Mordeson [30] has defined the complement of a fuzzy graph $G : (\sigma, \mu)$ as a fuzzy graph $G^c : (\sigma^c, \mu^c)$ where $\sigma^c = \sigma$ and $\mu^c(u, v) = 0$ if $\mu(u, v) > 0$ and $\mu^c(u, v) = \sigma(u) \land \sigma(v)$ otherwise. It follows from this definition that $G^c$ is a fuzzy graph even if $G$ is not and that $(G^c)^c = G$ if and only if $G$ is a strong fuzzy graph [Definition 1.16]. Also, automorphism group of $G$ and $G^c$ are not identical. These observations motivate us to modify the notion of complement of a fuzzy graph. Some properties of self complementary fuzzy graphs are also studied. We also show that automorphism group of $G$ and its complement $\overline{G}$ are identical.

In the second part of this chapter we consider some operations on fuzzy graphs and prove that complement of the union of two fuzzy graphs is the join of their complements and the complement of the join of two fuzzy graphs is the union of their complements. Finally we prove that complement of the composition of two strong fuzzy graphs is the composition of their complements. We conclude this chapter with a discussion on some open problems.
5.1 Complement of a Fuzzy Graph

We first illustrate the drawbacks in the definition of complement of a fuzzy graph mentioned above. In Fig. 5.1, \((G^c)^c \neq G\) and note that they are identical provided \(G\) is a strong fuzzy graph [Definition 1.16].

![Fig 5.1](image1)

Now, consider the fuzzy graph \(G\) and \(G^c\) in Fig. 5.2. The automorphism group of \(G\) consists of two maps \(h_1\) and \(h_2\) where \(h_1\) is the identity map and \(h_2\) is given by the permutation \((v_1) (v_2 v_4) (v_3)\). But the automorphism group of \(G^c\) consists of four maps \(h_1, h_2, h_3\) and \(h_4\) where \(h_1\) and \(h_2\) are automorphisms of \(G\) and \(h_3\) and \(h_4\) are given by 
\[h_3 = (v_1 v_3) (v_2) (v_4)\] and 
\[h_4 = (v_1 v_3) (v_2 v_4).\]

![Fig 5.2](image2)
Definition 5.1. The complement of a fuzzy graph \( G : (\sigma, \mu) \) is the fuzzy graph \( \overline{G} : (\overline{\sigma}, \overline{\mu}) \) where \( \overline{\sigma} = \sigma \) and \( \overline{\mu}(u, v) = \sigma(u) \land \sigma(v) - \mu(u, v) \) for all \( u, v \) in \( V \).

Example.

![Graph G](image1)

![Graph \( \overline{G} \)](image2)

Fig 5.3

We have

\[
\overline{\sigma} = \sigma = \sigma \quad \text{and} \quad \overline{\mu}(u, v) = \overline{\sigma}(u) \land \overline{\sigma}(v) - \overline{\mu}(u, v) \\
= \sigma(u) \land \sigma(v) - (\sigma(u) \land \sigma(v)) - \mu(u, v)) \\
= \mu(u, v) \ \forall u, v.
\]

Hence \( \overline{\overline{G}} = G \).

Remark 5.1. A node can be a fuzzy cut node of both \( G \) and \( \overline{G} \).

In Fig. 5.4, \( w \) is a fuzzy cut node of \( G \) and \( \overline{G} \).

![Graph G](image3)

![Graph \( \overline{G} \)](image4)

Fig 5.4
Definition 5.2. A fuzzy graph $G$ is self complementary if $G \approx \overline{G}$.

In the following theorems we present a necessary and then a sufficient condition for a fuzzy graph to be self complementary.

