CHAPTER III

SOME FIXED POINT THEOREMS FOR ORBITALLY CONTINUOUS MAPPINGS IN METRIC SPACES
CHAPTER III

SOME FIXED POINT THEOREMS FOR ORBITALLY CONTINUOUS MAPPINGS IN METRIC SPACES

3.1. In 1974, Cirić ([6], Theorem 1) obtained a non-unique fixed point theorem for an orbitally continuous self-mapping $T$ of orbitally complete metric space $(X, d)$ satisfying

\[(3.1.1) \quad \min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} \leq \min\{d(x, Ty), d(y, Tx)\} \leq p d(x, y)\]

for all $x, y \in X$ and for some $p \in (0, 1)$.

Recently, Pačpatte [5] established the following non-unique fixed point theorem for Cirić type map.

**THEOREM A.** Let $(X, d)$ be orbitally complete metric space and $T$ be an orbitally continuous self-mapping of $X$ satisfying

\[(3.1.2) \quad \min\{d(Tx, Ty), d(x, Tx), d(y, Ty), \frac{d(y, Ty)[1+d(x, Tx)]}{1+d(x, y)}\} \leq \min\{d(x, Ty), d(y, Tx), \frac{d(x, Ty)[1+d(x, Tx)+d(y, Tx)]}{1+d(x, y)}\} \leq p d(x, y)\]

for every $x, y \in X$ and $p \in (0, 1)$. Then $T$ has a fixed point in $X$. 
In the present section, we obtain a fixed point theorem involving four points of orbitally complete metric space which improves upon Theorem A.

The concept of contractive type mappings involving four points of the space under consideration has been first studied by Pittnauer [1] and then more general results have been established by Ćheri [4]. Our result is inspired by the results of Dhage [1] and Ćheri [4].

**Theorem 1.** Let \((X,d)\) be orbitally complete metric space and \(T\) be an orbitally continuous self-mapping of \(X\) satisfying

\[
(3.1.3) \quad \min\left\{d(Tu_1, Tu_2), d(u_1, Tu_3), d(u_2, Tu_4)\right\} \leq \frac{d(u_2, Tu_4)[1+d(u_1, Tu_3)]}{[1+d(u_1, u_2)]} \\
+ \alpha \min\left\{d(u_1, Tu_4), d(u_3, Tu_3)\right\} \frac{d(u_1, Tu_3)[1+d(u_1, Tu_3)+d(u_2, Tu_3)]}{[1+d(u_1, u_2)]} \\
\leq p d(u_1, u_2) + q d(u_1, Tu_3)
\]

for all \(u_1, u_2, u_3, u_4 \in X\) and \(\alpha, p, q\) are real numbers such that \(0 < p + q < 1\). Then \(T\) has a fixed point in \(X\).

**Proof.** Let \(x, y \in X\) and define \(u_1 = Ty, u_2 = Tx, u_3 = x, u_4 = y\), then (3.1.3) gives
\[(3.1.4) \quad \min \{d(T^2y, T^2x), d(Ty, Tx), d(Tx, Ty), \]
\[\frac{d(Tx, Ty) [1+d(Ty, Tx)]}{[1 + d(Ty, Tx)]} \}
\[+ a \min \{d(Ty, Ty), d(Tx, Tx), \frac{d(Ty, Ty)[1+d(Ty, Tx)+d(Tx, Tx)]}{[1 + d(Ty, Tx)]} \} \]
\[\leq p d(Ty, Tx) + q d(Ty, Tx). \]

Let \( x \in X \) be arbitrary. We define a sequence \( \{x_n\} \) by \( x_0 = x, x_1 = Tx_0, x_2 = Tx_1, \ldots, x_n = Tx_{n-1} \). If for some \( n \in \mathbb{N} \), \( x_n = x_{n+1} \), then \( \{x_n\} \) is a Cauchy sequence and the limit of \( \{x_n\} \) is a fixed point of \( T \). Suppose that \( x_n \neq x_{n+1} \) for each \( n = 0, 1, 2, \ldots \). Then by (3.1.4), for some \( x = x_{n-1}, y = x_{n-2} \), we have
\[\min \{d(T^2x_{n-2}, T^2x_{n-1}), d(Tx_{n-2}, Tx_{n-1})\} \leq (p+q)d(Tx_{n-2}, Tx_{n-1})\]
i.e. \( \min \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \leq h d(x_{n-1}, x_n) \)
i.e. \( d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \)
where \( p+q = h < (0,1) \), since \( d(x_{n-1}, x_n) \leq h \min d(x_{n-1}, x_n) \), \( h \in (0,1) \) is impossible.

