CHAPTER IV

COMMON FIXED POINT THEOREMS ON MULTIFUNCTIONS
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4.1. Dealing with a complete metric space, Pachpatte established the following theorems in [3] and [7] respectively.

**Theorem A.** Let $S$ and $T$ be mappings of a complete metric space $(X, d)$ itself satisfying the inequality

$$
(4.1.1) \quad d(sx, ty) \leq c \max \left\{ \frac{[d(x, sx)]^2 + [d(y, ty)]^2}{d(x, sx) + d(y, ty)} \right\},
$$

$$
\frac{1}{2} \frac{[d(x, ty)]^2 + [d(y, sx)]^2}{d(x, sx) + d(y, ty)}
$$

for all $x, y$ in $X$ for which $d(x, sx) + d(y, ty) \neq 0$, where $0 < c < 1$. Then $S$ and $T$ have a common fixed point. Further, if $d(x, sx) + d(y, ty) = 0$ implies $d(sx, ty) = 0$, the fixed point is unique.

**Theorem B.** Let $S$ and $T$ be mappings of a complete metric space $(X, d)$ into itself such that

$$
(4.1.2) \quad \{d(sx, ty)\}^2 \leq c \max \left\{ [d(x, sx) d(y, ty) + d(x, ty) d(y, sx)] \right\},
$$

$$
\frac{1}{2} [d(x, ty) d(y, ty) + d(x, sx) d(y, sx)]
$$
for all \( x, y \) in \( X \) where \( 0 < c < 1 \). Then \( S \) and \( T \) have a unique
common fixed point.

In the present section, we give similar common
fixed point theorems for two multifunctions \( T_1 \) and \( T_2 \) which
generalize Theorem A and B. Before the statement of our
theorems we mention some lemmas which will be required in
the sequel.

**Lemma 1.** Let \((X, d)\) be a metric space and \( T_1, T_2 : (X, d) \rightarrow CB(X) \)
be two multifunctions. If

\[
H^m(T_1, T_2) \leq c \max \left\{ \frac{d^p(x, T_1 x) + d^p(y, T_2 y)}{d^{p-m}(x, T_1 x) + d^{p-m}(y, T_2 y)}, \right. \\
\left. \frac{1}{2^{p-1}} \frac{d^p(x, T_2 y) + d^p(y, T_1 x)}{d^{p-m}(x, T_1 x) + d^{p-m}(y, T_2 y)} \right\}
\]

holds for all \( x, y \) in \( X \) for which \( d^{p-m}(x, T_1 x) + d^{p-m}(y, T_2 y) \neq 0 \),
where \( 0 < c < 1, m \geq 1, p \geq 2, m < p \) and \( F(T_1) \neq \emptyset \), then
\( F(T_2) \neq \emptyset \) and \( F(T_1) = F(T_2) \).

**Proof.** Let \( u \in F(T_1) \), then \( u \in T_1 u \) and if \( d(u, T_2 u) \neq 0 \),
then applying (4.1.3), we have

\[
d^m(u, T_2 u) \leq H^m(T_1 u, T_2 u) \leq c \max \left\{ \frac{d^p(u, T_1 u) + d^p(u, T_2 u)}{d^{p-m}(u, T_1 u) + d^{p-m}(u, T_2 u)} \right\}
\]
\[
\frac{1}{2^{P-1}} \frac{d_P(u_0 T_2 u) + d_P(u_0 T_1 u)}{d_P^m(u_0 T_1 u) + d_P^m(u_0 T_2 u)} \}
\]

\[
\frac{c d_P(u_0 T_2 u)}{d_P^m(u_0 T_1 u) + d_P^m(u_0 T_2 u)} \]

which implies \(d(u_0 T_2 u) = 0\). Since \(T_2 u\) is closed, this shows that \(u \in T_2 u\), which implies \(F(T_2) \subset F(T_1)\). Analogously \(F(T_2) \subset F(T_1)\).