Theorem 5.1. Let $G : (\sigma, \mu)$ be a selfcomplementary fuzzy graph. Then

$$\sum_{u \neq v} \mu(u, v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \land \sigma(v)).$$

Proof: Let $G : (\sigma, \mu)$ be a selfcomplementary fuzzy graph. Then there exists an isomorphism $h : V \to V$ such that

\[ \overline{\sigma}(h(u)) = \sigma(u) \land \sigma(v) \land \overline{\mu}(h(u), h(v)) = \mu(u, v) \land \overline{\mu}(u, v) \land \overline{\sigma}(h(u)) \land \overline{\sigma}(h(v)). \]

Now by definition of $G$, we have,

\[ \overline{\mu}(h(u), h(v)) = \overline{\sigma}(h(u)) \land \overline{\sigma}(h(v)) - \mu(h(u), h(v)) \]

ie., \[ \mu(u, v) = \sigma(u) \land \sigma(v) - \mu(h(u), h(v)) \]

ie., \[ \sum_{u \neq v} \mu(u, v) + \sum_{u \neq v} \overline{\mu}(h(u), h(v)) = \sum_{u \neq v} \sigma(u) \land \sigma(v) \]

ie., \[ 2 \sum_{u \neq v} \mu(u, v) = \sum_{u \neq v} \sigma(u) \land \sigma(v) \]

ie., \[ \sum_{u \neq v} \mu(u, v) = \frac{1}{2} \sum_{u \neq v} (\sigma(u) \land \sigma(v)). \]
Hence the theorem.

**Remark 5.2.** If $G : (V,E)$ is a self complementary (crisp) graph, then from Theorem 5.1, it follows that $2m = \frac{n(n-1)}{2}$ where $m = |E|$ and $n = |V|$ which is equivalent to the result that every self complementary (crisp) graph has $4k$ or $4k + 1$ nodes for some $k$.

**Remark 5.3.** The condition given in Theorem 5.1 is not sufficient. In the following example (Fig. 5.5), $G$ is not isomorphic to $\overline{G}$. But,

$$\sum_{u,v} \mu(u,v) = \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1 \text{ and } \frac{1}{2} \sum_{u,v} \sigma(u) \wedge \sigma(v) = \frac{1}{2} \left[ \frac{1}{2} + \frac{1}{2} + 1 \right] = 1.$$

![Fig. 5.5](image-url)

**Theorem 5.2.** Let $G : (\sigma, \mu)$ be a fuzzy graph. If $\mu(u,v) = \frac{1}{2} (\sigma(u) \wedge \sigma(v)) \forall u, v \in V$, then $G$ is self complementary.
Proof: Let $G : (\sigma, \mu)$ be a fuzzy graph such that $\mu(u,v) = \frac{1}{2}(\sigma(u) \wedge \sigma(v)) \forall u,v \in V$. Then $G \approx \overline{G}$ under the identity map on $V$.

Remark 5.4. The condition in Theorem 5.2 is not necessary. In the following example (Fig. 5.6) $G \approx \overline{G}$ where the isomorphism $h : V \rightarrow V$ is given by $h(u) = v$, $h(v) = x$, $h(w) = u$, $h(x) = w$, $h(y) = y$.

![Diagram 1](image1)

Fig. 5.6

Note that if $h$ is an automorphism of $G$ then $h^{-1}$ is also an automorphism of $G$, for

$\sigma(h^{-1}(u)) = \sigma(u') = \sigma(h(u')) = \sigma(u)$ and

$\mu(h^{-1}(u), h^{-1}(v)) = \mu(u', v') = \mu(h(u), h(v)) = \mu(u, v)$.

Similarly if $h$ and $g$ are automorphism of $G$ then their composition $h \circ g$ is also an automorphism of $G$ [3] and we have,
**Theorem 5.3**[3]. The set of all automorphisms of a fuzzy graph $G$ is a group under set-theoretic composition of maps as the binary operation.

**Notation:** We denote by $IG$, the group of automorphisms of the fuzzy graph $G$.

**Theorem 5.4.** Let $G : (\sigma, \mu)$ be a fuzzy graph. Then the automorphism groups of $G$ and $\overline{G}$ are identical.

**Proof:** Note that if $h \in IG$, then $h \in I\overline{G}$, for;

$h : V \rightarrow V$ is bijective,

$\overline{\sigma}(h(u)) = \sigma(h(u)) = \sigma(u) = \overline{\sigma}(u)$ and

$\overline{\mu}(h(u), h(v)) = (\sigma(h(u)) \wedge \sigma(h(v))) - \mu(h(u), h(v))$

$= (\sigma(u) \wedge \sigma(v)) - \mu(u, v)$

$= \overline{\mu}(u, v) \ \forall \ u, v \in V.$

Hence the theorem.