Proceeding in this manner, we obtain
\[d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \leq \cdots \leq h^nd(x_0, Tx).\]
Hence for any $p \in \mathbb{N}$, one has
\[
d(x_n, x_{n+p}) \leq \frac{h^n}{1-h} d(x_0, Tx).
\]

Since $\lim_{n \to \infty} \frac{h^n}{1-h} = 0$, it follows that $\{x_n\}$ is a Cauchy sequence. $X$ being $T$-orbitally complete, there is some $z \in X$ such that $z = \lim_{n \to \infty} T^n x$.

By the orbital continuity of $T$, we have
\[
Tx = \lim_{n \to \infty} T^n x = z.
\]

Thus $z$ is a fixed point of $T$. This completes the proof of the theorem.

**Remark 1.** If $a = -1$ and $u_1 = u_2 = x$, $u_2 = u_4 = y$ in (3.1.3), then we get

**Corollary 1.** Let $(X, d)$ be orbitally complete metric space and $T$ be an orbitally continuous self-mapping of $X$ satisfying

\[
(3.1.5) \quad \min \{d(Tx, Ty), d(x, Tx), d(y, Ty), \frac{d(y, Ty)[1+d(x, Tx)]}{1+d(x, y)} \}
\]

\[
- \min \{d(x, Ty), d(y, Tx), \frac{d(x, Ty)[1+d(x, Tx)+d(y, Tx)]}{1+d(x, y)} \}
\]

\[
\leq pd(x, y) + qd(x, Tx)
\]

for all $x, y \in X$ and $p$ and $q$ are real numbers satisfying $0 < p + q < 1$. Then $T$ has a fixed point in $X$. 
Remark 2. In the special case $q = 0$, inequality (3.1.5) reduces to (3.1.2) considered by Pachpatte [5].

The following example shows the advantage of our condition over (3.1.2).

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be a mapping defined by $f(x) = x^2 + 2x - 2$ for all $x \in \mathbb{R}$, where the metric $d$ is defined in the usual sense i.e. $d(x, y) = |x - y|$. Then for $x = 2$ and $y = 3$, by condition (3.1.2), we get

\[
\min \left\{ \frac{d(3, f(3))}{1 + d(2, 3)}, \frac{d(3, f(3))[1 + d(2, 2)]}{1 + d(2, 3)} \right\}
\]

\[
\min \left\{ \frac{d(2, f(3))}{1 + d(3, 3)}, \frac{d(2, f(3))[1 + d(2, 2)]}{1 + d(2, 3)} \right\}
\]

\[
\leq p \ d(2, 3)
\]

i.e., \[
\min \left\{ \frac{d(2, 13)}{1 + d(2, 2)}, \frac{d(3, 13)[1 + d(2, 6)]}{1 + d(2, 3)} \right\}
\]

\[
\min \left\{ \frac{d(2, 13)}{1 + d(2, 6)}, \frac{d(2, 13)[1 + d(2, 6)+d(3, 6)]}{1 + d(2, 3)} \right\}
\]

\[
\leq p \ d(2, 3)
\]

i.e., $1 \leq p$

which is a contradiction since $p < 1$ and hence condition (3.1.2) fails to prove the existence of the fixed point of the map $f$. But this mapping $f$ is contained in our class of mappings (3.1.5) and hence it guarantees the fixed point of map $f$. In fact $f$ has two fixed points, namely, $-2$ and $1$. 
3.2. Mishra [2] proved the following non-unique fixed point theorem for Cirić type mapping.

**Theorem B.** Let \((X,d)\) be orbitally complete metric space and \(T\) be an orbitally continuous self-mapping of \(X\) satisfying

\[
(3.2.1) \quad \min \{d(Tx, Ty), d(x, Tx), d(y, Ty), d(Tx, T^2 x), d(y, T^2 y)\} \\
- \min \{d(x, Ty), d(y, Tx), d(x, T^2 x), d(Ty, T^2 y)\} \\
\leq p d(x, y),
\]

where \(x, y \in X\) and \(p \in (0, 1)\). Then \(T\) has a fixed point in \(X\).

Our objective here is to improve upon Theorem B by obtaining another fixed point theorem involving four points of orbitally complete metric space.

**Theorem 2.** Let \((X,d)\) be orbitally complete metric space and \(T\) be an orbitally continuous self-mapping of \(X\) satisfying

\[
(3.2.2) \quad \min \left\{ d(Tu_1, Tu_2), d(u_1, Tu_3), d(u_2, Tu_4), d(Tu_1, T^2 u_3), \\
+ a \min \left\{ d(u_1, Tu_4), d(u_2, Tu_3), d(u_1, T^2 u_3), d(Tu_4, T^2 u_3) \right\} \right. \\
\leq p d(u_1, u_2) + q d(u_1, Tu_3)
\]

for all \(u_1, u_2, u_3, u_4 \in X\), where \(a, p\) and \(q\) are real numbers such that \(0 < p + q < 1\). Then \(T\) has a fixed point in \(X\).
\textbf{Proof.} Let }x, y \in X \text{ and define } u_1 = Ty, u_2 = Tx, u_3 = x, u_4 = y, \text{ then applying (3.2.2), we have}

\[
\min \left\{ d(T^2y, T^2x), \quad d(Ty, Tx), \quad d(Tx, Ty), \right. \\
\left. \quad d(T^2y, T^2x), \quad d(Tx, T^2x) \right\} \\
+ \min \left\{ d(Ty, Ty), \quad d(Tx, Tx), \quad d(Ty, T^2x), \quad d(Ty, T^2x) \right\} \\
\leq p \cdot d(Ty, Tx) + q \cdot d(Ty, Tx) \\
i.e.,
\]