**Lemma 2.** Let \((X, d)\) be a metric space and \(T_1, T_2 : (X, d) \to CB(X)\) be two multifunctions. If

\[
(4.1.4) \ \{H(T_1 x, T_2 y)\}^2 \leq c \max \left\{ [d(x, T_1 x) d(y, T_2 y) + d(x, T_2 y) d(y, T_1 x)] \right\}
\]

holds for all \(x, y\) in \(X\), where \(c < c < 1\) and \(F(T_1) \neq \emptyset\), then \(F(T_2) \neq \emptyset\) and \(F(T_1) = F(T_2)\).

**Proof.** Let \(u \in F(T_1)\), then \(u \in T_1 u\). By \((4.1.4)\), we have

\[
d^2(u, T_2 u) \leq H^2(T_1 u, T_2 u)
\]

\[
\leq c \max \left\{ [d(u, T_1 u) d(u, T_2 u) + d(u, T_2 u) d(u, T_1 u)] \right\}
\]
\[ \frac{1}{2} \left\{ d(u, T_2u) \ d(u, T_1u) + d(u, T_1u) d(u, T_1u) \right\} \]

i.e.,
\[ d^2(u, T_2u) \leq \frac{c}{2} \ d^2(u, T_1u) \]

which implies \( d(u, T_2u) = 0 \). Since \( T_2u \) is closed, this shows that \( u \in T_2 u \) which implies \( F(T_1) \subset F(T_2) \). Analogously \( F(T_2) \subset F(T_1) \).

**Theorem 1.** Let \((X, d)\) be a complete metric space and let
\( T_1 \circ T_2 : (X, d) \rightarrow CB(X) \) be two multifunctions such that (4.1.3) holds for all \( x, y \) in \( X \) for which \( \delta^{P_m}(x, T_1 x) + \delta^{P_m}(y, T_2 y) \neq 0 \).

where \( 0 < c < 1, m \geq 1, p \geq 2 \) and \( m < p \). Then \( T_1 \) and \( T_2 \) have common fixed points and \( F(T_1) = F(T_2) \).

**Proof.** Choose a real number \( q \) such that

\[ 1 < q < (c)^{-1/m} \]

Let \( x_0 \in X \) and \( x_1 \in T_1 x_0 \). Then there is an \( x_2 \in T_2 x_1 \) so that \( d(x_1, x_2) \leq q H(T_1 x_0, T_2 x_1) \). Suppose \( x_3, x_4, \ldots, x_{2n-1}, x_{2n}, \ldots \) could be chosen so that \( x_{2n-1} \in T_1 x_{2n-2}, x_{2n} \in T_2 x_{2n-1} \) and

\[ d(x_{2n-1}, x_{2n}) \leq q H(T_1 x_{2n-2}, T_2 x_{2n-1}) \]

\[ d(x_{2n-2}, x_{2n-1}) \leq q H(T_1 x_{2n-2}, T_2 x_{2n-3}) \]
Suppose first of all that
\[ \delta \rho^n(x_{2n-2}, T_1 x_{2n-2}) + \delta \rho^n(x_{2n-1}, T_2 x_{2n-1}) = 0 \]
for some \( n \). Then \( x_{2n-2} = T_1 x_{2n-2} = x_{2n-1} = T_2 x_{2n-1} \) and \( x_{2n-2} = x_{2n-1} \) is a common fixed point of \( T_1 \) and \( T_2 \). Similarly,
\[ \delta \rho^n(x_{2n-1}, T_2 x_{2n-1}) + \delta \rho^n(x_{2n}, T_1 x_{2n}) = 0 \]
for some \( n \) implies that \( x_{2n-1} = x_{2n} \) is a common fixed point of \( T_1 \) and \( T_2 \). Now suppose that
\[ \delta \rho^n(x_{2n-2}, T_1 x_{2n-2}) + \delta \rho^n(x_{2n-1}, T_2 x_{2n-1}) \neq 0 \]
and
\[ \delta \rho^n(x_{2n-1}, T_2 x_{2n-1}) + \delta \rho^n(x_{2n}, T_1 x_{2n}) \neq 0 \]
for \( n = 1, 2, \ldots \).

