CHAPTER VI

FIXED POINT THEOREMS FOR PAIR OF MAPPINGS
6.1. In 1984, Wang and his co-authors [1] established some fixed point theorems for certain expansion mappings in complete metric space. Recently in this direction, Popa ([4],[5], Theorem 1) obtained the following:

**Theorem A.** Let \((X,d)\) be a complete metric space and \(f : (X,d) \rightarrow (X,d)\) a surjective mapping. If there exists non-negative reals \(a,b,c\) with \(a > 0\) or \(b > 0\) and \(a + b + c > 1\) such that
\[
d^2(fx,fy) \geq a d(x,fx)d(x,y) + bd(y,fy)d(x,y) + cd(x,fx)d(y,fy)
\]
for each \(x,y\) in \(X\) with \(x \neq y\), then \(f\) has a fixed point.

**Theorem B.** Let \((X,d)\) be a complete metric space and \(f : (X,d) \rightarrow (X,d)\) a surjective mapping. If there exists non-negative reals \(a,b,c\) with \(a < 1\) and \(a + b + c > 1\) such that
\[
d^2(fx,fy) \geq a d^2(x,fx) + bd^2(y,fy) + cd^2(x,y)
\]
for each \(x,y\) in \(X\) with \(x \neq y\), then \(f\) has a fixed point. Further, if \(c > 1\), then the fixed point is unique.

Our objective here is to generalize the above results for pairs of mappings.
THEOREM 1. Let $f, g$ be surjective self-maps of a complete metric space $(X,d)$. Suppose there exists non-negative reals $a, b, c$ with $a, b > 0$ and $a + b + c > 1$ such that

$$d^2(fx, gy) \geq ad(x, fx) \cdot d(x, y) + bd(y, gy) \cdot d(x, y)$$

$$+ cd(x, fx) \cdot d(y, gy)$$

for each $x, y$ in $X$ with $x \neq y$, then $f$ and $g$ have a common fixed point.

PROOF. Let $x_0 \in X$. Since $f$ is surjective there exists a point $x_1 \in f^{-1}x_0$. Since $g$ is surjective there exists a point $x_2 \in g^{-1}x_1$. Continuing in this manner, we obtain a sequence $\{x_n\}$ with $x_{2n+1} \in f^{-1}x_{2n}$ and $x_{2n+2} \in g^{-1}x_{2n+1}$.

Assume $x_{2n} = x_{2n+1}$ for some $n$. If $x_{2n+1} \neq x_{2n+2}$, then applying (6.1.1), we have

$$d^2(x_{2n}, x_{2n+1}) = d^2(fx_{2n+1}, gx_{2n+2})$$

$$\geq a \cdot d(x_{2n+1}, x_{2n}) \cdot d(x_{2n+1}, x_{2n+2})$$

$$+ bd(x_{2n+2}, x_{2n+1}) \cdot d(x_{2n+2}, x_{2n+2})$$

$$+ cd(x_{2n+1}, x_{2n}) \cdot d(x_{2n+2}, x_{2n+1}) \cdot$$

Thus

$$0 \geq bd^2(x_{2n+1}, x_{2n+2})$$
which implies $x_{2n+1} = x_{2n+2}$ and $x_{2n}$ is a common fixed point of $f$ and $g$. Similarly, $x_{2n+1} = x_{2n+2}$ for some $n$ leads to $x_{2n+1}$ being a common fixed point of $f$ and $g$.

Suppose $x_n \neq x_{n+1}$ for each $n$. Then, by (6.1.1), we get

$$d^2(x_{2n}, x_{2n+1}) = d^2(fx_{2n+1}, gx_{2n+2})$$

$$\leq ad(x_{2n+1}, x_{2n})d(x_{2n+1}, x_{2n+2})$$

$$+ bd(x_{2n+2}, x_{2n+1})d(x_{2n+1}, x_{2n+2})$$

$$+ cd(x_{2n+1}, x_{2n})d(x_{2n+2}, x_{2n+1})$$

or,

$$bd^2(x_{2n+2}, x_{2n+1}) + (a+c)d(x_{2n+2}, x_{2n+1})d(x_{2n+1}, x_{2n})$$

$$-d^2(x_{2n}, x_{2n+1}) \leq 0$$

or,

$$bt_1^2 + (a+c)t_1 - 1 \leq 0$$

where $t_1 = \frac{d(x_{2n+2}, x_{2n+1})}{d(x_{2n+1}, x_{2n})}$.