In the following example (Fig. 5.7), $IG = I\overline{G} = \{ I, h \}$, where $I$ is the identity map and $h$ is given by the permutation $(v_1, v_3) (v_2, v_4)$.

**Fig. 5.7**

![Diagram](image-url)
5.2 Operations on Fuzzy Graphs

The operations on (crisp) graphs such as union join, cartesian product and composition are extended to fuzzy graphs [Definitions 1.29 – 1.32] and some of their properties are studied in [30]. In the following discussions an arc between two nodes \( u \) and \( v \) is denoted by \( uv \) rather than \((u, v)\), because in the cartesian product of two graphs, a node of the graph is in fact, an ordered pair.

**Theorem 5.5** Let \( G_1: (\sigma_1, \mu_1) \) and \( G_2: (\sigma_2, \mu_2) \) be two fuzzy graphs. Then

1) \( \bar{G}_1 + \bar{G}_2 \approx \bar{G}_1 \cup \bar{G}_2 \) and

2) \( \bar{G}_1 \cup \bar{G}_2 \approx \bar{G}_1 + \bar{G}_2 \)

**Proof**: We shall prove that the identity map is the required isomorphism.

1) Let \( I : V_1 \cup V_2 \rightarrow V_1 \cup V_2 \) be the identity map.

To prove \( \bar{\sigma}_1 + \bar{\sigma}_2 (u) = \bar{\sigma}_1 \cup \bar{\sigma}_2 (u) \)

and \( \mu_1 + \mu_2 (uv) = \bar{\mu}_1 \cup \bar{\mu}_2 (uv) \)

\( \bar{\sigma}_1 + \bar{\sigma}_2 (u) = (\sigma_1 + \sigma_2) (u) \) by definition of complement

\[
\bar{\sigma}_1 + \bar{\sigma}_2 (u) = \begin{cases} 
\sigma_1(u) & \text{if } u \in V_1 \\
\sigma_2(u) & \text{if } u \in V_2 
\end{cases}
\]

\[
= \begin{cases} 
\bar{\sigma}_1(u) & \text{if } u \in V_1 \\
\bar{\sigma}_2(u) & \text{if } u \in V_2 
\end{cases}
\]

\[
= \bar{\sigma}_1 \cup \bar{\sigma}_2 (u).
\]
$$\mu_1 + \mu_2 (uv) = (\sigma_1 + \sigma_2) u \land (\sigma_1 + \sigma_2)v - (\mu_1 + \mu_2)uv$$

$$= \begin{cases} (\sigma_1 \cup \sigma_2) u \land (\sigma_1 \cup \sigma_2) v - (\mu_1 \cup \mu_2)uv & \text{if } uv \in E_1 \cup E_2 \\ (\sigma_1 \cup \sigma_2) u \land (\sigma_1 \cup \sigma_2) v - \sigma_1(u) \land \sigma_2(v), & uv \in E' \text{ (Definition 1.)} \end{cases}$$

$$= \begin{cases} \sigma_1(u) \land \sigma_1(v) - \mu_1(uv), & \text{if } uv \in E_1 \\ \sigma_2(u) \land \sigma_2(v) - \mu_2(uv), & \text{if } uv \in E_2 \\ \sigma_1(u) \land \sigma_2(v) - \sigma_1(u) \land \sigma_2(v) & \text{if } uv \in E' \text{ where } u \in V_1, v \in V_2 \end{cases}$$

$$= \begin{cases} \mu_1(uv), & uv \in E_1 \\ \mu_2(uv), & uv \in E_2 \\ 0, & uv \in E' \end{cases}$$

$$= \mu_1 \cup \mu_2 (uv).$$

2) \( G_1 \cup G_2 \approx G_1 + G_2. \)

Let \( I : V_1 \cup V_2 \to V_1 \cup V_2 \) be the identity map.