Then, by (4.1.3), we have successively
\[
d^m(x_{2n-1}, x_{2n}) \leq c \rho^m H^m(T_1 x_{2n-2}, T_2 x_{2n-1})
\]
\[
\leq c \rho^m \max \left\{ \frac{d^p(x_{2n-2}, T_1 x_{2n-2}) + d^p(x_{2n-1}, T_2 x_{2n-1})}{\delta \rho^m(x_{2n-2}, T_1 x_{2n-2}) + \delta \rho^m(x_{2n-1}, T_2 x_{2n-1})}, \frac{d^p(x_{2n-2}, T_2 x_{2n-1}) + d^p(x_{2n-1}, T_1 x_{2n})}{\delta \rho^m(x_{2n-2}, T_1 x_{2n-2}) + \delta \rho^m(x_{2n-1}, T_2 x_{2n-1})} \right\}
\]
\[
\leq \frac{1}{2^{p-1}} \max \left\{ \frac{d^p(x_{2n-2}, x_{2n-2}) + d^p(x_{2n}, x_{2n})}{\delta \rho^m(x_{2n-2}, T_1 x_{2n-2}) + \delta \rho^m(x_{2n-1}, T_2 x_{2n-1})}, \frac{d^p(x_{2n-2}, x_{2n-2}) + d^p(x_{2n-1}, x_{2n})}{\delta \rho^m(x_{2n-2}, T_1 x_{2n-2}) + \delta \rho^m(x_{2n-1}, T_2 x_{2n-1})} \right\}
\]
\[
\frac{1}{2^{p-1}} \frac{d^P(x_{2n-2}^* x_{2n})}{d^{P_m}(x_{2n-2}^* x_{2n-1}) + d^{P_m}(x_{2n-1}^* x_{2n})}
\]

\[
\leq \text{eq}^m \max \left\{ \frac{d^P(x_{2n-2}^* x_{2n-1}) + d^P(x_{2n-1}^* x_{2n})}{d^{P_m}(x_{2n-2}^* x_{2n-1}) + d^{P_m}(x_{2n-1}^* x_{2n})} \right\}
\]

\[
\leq \text{eq}^m \frac{d^P(x_{2n-2}^* x_{2n-1}) + d^P(x_{2n-1}^* x_{2n})}{d^{P_m}(x_{2n-2}^* x_{2n-1}) + d^{P_m}(x_{2n-1}^* x_{2n})}
\]

\[
\leq \frac{1}{2^{p-1}} \left[ \frac{d(x_{2n-2}^* x_{2n-1}) + d(x_{2n-1}^* x_{2n})}{d^{P_m}(x_{2n-2}^* x_{2n-1}) + d^{P_m}(x_{2n-1}^* x_{2n})} \right]^P
\]

\[
\left. \begin{align*}
\leq \text{eq}^m &+ \frac{1}{2} \left\{ d^P(x_{2n-2}^* x_{2n-1}) + d^P(x_{2n-1}^* x_{2n}) \right\} \\
> \frac{1}{2} \left\{ d(x_{2n-2}^* x_{2n-1}) + d(x_{2n-1}^* x_{2n}) \right\}^P
\end{align*} \right\}
\]

If \( x_{2n-1} \neq x_{2n} \), then

\[
d^P(x_{2n-1}^* x_{2n}) \cdot (1 - \text{eq}^m) + d^m(x_{2n-1}^* x_{2n}) \cdot d^{P_m}(x_{2n-2}^* x_{2n-1})
\]

\[
-\text{eq}^m \cdot d^P(x_{2n-2}^* x_{2n-1}) \leq 0
\]

and

\[
t^P(1 - \text{eq}^m) + t^m - \text{eq}^m \leq 0, \text{ where } t = \frac{d(x_{2n-1}^* x_{2n})}{d(x_{2n-2}^* x_{2n-1})}.
\]
Let \( f : [0, \infty) \to \mathbb{R} \) be the function \( f(t) = t^p(1-e^{q^m}) + t^m - e^{q^m} \).