Now, let $h_1 : (0, \infty) \to \mathbb{R}$ be the function

$$h_1(t_1) = bt_1^2 + (a+c)t_1 - 1.$$ Then $h_1(c) = -1$ and

$$h_1(1) = a + b + c - 1 > 0$$

from the hypothesis. Let $k \in (0,1)$ be the root of the equation $h_1(t_1) = 0$, then $h_1(t_1) \leq 0$ for $t_1 \leq k$ and thus
Similarly, from (6.1.1), we have

\[ a d^2(x_{2n+3} \cdot x_{2n+2}) + (b+c) d(x_{2n+2} \cdot x_{2n+1}) d(x_{2n+3} \cdot x_{2n+2}) = d^2(x_{2n+2} \cdot x_{2n+1}) \leq 0 \]

or,

\[ a t_2^2 + (b+c) t_2 - 1 \leq 0 \]

where

\[ t_2 = \frac{d(x_{2n+3} \cdot x_{2n+2})}{d(x_{2n+2} \cdot x_{2n+1})} \]

Again, let \( h_2 : [0, \infty) \to \mathbb{R} \) be the function
\[
h_2(t_2) = a t_2^2 + (b+c) t_2 - 1.\]

Then \( h_2(0) = -1 \) and

\( h_2(1) = a + b + c - 1 > 0 \) from the hypothesis. Let \( r \in (0, 1) \) be the root of the equation \( h_2(t_2) = 0 \), then \( h_2(t_2) \leq 0 \) for \( t_2 \leq r \) and thus

\[ d(x_{2n+3} \cdot x_{2n+2}) \leq r d(x_{2n+2} \cdot x_{2n+1}) \]

Now by putting max \( \{k, r\} = 1 \in (0, 1) \), we have from (6.1.2) and (6.1.3),

\[ d(x_{2n+1} \cdot x_{2n+2}) \leq \cdots \leq \cdots \]

Then by a routine calculation we can show that \( \{x_n\} \) is a Cauchy sequence and since \( X \) is complete, it converges to a point \( x \in X \).
Without loss of generality we may assume that \( x_n \neq x \) for infinitely many \( n \) since, otherwise, \( f \) and \( g \) have a common fixed point. If there exists an infinite number of integers \( n \) such that \( x_{2n} \neq x \), define \( y \in g^{-1}x \). Then, we have from (6.1.1),

\[
d^2(x_{2n}, x) = d^2(f x_{2n+1}, g y) \\
\geq a d(x_{2n+1}, x_{2n}) d(x_{2n+1}, y) \\
+ b d(y, g y) d(x_{2n+1}, y) \\
+ c d(x_{2n+1}, x_{2n}) d(y, g y).
\]

Letting \( n \to \infty \), we get

\[
o \geq b d^2(x, y)
\]

which implies \( x = y \). Since \( x = g y \), then \( x = g x \).

If \( x_{2n+1} \neq x \) for all \( n \) sufficiently large, then \( x_{2n} = f x_{2n+1} = x \). Taking the limit as \( n \to \infty \) yields \( x \) as a fixed point of \( f \).

If \( x_{2n+1} \neq x \) infinitely many \( n \), define \( z \in f^{-1}x \). Then, applying (6.1.1), we have

\[
d^2(x_{2n+1}, x) = d^2(g x_{2n+2}, f z) \\
\geq a d(z, f z) d(z, x_{2n+2}).
\]
\[ + bd \left( x_{2n+2} \ x_{2n+1} \right) \ d(z_2 x_{2n+2}) \]
\[ + cd \left( z_2 x_2 \right) \ d( x_{2n+2} \ x_{2n+1}) \cdot \]

Letting \( n \to \infty \), we have

\[ \circ \geq a \ \ d^2(x, z) \]

which simply implies \( z = x \), and \( x \) is a common fixed point of \( f \) and \( g \). This completes the proof.