To prove \( \sigma_1 \cup \sigma_2 (u) = (\sigma_1 + \sigma_2) (uv) \)
and \( \mu_1 \cup \mu_2 (uv) = (\mu_1 + \mu_2) (uv). \)

$$\overline{\sigma_1 \cup \sigma_2 (u)} = (\overline{\sigma_1 \cup \sigma_2 (u)})$$

$$= \begin{cases} \overline{\sigma_1(u)}, & \text{if } u \in V_1 \\ \overline{\sigma_2(u)}, & \text{if } u \in V_2 \end{cases}$$

$$= \begin{cases} \overline{\sigma_1(u)}, & \text{if } u \in V_1 \\ \overline{\sigma_2(u)}, & \text{if } u \in V_2 \end{cases}$$

$$= (\overline{\sigma_1 \cup \sigma_2}) (u)$$

$$= (\sigma_1 + \sigma_2) (u).$$
\[ \mu_1 \cup \mu_2 (uv) = (\sigma_1 \cup \sigma_2 )u \land (\sigma_1 \cup \sigma_2 )v - (\mu_1 \cup \mu_2 )(uv) \]

\[ = \begin{cases} 
\sigma_1(u) \land \sigma_1(v) - \mu_1(uv), & \text{if } uv \in E_1 \\
\sigma_2(u) \land \sigma_2(v) - \mu_2(uv), & \text{if } uv \in E_2 \\
\sigma_1(u) \land \sigma_2(v) - 0, & \text{when } u \in V_1, v \in V_2 
\end{cases} \]

\[ = \begin{cases} 
\bar{\mu}_1(uv), & \text{if } uv \in E_1 \\
\bar{\mu}_2(uv), & \text{if } uv \in E_2 \\
\sigma_1(u) \land \sigma_2(v), & \text{if } u \in V_1, v \in V_2 
\end{cases} \]

\[ = \begin{cases} 
((\bar{\mu}_1 \cup \bar{\mu}_2)uv & \text{if } uv \in E_1 \text{ or } E_2 \\
\sigma_1(u) \land \sigma_2(v) & \text{if } uv \in E^1 
\end{cases} \]

\[ = (\bar{\mu}_1 + \bar{\mu}_2)(uv). \]

**Remark 5.5.** Note that if \( G \) is a strong fuzzy graph, then \( \overline{G} \) is also strong, for, let \( uv \in \mu^* \), then

\[ \bar{\mu}(uv) = \sigma(u) \land \sigma(v) - \mu(uv) = \sigma(u) \land \sigma(v) - \sigma(u) \land \sigma(v) = 0 \]

and if \( uv \notin \mu^* \) then

\[ \bar{\mu}(uv) = \sigma(u) \land \sigma(v) - \mu(uv) = \sigma(u) \land \sigma(v) - 0 = \sigma(u) \land \sigma(v). \]

**Theorem 5.6.** Let \( G_1: (\sigma_1, \mu_1) \) and \( G_2: (\sigma_2, \mu_2) \) be two strong fuzzy graphs. Then \( G_1 \circ G_2 \)

is a strong fuzzy graph and \( \overline{G_1 \circ G_2} \approx \overline{G_1} \circ \overline{G_2} \).

**Proof:** Let \( G_1 \circ G_2 = G: (\sigma, \mu) \) where \( \sigma = \sigma_1 \circ \sigma_2, \mu = \mu_1 \circ \mu_2 \)

and \( G^* = (V, E) \) where \( V = V_1 \times V_2, E = \{ (u, u_2)(u, v_2) : u \in V_1, u_2v_2 \in E_2 \} \cup \{ (u_1, w)(v_1, w) : w \in V_2, u_1v_1 \in E_1 \} \cup \{ (u_1, u_2)(v_1, v_2) : u_1v_1 \in E_1, u_2 \neq v_2 \}. \)