Then \( f'(t) > 0 \) for any \( t > 0 \), \( f(0) < 0 \) and \( f(1) = 2(1-e^{q^m}) > 0 \) by (4.1.3). Let \( k \in (0,1) \) be the root of the equation \( f(t) = 0 \), then \( f(t) \leq 0 \) for \( t \leq k \) and thus

\[
d(x_{2n-1}, x_{2n}) \leq kd(x_{2n-2}, x_{2n-1}) \quad \text{for } n = 1, 2, \ldots
\]

Similarly we have

\[
d(x_{2n}, x_{2n+1}) \leq kd(x_{2n-1}, x_{2n}) \quad \text{for } n = 1, 2, \ldots
\]

Repeating the above argument, we obtain

\[
d(x_n, x_{n-1}) \leq k^n d(x_0, x_1).
\]

Then, by a routine calculation, we can show that \( \{x_n\} \) is a Cauchy sequence and since \( X \) is complete, we have \( \lim x_n = x \) for some \( x \in X \). If we now suppose that \( d(u, \bar{u}) > 0 \), then

\[
d^m(x_{2n}, \bar{u}) \leq \left( \frac{d^p(u, \bar{u}) + d^p(x_{2n-1}, T_2x_{2n-1})}{\delta^p(u, \bar{u})+\delta^p(x_{2n-1}, T_2x_{2n-1})} \right)_{\frac{1}{2^{n-1}}} \]

\[
\leq \max \left\{ \frac{d^p(u, \bar{u}) + d^p(x_{2n-1}, T_2x_{2n-1})}{\delta^p(u, \bar{u})+\delta^p(x_{2n-1}, T_2x_{2n-1})}, \frac{d^p(x_{2n-1}, \bar{T}_1u)+d(x_{2n-1}, \bar{T}_1u)}{\delta^p(u, \bar{u})+\delta^p(x_{2n-1}, T_2x_{2n-1})} \right\}
\]
\[ \leq c \max \left\{ \frac{d^p(u, T_1 u) + d^p(x_{2n-1}, x_{2n})}{d^{p^m}(u, T_1 u) + d^{p^m}(x_{2n-1}, x_{2n})}, \frac{1}{2^{p-1}} \frac{d^p(u, T_1 u) + d^p(x_{2n-1}, T_1 u)}{d^{p^m}(u, T_1 u) + d^{p^m}(x_{2n-1}, x_{2n})} \right\} \]

and on letting \( n \) tends to infinity, we have

\[ d(u, T_1 u) \leq (c)^{1/m} d(u, T_1 u), \] giving a contradiction, since

\( (c)^{1/m} < 1 \). It follows that \( d(u, T_1 u) = 0 \). Since \( T_1 u \) is closed,
this shows that \( u \in T_1 u \). By Lemma 1, \( u \in T_2 u \) and \( F(T_1) = F(T_2) \).