**Theorem 2.** Let \( f, g \) be surjective self-maps of a complete metric space \((X, d)\). Suppose there exists non-negative functions \( a, b, c \) satisfying

\[ (6.1.4) \quad \inf_{x, y \in X} (a(x, y) + b(x, y) + c(x, y)) > 1, \]

\[ (6.1.5) \quad \sup_{x, y \in X} \left\{ a(x, y), b(x, y) \right\} < 1, \quad \text{and} \]

\[ (6.1.6) \quad d^2(fx, gy) \geq a(x, y) \ d^2(x, fx) + b(x, y) \ d^2(y, gy) \]

\[ + c(x, y) \ d^2(x, y) \]

for each \( x, y \) in \( X \) with \( x \neq y \), then \( f \) and \( g \) have common fixed point. Further if \( c(x, y) > 1 \) then the fixed point is unique.
**Proof.** Define \( \{x_n\} \) as in Theorem 1. Suppose \( x_{2n} = x_{2n+1} \) for some \( n \). If \( x_{2n+1} \neq x_{2n+2} \), then from (6.1.6), we have

\[
d^2(x_{2n}, x_{2n+1}) = d^2(fx_{2n+1}, gx_{2n+2})
\]

\[
\geq a d^2(x_{2n+1}, x_{2n}) + bd^2(x_{2n+2}, x_{2n+1})
\]

\[
+ c d^2(x_{2n+1}, x_{2n+2})
\]

where \( a, b \) and \( c \) are evaluated at \( (x_{2n+1}, x_{2n+2}) \). Thus,

\[
0 \geq (b+c) d^2(x_{2n+2}, x_{2n+1})
\]

If \( (b+c) = 0 \), then it, since \( a < 1 \), contradicts (6.1.4). Therefore \( x_{2n+1} = x_{2n+2} \) and \( x_{2n} \) is a common fixed point of \( f \) and \( g \). Similarly, \( x_{2n+1} = x_{2n+2} \) for some \( n \) leads to \( x_{2n+1} \) being a common fixed point of \( f \) and \( g \).

Suppose \( x_n \neq x_{n+1} \) for each \( n \). Applying (6.1.6) we have

\[
d^2(x_{2n}, x_{2n+1}) = d^2(fx_{2n+1}, gx_{2n+2})
\]

\[
\geq a d^2(x_{2n+1}, x_{2n}) + bd^2(x_{2n+2}, x_{2n+1})
\]

\[
+ cd^2(x_{2n+1}, x_{2n+2})
\]

or,

(6.1.7) \( (1-a) d^2(x_{2n}, x_{2n+1}) \geq (b+c) d^2(x_{2n+2}, x_{2n+1}) \).
where \( a, b, c \) are evaluated at \((x_{2n+1}, x_{2n+2})\).

Again (6.1.6) yields

\[
d^2(x_{2n+1}, x_{2n+2}) = d^2(gx_{2n+2}, fx_{2n+3})
\]

\[
\geq a'\, d^2(x_{2n+3}, x_{2n+2}) + \\
+ b'\, d^2(x_{2n+2}, x_{2n+1}) \]

\[
+ c'\, d^2(x_{2n+2}, x_{2n+3})
\]

or,

\[
(1-b')\, d^2(x_{2n+1}, x_{2n+2}) \geq (a'+c')\, d^2(x_{2n+2}, x_{2n+3}),
\]

where \( a', b', c' \) are evaluated at \((x_{2n+3}, x_{2n+2})\).

Inequalities (6.1.7) and (6.1.8) along with conditions (6.1.4) and (6.1.5), imply that \( \{x_n\} \) is Cauchy, hence convergent to some \( x \) in \( X \).