Now,
1) \( \mu(u, u_2)(u, v_2) = \sigma_1(u) \wedge \mu_2(u_2, v_2) \) [Definition 1.32]
\[= \sigma_1(u) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), \text{ } G_2 \text{ being strong} \]
\[= (\sigma_1(u) \wedge \sigma_2(u_2)) \wedge (\sigma_1(u) \wedge \sigma_2(v_2)) \]
\[= (\sigma_1 \circ \sigma_2)(u, u_2) \wedge (\sigma_1 \circ \sigma_2)(u, v_2) \]
\[= \sigma(u, u_2) \wedge \sigma(u, v_2) \]

2) \( \mu(u_1, w)(v_1, w) = \sigma_2(w) \wedge \mu_1(u_1, v_1) \) [Definition 1.32]
\[= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(w), \text{ } G_1 \text{ being strong} \]
\[= (\sigma_1(u_1) \wedge \sigma_2(w)) \wedge (\sigma_1(v_1) \wedge \sigma_2(w)) \]
\[= (\sigma_1 \circ \sigma_2)(u_1, w) \wedge (\sigma_1 \circ \sigma_2)(v_1, w) \]
\[= \sigma(u_1, w) \wedge \sigma(v_1, w) \]

and

3) \( \mu(u_1, u_2)(v_1, v_2) = \mu_1(u_1, v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2) \) [Definition 1.32]
\[= \sigma_1(u_1) \wedge \sigma_1(v_1) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), \text{ } G_1 \text{ being strong} \]
\[= (\sigma_1(u_1) \wedge \sigma_2(u_2)) \wedge (\sigma_1(v_1) \wedge \sigma_2(v_2)) \]
\[= (\sigma_1 \circ \sigma_2)(u_1, u_2) \wedge (\sigma_1 \circ \sigma_2)(v_1, v_2) \]
\[= \sigma(u_1, u_2) \wedge \sigma(v_1, v_2). \]

Thus from 1, 2 and 3 it follows that \( G \) is a strong fuzzy graph.

Next, to prove \( \overline{G_1 \circ G_2} = \overline{G_1} \circ \overline{G_2} \).
Now, the various types of arcs (say) \( e \), joining the nodes of \( V \) are the following and it suffices to prove that \( \overline{\mu_1 \circ \mu_2} = \overline{\mu_1 \circ \mu_2} \) in each case.

Case I. \( e = (u, u_2)(u, v_2) \), \( u_2 v_2 \in E_2 \)

Then \( e \in E \) and \( G \) being strong \( \overline{\mu}(e) = 0 \). Also, \( \overline{\mu_1 \circ \mu_2}(e) = 0 \), since \( u_2 v_2 \not\in \overline{E}_2 \).

Case II. \( e = (u, u_2)(u, v_2) \), \( u_2 \neq v_2 \) and \( u_2 v_2 \not\in E_2 \)

Here \( e \not\in E \), so \( \mu(e) = 0 \).

Now,
\[
\overline{\mu}(e) = \sigma(u, u_2) \wedge \sigma(u, v_2)
\]
\[
= (\sigma_1(u) \wedge \sigma_2(u_2)) \wedge (\sigma_1(u) \wedge \sigma_2(v_2)) \quad \text{and since}
\]
\[
u_2 v_2 \in \overline{E}_2 \quad \text{we have,}
\]
\[
\overline{\mu_1 \circ \mu_2}(e) = \sigma_1(u) \wedge \overline{\mu_2}(u_2 v_2)
\]
\[
= \sigma_1(u) \wedge \sigma_2(u_2) \wedge \sigma_2(v_2), \overline{G}_2 \text{ being strong}
\]
\[
= \overline{\mu}(e).
\]

Case III. \( e = (u_1, w)(v_1, w) \), \( u_1 v_1 \in E_1 \)

Then \( e \in E \), so \( \overline{\mu}(e) = 0 \) as in Case 1.

Also, since \( u_1 v_1 \not\in \overline{E}_1 \), we have \( \overline{\mu_1 \circ \mu_2}(e) = 0 \).