If \( T_1 \) and \( T_2 \) are single valued mappings, we have the following:

**Theorem 2.** Let \( T_1 \) and \( T_2 \) be mappings of a complete metric
space into itself such that

\[ d^m(T_1 x, T_2 y) \leq c \max \left\{ \frac{d^p(x, T_1 x) + d^p(y, T_2 y)}{d^{m^p}(x, T_1 x) + d^{m^p}(y, T_2 y)}, \frac{1}{2^{p-1}} \frac{d^p(x, T_2 y) + d^p(y, T_1 x)}{d^{m^p}(x, T_1 x) + d^{m^p}(y, T_2 y)} \right\} \]

for all \( x, y \) in \( X \) for which \( d^{m^p}(x, T_1 x) + d^{m^p}(y, T_2 y) \neq 0 \),

where \( 0 < c < 1, \ m \geq 1, \ p \geq 2, \ m < p \), then \( T_1 \) and \( T_2 \) have
common fixed points and \( F(T_1) = F(T_2) \). Further, if
$d^{p-m}(x, T_1x) + d^{p-m}(y, T_2y) = 0$ implies $d(T_1x, T_2y) = 0$, the fixed point $u$ is unique.

The existence follows from Theorem 1. Now suppose that $T_1$ and $T_2$ have a second fixed point $u'$. Then $\{d(u, T_1u)\}^{p-m} + \{d(u', T_2u')\}^{p-m} = 0$ implies $d(T_1u, T_2u') = 0$ and therefore $u = T_1u$, $u' = T_2u'$ and $T_1u = T_2u'$. Thus the common fixed point of $T_1$ and $T_2$ is, in this case, unique.

**Remark.** On taking $p = 2$ and $m = 1$ in Theorem 2, we obtain Theorem A.

**Theorem 3.** Let $(X, d)$ be a complete metric space and let $T_1, T_2 : (X, d) \to CB(X)$ be two multifunctions such that (4.1.4) holds for all $x, y$ in $X$, where $0 < c < 1$. Then $T_1$ and $T_2$ have common fixed points and $F(T_1) = F(T_2)$.

**Proof.** Choose a real number $q$ such that

\[(4.1.7) \quad 1 < q < \sqrt{1/c}.

Let $x_0 \in X$ and $x_1 \in T_1x_0$. Then there is an $x_2 \in T_2x_1$ so that $d(x_1, x_2) \leq qH(T_1x_0, T_2x_1)$. Suppose $x_3, x_4, \ldots, x_{2n-1}, x_{2n}, \ldots$ could be chosen so that $x_{2n-1} \in T_1x_{2n-2}$, $x_{2n} \in T_2x_{2n-1}$ and

\[d(x_{2n-1}, x_{2n}) \leq qH(T_1x_{2n-2}, T_2x_{2n-1})\]

\[d(x_{2n-2}, x_{2n-1}) \leq qH(T_1x_{2n-2}, T_2x_{2n-3}).\]
By (4.1.4), for $x = x_{2n-2}$ and $y = x_{2n-1}$ we have

$$d^2(x_{2n-1}, x_{2n}) \leq q^2 d^2(T_1 x_{2n-2}, T_2 x_{2n-1})$$

$$\leq q^2 c \max \left\{ d(x_{2n-2}, T_1 x_{2n-2}) d(x_{2n-1}, T_2 x_{2n-1}) + d(x_{2n-2}, T_2 x_{2n-1}) d(x_{2n-1}, T_1 x_{2n-2}) \right\}$$

$$\leq q^2 c \max \left\{ d(x_{2n-2}, x_{2n-2}) d(x_{2n-1}, x_{2n}) + \frac{1}{2} d(x_{2n-2}, x_{2n}) d(x_{2n-1}, x_{2n}) \right\}$$

which simply implies

$$d(x_{2n-1}, x_{2n}) \leq q^2 c d(x_{2n-2}, x_{2n-1})$$

Analogously, we have

$$d(x_{2n-2}, x_{2n-1}) \leq q^2 c d(x_{2n-3}, x_{2n-2})$$

Repeating the above argument, we get

$$d(x_n, x_{n-1}) \leq (q^2 c)^n d(x_0, x_1), \text{ where } q^2 c < 1 \text{ from (4.1.7)}.$$  