Without loss of generality we assume that \( x_n \neq x \) for infinitely many \( n \) since, otherwise, \( f \) and \( g \) have a common fixed point. If there exists an infinite number of integers \( n \) such that \( x_{2n} \neq x \), we define \( y \in g^{-1}x \). Then, applying (6.1.6), we obtain

\[
d^2(x_{2n}, x) = d^2(fx_{2n+1}, gy)
\]
\[ a \geq \frac{d^2(x_{2n+1}, x_{2n}) + bd^2(y, gy) + c d^2(x_{2n+1}, y)}{\min \{d^2(y, x), d^2(x_{2n+1}, y)\}}, \]

where \( a, b, c \) are evaluated at \((x_{2n+1}, y)\). The above inequality implies that

\[ d^2(x_{2n}, x) \geq (b+c) \min \{d^2(y, x), d^2(x_{2n+1}, y)\}, \]

\[ \geq \inf_{x, y \in X} (b+c) \min \{d^2(y, x), d^2(x_{2n+1}, y)\}. \]

Letting \( n \to \infty \), we have

\[ \infty \geq \inf_{x, y \in X} (b+c) d(x, y) \]

which implies that either \( x = y \) or \( \inf_{x, y \in X} (b+c) = 0 \).

But the latter condition, along with (6.1.5), contradicts (6.1.4). Therefore \( x = y \). Since \( x = gy \), then \( x = gx \).

If \( x_{2n+1} \neq x \) for all \( n \) sufficiently large, then

\[ x_{2n} = fx_{2n+1} = x. \]

Letting \( n \to \infty \), we obtain \( x \) as a fixed point of \( f \).

If \( x_{2n+1} \neq x \) infinitely many \( n \), define \( z \in f^{-1}x \).

Then from (6.1.6) with \( a, b, c \) evaluated at \((z, x_{2n+2})\), we get

\[ d^2(x_{2n+1}, x) = d^2(gx_{2n+2}, fz) \]

\[ \geq a d^2(z, fz) + bd^2(x_{2n+2}, x_{2n+1}) \]

\[ + c d^2(z, z_{2n+2}). \]
\[ \inf_{x,y \in X} (a+c) \min \{d(x,x_n), d(z,x_{2n+2}) \}. \]

Letting \( n \to \infty \), we get \( c \geq \inf_{x,y \in X} (a+c) d(x,z) \), which, in light of (6.1.5) and (6.1.4), implies \( z = x \) and \( x \) is a common fixed point of \( f \) and \( g \).

Further, let \( c(x,y) > 1 \) and \( f \) and \( g \) have a second fixed point \( x' (\neq x) \). Then, from (6.1.6) with \( a,b,c \) evaluated at \( (x,x') \),

\[ d^2(x,x') = d^2(fx, gx') \]

\[ \geq a d^2(x,fx) + b d^2(x',gx') + c d^2(x,x') \]

\[ = c d^2(x,x') \]

which simply implies \( d(x,x') = 0 \) and therefore \( x = x' \). This completes the proof.

**Remarks.**

1. Setting \( f = g \) in Theorem 1, we get Theorem A.
2. Taking \( f = g \) and \( a,b,c \) constants in Theorem 2, it reduces to Theorem B.

6.2. Popa ([4],[5], Theorem 2) also proved the following fixed point theorems for certain expansion mappings satisfying rational inequalities.
**THEOREM C.** Let \((X,d)\) be a complete metric space and \(f : (X,d) \rightarrow (X,d)\) a surjective mapping. If there exists a real constant \(k \in \left(\frac{2}{3}, 1\right)\) such that
\[
d(fx, fy) \geq k \frac{d^2(x, fx) + d^2(y, fy) + d(x, fx) d(y, fy)}{d(x, fx) + d(y, fy)}
\]
for each \(x \neq y\) in \(X\) for which \(d(x, fx) + d(y, fy) \neq 0\), then \(f\) has a fixed point.

**THEOREM D.** Let \((X,d)\) be a complete metric space and \(f : (X,d) \rightarrow (X,d)\) a surjective mapping. If there exist non-negative reals \(a, b, c\) with \(a + b + c > 2\) such that
\[
d(fx, fy) \geq \frac{ad(x, fx) d(x, y) + bd(y, fy) d(x, y) + cd(x, fx) d(y, fy)}{d(x, fx) + d(y, fy)}
\]
for each \(x \neq y\) in \(X\) for which \(d(x, fx) + d(y, fy) \neq 0\), then \(f\) has a fixed point.