Case IV. \( e = (u_1, w)(v_1, w), u_1 v_1 \not\in E_1 \)
\( \overline{\mu}(e) = \sigma(u, w) \land \sigma(v, w) \)

\[ \begin{align*}
\sigma_1(u) \land \sigma_1(v) \land \sigma_2(w) \text{ and since } u, v \in \overline{E}_1 \\
\text{we have,} \\
\overline{\mu}_1 \circ \overline{\mu}_2(e) = \sigma_2(w) \land \overline{\mu}_1(u, v_1) \\
= \sigma_1(u_1) \land \sigma_1(v_1), \text{ } \overline{G}_1 \text{ being strong} \\
= \overline{\mu}(e). 
\end{align*} \]

Case V. \( e = (u_1, u_2)(v_1, v_2), u_1, v_1 \in E_1 \text{ and } u_2 \neq v_2 \)

Here \( e \notin E \), so \( \overline{\mu}(e) = 0 \) as in Case 1.

Also, since \( u_1, v_1 \notin \overline{E}_1 \), we have \( \overline{\mu}_1 \circ \overline{\mu}_2(e) = 0 \).

Case VI. \( e = (u_1, u_2)(v_1, v_2), u_1, v_1 \notin E_1 \text{ and } u_2 \neq v_2 \)

Then \( e \notin E \), hence \( \mu(e) = 0 \)

Thus \( \overline{\mu}(e) = \sigma(u_1, u_2) \land \sigma(v_1, v_2) \)

\[ \begin{align*}
= \sigma_1(u_1) \land \sigma_1(v_1) \land \sigma_2(u_2) \land \sigma_2(v_2) 
\end{align*} \]

and since \( u_1, v_1 \in \overline{E}_1 \), we have,

\( \overline{\mu}_1 \circ \overline{\mu}_2 = \overline{\mu}_1(u_1, v_1) \land \sigma_2(u_2) \land \sigma_2(v_2) \)

\[ \begin{align*}
= \sigma_1(u_1) \land \sigma_1(v_1) \land \sigma_2(u_2) \land \sigma_2(v_2), \text{ } \overline{G}_1 \text{ being strong} \\
= \overline{\mu}(e). 
\end{align*} \]

Case VII. \( e = (u_1, u_2)(v_1, v_2), u_1, v_1 \notin E_1, u_2, v_2 \notin E_2 \)

Here \( e \notin E \), hence \( \mu(e) = 0 \)

Thus \( \overline{\mu}(e) = \sigma(u_1, u_2) \land \sigma(v_1, v_2) \)
Now, $u_iv_i \in \overline{E}_1$ and if $u_2 = v_2 = w$, then we have Case IV.

Next, if $u_iv_i \in \overline{E}_1$ and if $u_2 \neq v_2$, then we have Case VI.

Thus from Cases I to VI, it follows that $\overline{G}_1 \circ \overline{G}_2 \approx \overline{G}_1 \circ \overline{G}_2$.

Remark 5.6. In general $\overline{G}_1 \circ \overline{G}_2 \neq \overline{G}_1 \circ \overline{G}_2$. Consider the following example in which $G_1$ and $G_2$ are not strong.
Thus $G_1 \circ G_2$

Thus $G_1 \circ G_2 \neq G_1 \circ G_2$.

5.3 Conclusion and Suggestions for Further Study

In this thesis an attempt to develop the properties of basic concepts in fuzzy graphs such as fuzzy bridges, fuzzy cutnodes, fuzzy trees and blocks in fuzzy graphs have been made.

The notion of complement of a fuzzy graph is modified and some of its properties are studied. Since the notion of complement has just been initiated, several properties of $G$ and $\overline{G}$ available for crisp graphs can be studied for fuzzy graphs also.

We have mainly focussed on fuzzy trees defined by Rosenfeld. In [8], several other types of fuzzy trees are defined depending on the acyclic level of a fuzzy graph. We have observed that there are selfcentered fuzzy trees. However, which fuzzy trees are selfcentered is yet to be analyzed. The center problems can also be carried over to other types of fuzzy trees mentioned above.
Identification of blocks of a fuzzy graph is still an open problem; solving which may lead to the study of fuzzy block graphs, fuzzy cut point graphs etc. The study of the parameter $C(G)$ – the connectedness level of a fuzzy graph [10] can be done on $G$ and $\overline{G}$. 