Then, by routine calculation, one can show that $\{x_n\}$ is Cauchy sequence and since $(X, d)$ is complete, we have
\[ \lim x_n = u \text{ for some } u \in X. \text{ If we now assume that} \]
\[ d(u, T_1 u) \neq 0, \text{ then (4.1.4) yields} \]
\[ d^2(x_{2n}, T_1 u) \leq H^2(T_2 x_{2n-1}, T_1 u) \]
\[ \leq c \max \{ [d(u, T_1 u) d(x_{2n-1}, T_2 x_{2n-1})] \]
\[ + d(u, T_2 x_{2n-1}) d(x_{2n-1}, T_1 u) \} \]
\[ - \frac{1}{2} [d(u, T_2 x_{2n-1}) d(x_{2n-1}, T_2 x_{2n-1}) + d(u, T_1 u) d(x_{2n-1}, T_1 u)] \]
\[ \leq c \max \{ [d(u, T_1 u) d(x_{2n-1}, x_{2n})] + d(u, x_{2n}) d(x_{2n-1}, T_1 u) \} \]
\[ - \frac{1}{2} [d(u, x_{2n}) d(x_{2n-1}, x_{2n}) + d(u, T_1 u) d(x_{2n-1}, T_1 u)] \}.
\]
Now letting \( n \to \infty \), we obtain \( d^2(u, T_1 u) \leq \frac{1}{2} d^2(u, T_1 u) \),
giving a contradiction since \( 0 < c < 1 \). It follows that
\( d(u, T_1 u) = 0 \). Since \( T_1 u \) is closed, this shows that \( u \in T_1 u \).
By Lemma 2, \( u \in T_2 u \) and \( F(T_1) = F(T_2) \).

4.2. The purpose of this section is to obtain similar
common fixed point theorems for a sequence of multifunctions
which generalize Theorem A and B.

**Theorem 4.** Let \((X, d)\) be a complete metric space and \( \{T_n\} \in \mathbb{N} \)
a sequence of multifunction of \( X \) into \( CB(X) \) such that
\[ (4.2.1) \quad H^p(T_1 x, T_n y) \leq c \max \left\{ \frac{d^p(x, T_1 x) + d^p(y, T_n y)}{d^p(x, T_1 x) + d^p(y, T_n y)} \right\}, \]
\[
\frac{1}{2^{p-1}} \frac{d^P(x, T_n y) + d^P(y, T_n x)}{\delta^{p-m}(x, T_1 x) + \delta^{p-m}(y, T_1 y)}
\]

holds for all \(x, y\) in \(X\), \(n \geq 2\), \(0 < c < 1\), \(m \geq 1\), \(p \geq 2\), \(m < p\) for which \(\delta^{p-m}(x, T_1 x) + \delta^{p-m}(y, T_1 y) = e\). Then \(\{ T_n \}_{n \in \mathbb{N}}\) has common fixed points and \(F(T_1) = F(T_n)\).

**Proof.** It follows by Theorem 1 and Lemma 1.

**Theorem 5.** Let \((X, d)\) be a complete metric space and \(\{ T_n \}_{n \in \mathbb{N}}\) a sequence of multifunctions of \((X, d)\) into \(CB(X)\) such that

\[
\tag{4.2.2}
\{ H(T_1 x, T_n y) \}^2 \leq c \max \left\{ d(x, T_1 x) d(y, T_n y) + d(x, T_n y) d(y, T_1 x) \right\}
\]

holds for all \(x, y \in X\) and \(n \geq 2\), where \(0 < c < 1\). Then \(\{ T_n \}_{n \in \mathbb{N}}\) has common fixed points and \(F(T_1) = F(T_n)\).

**Proof.** It follows by Theorem 3 and Lemma 2.

4.3. We conclude this chapter by considering a metric space with two metrics \(e\) and \(d\) and by obtaining fixed point theorems for sequence of multifunctions. For the proof of the theorems we require the following lemma.
**Lemma 3.** (Popa [2], Theorem 4) Let $X$ and $Y$ be two topological spaces. If the multifunction $F : X \rightarrow Y$ with $Y$, a $T_3$ space, is u.s.c. and $F(x)$ is closed subset of $X$ for all $x \in X$, then $F$ has the closed graph.