Now we generalize these results as follows:

**THEOREM 3.** Let \(f, g\) be surjective self-maps of a complete metric space \((X,d)\). If there exists a real constant \(k \in \left(\frac{2}{3}, 1\right)\) such that
\[
d(fx, gy) \geq k \frac{d^2(x, fx) + d^2(y, gy) + d(x, fx) d(y, gy)}{d(x, fx) + d(y, gy)}
\]
for each \(x \neq y\) in \(X\) for which \(d(x, fx) + d(y, gy) \neq 0\), then \(f\) and \(g\) have a common fixed point.
**Proof.** Define a sequence \( \{x_n\} \) as in Theorem 1. If \( x_n = x_{n+1} \) for any \( n \), it is easily to see that \( f \) and \( g \) have a common fixed point.

Suppose \( x_n \neq x_{n+1} \) for each \( n \).

Applying (6.2.1), we have

\[
d(x_{2n}, x_{2n+1}) = d(fx_{2n+1}, gx_{2n+2})
\]

\[
\leq \frac{d^2(x_{2n+1}, x_{2n}) + d^2(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n})d(x_{2n+2}, x_{2n+1})}{d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1})}
\]

or,

\[
k d^2(x_{2n+2}, x_{2n+1}) + (k-1)d(x_{2n+1}, x_{2n})d(x_{2n+2}, x_{2n})
\]

\[
+ (k-1) d^2(x_{2n+1}, x_{2n}) \leq 0
\]

or,

\[
k t_1^2 + (k-1) t_1 + (k-1) \leq 0, \quad \text{where} \quad t_1 = \frac{d(x_{2n+2}, x_{2n+1})}{d(x_{2n+1}, x_{2n})}
\]

Now let \( h_1 : (0, \infty) \to \mathbb{R} \) be the function

\[
h_1(t_1) = kt_1^2 + (k-1) t_1 + (k-1) \cdot \text{Then} \quad h_1(0) = k-1 < 0 \quad \text{and} \quad h_1(1) = 3k-2 > 0 \quad \text{from the hypothesis. Let} \quad p \in (0, 1) \quad \text{be the root of the equation} \quad h_1(t_1) = 0, \text{then} \quad h_1(t_1) \leq 0 \quad \text{for} \quad t_1 \leq p. \text{ Thus} \]
(6.2.2) \[ d(x_{2n+1}, x_{2n+2}) \leq pd(x_{2n+1}, x_2n) \cdot \]

Similarly, from (6.2.1), we have

\[ kd^2(x_{2n+3}, x_{2n+2}) + (k-1) d(x_{2n+3}, x_{2n+2})d(x_{2n+2}, x_2n+1) \]

\[ + (k-1) d^2(x_{2n+1}, x_{2n+2}) \leq 0 \]

or,

\[ kt_2^2 + (k-1) t_2 + (k-1) \leq 0, \text{ where } t_2 = \frac{d(x_{2n+3}, x_{2n+2})}{d(x_{2n+2}, x_2n+1)} \cdot \]

Again let \( h_2 \in (0, \infty) \rightarrow \mathbb{R} \) be the function

\[ h_2(t_2) = kt_2^2 + (k-1) t_2 + (k-1). \]

Then \( h_2(0) = k-1 < 0 \) and

\[ h_2(1) = 3k-2 > 0 \] from the hypothesis. Let \( q \in (0, 1) \) be the root of the equation \( h_2(t_2) = 0 \), then \( h_2(t_2) \leq 0 \) for \( t_2 < q \)

and thus

(6.2.3) \[ d(x_{2n+3}, x_{2n+2}) \leq qd(x_{2n+2}, x_2n) \cdot \]

Taking max \( \{p, q\} = \lambda \in (0, 1) \), we have from (6.2.2) and (6.2.3),

\[ d(x_{2n+1}, x_{2n+2}) \leq \lambda d(x_{2n}, x_{2n+1}) \leq \lambda^2 d(x_{2n-1}, x_2n) \leq \cdots \]

\[ \cdots \leq \lambda^{2n+1} d(x_0, x_1) \cdot \]

