| \| \mathbf{t} - \mathbf{e} \| + \| \mathbf{T}(\mathbf{e}) - F(\mathbf{e}) \| + \| \mathbf{T}(\mathbf{e}) - \mathbf{e} \|}
\]

\[
\cdot \cdot \cdot \quad \| \mathbf{e}_x - F(\mathbf{e}) \| \leq 0.
\]
Thus

\[(6.3.7) \quad E x = F (E x) = F (E x)\]

Thus \(E x\) is the common fixed point of \(E, F\) and \(T\)

Let \(u, v\) be two point of \(X\) such that

\[E u = F u = T u = u\]

and \[E v = F v = T v = v\]

\[||u - v|| = ||E u - F v||\]

\[\left\{ \frac{||T u - E u|| + ||T v - E v||}{||T u - F u|| + ||T v - E u||} \right\}\]

Hence \(u = v\). Therefore the proof of the theorem is complete.

6.4 We shall also prove the following theorem.

**Theorem 3.** Let \(K\) be a nonempty, bounded, closed convex subset of a reflexive Banach space \(X\) and let \(K\) have
normal structure, $E, F$ and $T$ be three continuous self mappings satisfying.

\[(6.4.1) \quad ET = TE, \ FT = TF, \ E(K)CT(K), \ F(K)CT(K)\]

\[(6.4.2) \quad ||Ex - Fy|| \leq \frac{||Tx - Ex|| + ||Ty - Fy|| + ||Ty - Ex||}{||Tx - Ty||} \]

for all $x, y \in K$ and $||Tx - Ty|| + ||Ty - Ex|| \neq 0$.

Then

$E, F, T$ have a unique common fixed point.

**Proof**  Let $x_0$ be any point in $K$, let $x_1 \in K$ be such that $Tx_1 = Ex_0$ and let $x_2 \in K$ be such that $Tx_2 = Fx_1$. In general, we can choose $x_{2n+1}$ and $x_{2n+2}$ such that

\[(6.4.3) \quad Tx_{2n+1} = Ex_{2n}, \ Tx_{2n+2} = Fx_{2n+1}\]
for $n=0, 1, \ldots$ we can do this since (6.4.1) holds.

We have

\[
||Tx_{2n+1} - Tx_{2n+2}|| = ||Ex_{2n} - Fx_{2n+1}||
\]

\[
\leq \left( ||Tx_{2n} - Fx_{2n+1}|| + ||Tx_{2n+1} - Ex_{2n}|| \right)
\]

\[
\leq ||Tx_{2n} - Fx_{2n+1}|| + ||Tx_{2n+1} - Ex_{2n}||
\]

\[
||Tx_{2n} - Tx_{2n+1}|| + ||Tx_{2n} - Fx_{2n+2}|| + ||Tx_{2n+1} - Tx_{2n+2}||
\]

\[
\leq ||Tx_{2n} - Tx_{2n+2}|| + ||Tx_{2n+1} - Tx_{2n+1}||
\]

\[
||Tx_{2n} - Tx_{2n+1}|| + ||Tx_{2n} - Tx_{2n+2}||
\]
Thus

$$\| T_{2n+1} - T_{2n+2}^2 \| \leq \| T_{2n} - T_{2n+1} \|$$

Further

$$\| T_{2n+1} - T_{2n} \| = \| E_{2n} - F_{2n-1} \|$$

$$\| T_{2n} - E_{2n} \| \cdot \| T_{2n} - F_{2n-1} \| + \| T_{2n-1} - F_{2n-2} \|$$

$$\leq \| T_{2n} - E_{2n} \|$$

$$\leq \| T_{2n} - F_{2n-1} \| + \| T_{2n-1} - E_{2n} \|$$

$$\| T_{2n} - T_{2n+1} \| + \| T_{2n} - T_{2n} \| + \| T_{2n-1} - T_{2n} \|$$

$$\leq \| T_{2n-1} - T_{2n+1} \|$$

$$\| T_{2n} - T_{2n} \| + \| T_{2n-1} - T_{2n+1} \|$$
\[ ||x_{2n} - x_{2n-1}|| + ||x_{2n} - x_{2m}|| + ||x_{2m-1} - x_{2n+1}|| \leq ||x_{2n} - x_{2m}|| + ||x_{2m-1} - x_{2n+1}|| \]

\[ ||x_{2n} - x_{2m}|| + ||x_{2m-1} - x_{2n+1}|| \leq \left\{ \begin{array}{l} ||x_{2n} - x_{2n-1}|| + ||x_{2n} - x_{2m}|| + ||x_{2m-1} - x_{2n+1}||, \\
||x_{2n} - x_{2m}|| \end{array} \right\} \]

\[ \leq ||x_{2n} - x_{2m}|| + ||x_{2m-1} - x_{2n+1}|| \]

\[ \therefore \quad ||x_{2n+1} - x_{2m}|| \leq ||x_{2n} - x_{2m-1}|| \]

Thus \( \{x_n\} \) is a Cauchy sequence.

The rest follows similar to theorem 2.