**Theorem 6.** Let $(X,d)$ be a metric space with two metrics $e$ and $d$. If $X$ satisfies the following conditions:

1. $e(x,y) \leq d(x,y)$ for any $x,y \in X$,
2. $X$ is complete with respect to $e$,
3. two multifunctions $T_1,T_2 : X \rightarrow X$ are punctually closed and punctually bounded with respect to both metrics,
4. $T_1$ or $T_2$ u.s.c. with respect to $e$,
5. The inequality (4.1.3) holds for all $x,y$ in $X$ for which $d^{P-m}(x,T_1x) + d^{P-m}(y,T_2y) < e$, where $0 < e < 1$, $m \geq 1$, $p \geq 2$, $m < p$.

Then $T_1$ and $T_2$ have common fixed points and $F(T_1) = F(T_2)$.

**Proof.** Analogously as in the proof of Theorem 1, for any $x_0 \in X$, we can construct a sequence $\{x_n\}$ such that $x_{2n+1} \in T_1x_{2n}$, $\{x_n\}$ is a Cauchy sequence with respect to $d$. Therefore, by $e \leq d$, $\{x_n\}$ is a Cauchy sequence with respect to $e$, and since $X$ is complete with respect to $e$, $x_n \rightarrow x$. 


As $T_1$ is u.s.c. from Lemma 3, $T_1$ has closed graph and then from $x_{2n+1} \notin T_1'x_{2n}$ results $x \notin T_1'x$ and from Lemma 1,
$F(T_1) = F(T_2)$.

**Theorem 7.** Let $X$ be a metric space with two metrics $e$ and $d$. If $X$ satisfies the conditions $(4.3.1)-(4.3.4)$ and the inequality $(4.1.4)$ holds for all $x, y \in X$, where $0 < c < 1$, then $T_1$ and $T_2$ have common fixed points and $F(T_1) = F(T_2)$.

**Proof.** Analogously as in the proof of Theorem 3, for any $x_0 \in X$, we can construct a sequence $\{x_n\}$ such that $x_{2n+1} \in T_1'x_{2n}$, $\{x_n\}$ is a Cauchy sequence with respect to $d$. Applying Lemma 2 in place of Lemma 1, the remaining proof is similar to the proof of Theorem 6.

**Theorem 8.** Let $X$ be a metric space with two metrics $e$ and $d$. If $X$ satisfies the following conditions:

(4.3.6) The sequence of multifunctions $\{T_n\}_{n \in \mathbb{N}}$ is formed by punctually closed and punctually bounded with respect to both metrics,

(4.3.7) $e, d$ and $T_1$ satisfy conditions $(4.3.1), (4.3.2)$ and $(4.3.4)$,

(4.3.8) the inequality $(4.2.1)$ holds for all $x, y \in X$ for which $d^p(x, T_1x) + d^p(y, T_1y) = 0$, where $n \geq 2$.,
\[ 0 < c < 1, \ m \geq 1, \ p \geq 2, \ m < p. \]

then \( \{T_n\}_{n \in \mathbb{N}} \) has common fixed points and \( F(T_1) = F(T_n) \).

**Proof.** It follows by Theorem 6 and Lemma 1.

**Theorem 9.** Let \( X \) be a metric space with two metrics \( e \) and \( d \). If \( X \) satisfies the conditions (4.3.6), (4.3.7) and the inequality (4.2.2) holds for all \( x, y \) in \( X \), \( n \geq 2 \) and \( 0 < c < 1 \), then \( \{T_n\}_{n \in \mathbb{N}} \) has common fixed points and \( F(T_1) = F(T_n) \).

**Proof.** It follows by Theorem 7 and Lemma 2.