Then by a routine calculation we can show that \( \{x_n\} \)

is a Cauchy sequence. By the completeness of \( X \), \( \{x_n\} \) converges to a point \( x \in X \).
Without loss of generality we may assume that $x_n \neq x$ for infinitely many $n$ since, otherwise, $f$ and $g$ have a common fixed point. If there exists an infinite number of integers $n$ such that $x_{2n} \neq x$, define $y \in g^{-1}x$. Then by (6.2.1),

$$d(x_{2n}, x) = d(fx_{2n}+1, gy) \geq k \frac{d^2(x_{2n+1}, x_{2n}) + d^2(y, gy) + d(x_{2n+1}, x_{2n})}{d(x_{2n+1}, x_{2n}) + d(y, gy)},$$

Letting $n \to \infty$, we have $0 \geq kd(y, x)$ which implies $x = y$.

Since $x = gy$, then $x = gx$.

If $x_{2n+1} \neq x$ for all $n$ sufficiently large, then $x_{2n} = fx_{2n+1} = x$. Taking the limit as $n \to \infty$ yields $x$ as a fixed point of $f$.

If $x_{2n+1} \neq x$ for infinitely many $n$, define $z = f^{-1}x$. Then from (6.2.1),

$$d(x_{2n+1}, x) = d(gx_{2n+2}, fz) \geq k \frac{d^2(z, fz) + d^2(x_{2n+1}, x_{2n+1}) + d(z, fz) + d(x_{2n+1}, x_{2n+1})}{d(z, fz) + d(x_{2n+2}, x_{2n+1})}.$$ 

Letting $n \to \infty$, we have $0 \geq kd(z, x)$ which implies $x = z$.

Since $x = fz$, then $x = fx$, and $x$ is a common fixed point of $f$ and $g$. This completes the proof.
**Theorem 4.** Let \( f, g \) be surjective self-maps of a complete metric space \((X, d)\). Suppose there exists non-negative reals \( a, b, c \) with \( a > 0, b > 0 \) and \( a + b + c > 2 \) such that

\[
(6.2.4) \quad d(fx, gy) \geq \frac{ad(x, fx)d(y, y) + bd(y, gy)d(x, y) + cd(x, fx)d(y, gy)}{d(x, fx) + d(y, gy)}
\]

for each \( x \neq y \) in \( X \) for which \( d(x, fx) + d(y, gy) > 0 \), then \( f \) and \( g \) have a common fixed point.

**Proof.** It is similar to the proof of Theorem 3.

**Remark 3.** If we take \( f = g \) in Theorem 3 and Theorem 4, we obtain Theorem C and Theorem D respectively.

6.3. Popa concluded his papers ([4], [5], Theorem 3) by proving:

**Theorem 6.** Let \((X, d)\) be a complete metric space and \( f : (X, d) \to (X, d) \) a surjective continuous mapping. If there exists a real constant \( k > 1 \) such that

\[
d^2(fx, fy) \geq k \min \{d^2(x, fx), d^2(y, fy), d(x, fx) \cdot d(x, y), d(y, fy) \cdot d(x, y) \}
\]

for any \( x, y \) in \( X \), then \( f \) has a fixed point.

**Theorem 7.** Let \((X, d)\) be a complete metric space and \( f : (X, d) \to (X, d) \) a surjective continuous mapping. If there exists a real constant \( k > 1 \) such that
\[ d^2(fx, fy) \leq k \min \left\{ d(x, fx) \cdot d(y, fy) ; \frac{d(y, fy)}{d(x, fx)} \right\} \]

for any \( x, y \) in \( X \), then \( f \) has a fixed point.

Finally, in this section, we present the following generalizations of the above results.

**Theorem 5.** Let \( f \) and \( g \) be surjective continuous self-maps of a complete metric space \((X, d)\). If there exists a real number \( k > 1 \) such that

\[ (6.3.1) \quad d^2(fx, gy) \leq k \min \left\{ d^2(x, fx) ; d^2(y, gy) ; d(x, fx) \cdot d(y, gy) ; d(x, fx) \cdot d(y, gy) \right\} \]

for each \( x, y \) in \( X \), then \( f \) or \( g \) has a fixed point or \( f \) and \( g \) have a common fixed point.

**Proof.** Define \( \{x_n\} \) as in Theorem 1. If \( x_n = x_{n+1} \) for any \( n \), then \( f \) or \( g \) has a fixed point.

Assume \( x_n \neq x_{n+1} \) for each \( n \). The application of (6.3.1) gives

\[ d^2(x_{2n}, x_{2n+1}) = d^2(fx_{2n+1}, gx_{2n+2}) \]

\[ \leq k \min \left\{ d^2(x_{2n+1}, x_{2n}) ; d^2(x_{2n+2}, x_{2n+1}) ; d(x_{2n+1}, x_{2n}) \cdot d(x_{2n+1}, x_{2n+2}) \right\} \]
\[
= kd(x_{2n+1}, x_{2n+2}) \min \left\{ d(x_{2n+1}, x_{2n+2}), \quad d(x_{2n}, x_{2n+1}) \right\}.
\]

We have either
\[
d(x_{2n}, x_{2n+1}) \geq kd(x_{2n+1}, x_{2n+2}) \geq \forall k \ d(x_{2n+1}, x_{2n+2})
\]
or,
\[
d^2(x_{2n}, x_{2n+1}) \geq kd^2(x_{2n+1}, x_{2n+2}), \text{ which implies}
\]
\[(6.3.2) \quad d(x_{2n}, x_{2n+1}) \geq \forall k \ d(x_{2n+1}, x_{2n+2}) \]
and,
\[
d^2(x_{2n+1}, x_{2n+2}) = d^2(gx_{2n+2}, fx_{2n+3})
\]
\[
\geq k \ min \left\{ d^2(x_{2n+3}, x_{2n+2}), d^2(x_{2n+2}, x_{2n+1}), \quad d(x_{2n+2}, x_{2n+1})d(x_{2n+2}, x_{2n+1}) \right\}.
\]
\[
= kd(x_{2n+2}, x_{2n+3}) \ min \left\{ d(x_{2n+2}, x_{2n+3}), \quad d(x_{2n+1}, x_{2n+2}) \right\}.
\]

We have either
\[
d(x_{2n+1}, x_{2n+2}) \geq kd(x_{2n+2}, x_{2n+3}) \geq \forall k \ d(x_{2n+2}, x_{2n+3})
\]
or,
\[
d^2(x_{2n+1}, x_{2n+2}) \geq kd^2(x_{2n+2}, x_{2n+3}), \text{ which implies}
\]
(6.3.3) \[ d(x_{2n+1}, x_{2n+2}) \leq \sqrt{k} d(x_{2n+2}, x_{2n+2}). \]

Therefore from (6.3.2) and (6.3.3), it follows that

\[ d(x_{2n+3}, x_{2n+2}) \leq \sqrt{k} d(x_{2n+2}, x_{2n+2}) \leq \cdots \leq \left( \frac{1}{\sqrt{k}} \right)^{2n+2} d(x_1, x_0), \]

which, since \( k > 1 \), implies that \( \{x_n\} \) is a Cauchy sequence, hence convergent to some \( x \) in \( X \). The condition \( x_{2n} = f x_{2n-1} \) and \( x_{2n+1} = g x_{2n+1} \) and the continuity of \( f \) and \( g \) imply that \( x \) is a common fixed point of \( f \) and \( g \). This completes the proof.

**THEOREM 6.** Let \( f \) and \( g \) be surjective continuous self-maps of a complete metric space \((X,d)\). If there exists a real number \( k > 1 \) such that

\begin{align*}
(6.3.4) \quad d^2(fx, gy) &\geq k \min \left\{ d(x, fx) d(x, y) ; d(y, gy) d(y, gy) ; d(x, fx) d(y, gy) \right\}
\end{align*}

for each \( x, y \) in \( X \), then \( f \) or \( g \) has a fixed point or \( f \) and \( g \) have a common fixed point.

**PROOF.** It is similar to the proof of Theorem 5.

**REMARK 4.** Setting \( f = g \) in Theorem 5 and Theorem 6, we get Theorem 5 and Theorem 6 respectively.