(3.2.3) \quad \min \left\{ d(T^2y, T^2x), \quad d(Ty, Tx), \quad d(Tx, T^2x) \right\} \leq (p+q) d(Ty, Tx).

Let }x \in X \text{ be arbitrary. We define a sequence } \{x_n\} \text{ by } x_0 = x, \quad x_1 = Tx_0, \quad x_2 = Tx_1, \ldots, \quad x_n = Tx_{n-1}. \text{ If for some } n \in \mathbb{N}, \quad x_n = x_{n+1}, \text{ then } \{x_n\} \text{ is a Cauchy sequence, and the limit of } \{x_n\} \text{ is a fixed point of } T. \text{ Suppose that } x_n \neq x_{n+1} \text{ for each } n = 0, 1, 2, \ldots. \text{ Then by (3.2.3), for } x = x_{n-1}, \quad y = x_{n-2}, \text{ we have}

\[
\min \left\{ d(T^2x_{n-0}, T^2x_{n-1}), \quad d(Tx_{n-2}, Tx_{n-1}), \quad (Tx_{n-1}, T^2x_{n-1}) \right\} \\
\leq (p+q) \cdot d(Tx_{n-2}, Tx_{n-1})
\]
i.e. \quad \min \left\{ d(x_n, x_{n+1}), \quad d(x_{n-1}, x_n), \quad d(x_n, x_{n+1}) \right\} \leq h d(x_{n-1}, x_n)

\text{i.e., } d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n),

\text{where } (p+q) = h \in (0, 1), \text{ since } d(x_{n-1}, x_n) \leq h \cdot d(x_{n-1}, x_n), \quad h \in (0, 1) \text{ is impossible.}
Proceeding in this manner, we obtain

\[ d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \leq \cdots \leq h^n d(x, Tx). \]

The rest of the proof is exactly the same as that of Theorem 1.

Remark 3. In case \( a = -1 \) and \( u_1 = u_3 = x, u_2 = u_4 = y \) in (3.2.2), then we obtain

Corollary 2. Let \( (X, d) \) be orbitally complete metric space and \( T \) be an orbitally continuous self-mapping of \( X \) satisfying

\[
\begin{align*}
\min & \{d(Tx, Ty), d(x, Tx), d(y, Ty), d(Tx, T^2x), d(y, T^2x)\} \\
-\min & \{d(x, Ty), d(y, Tx), d(x, T^2x), d(Ty, T^2x)\} \\
\leq & \ p d(x, y) + q d(x, Tx)
\end{align*}
\]

for all \( x, y \in X \) and \( p \) and \( q \) are real numbers satisfying \( 0 < p + q < 1 \). Then \( T \) has a fixed point in \( X \).

Remark 4. On taking \( q = 0 \), inequality (3.2.4) reduces to (3.2.1) considered by Mishra [2].

We now give an example showing the advantage of our condition over (3.2.1).

Example 2. Let \( f : \mathbb{R} \to \mathbb{R} \) be a mapping defined by \( f(x) = x^2 + x + 1 \) for all \( x \in \mathbb{R} \), where the metric \( d \) is defined in the usual sense, i.e., \( d(x, y) = |x - y| \). Then for \( x = 2 \) and \( y = 3 \) condition (3.2.1) yields
\[
\min \left\{ d(f(2), f(3)), d(2, f(2)), d(3, f(3)), d(f(2), f^2(2)), d(3, f^2(2)) \right\} \\
- \min \left\{ d(2, f(3)), d(3, f(2)), d(2, f^2(2)), d(f(3), f^2(2)) \right\} \\
\leq pd(2, 3)
\]

i.e., \( \min \{d(3, 11), d(2, 5), d(3, 11), d(5, 29), d(3, 29)\} \)

\[-\min \{d(2, 11), d(3, 5), d(2, 29), d(11, 29)\} \leq pd(2, 3)\]

i.e., \( 1 \leq p \)

which is a contradiction since \( p < 1 \) and hence condition (3.2.1) fails to prove the existence of the fixed point of the map \( f \). But this mapping \( f \) is contained in our class of mapping (3.2.4) and hence it guarantees the fixed point of the map \( f \). In fact \( f \) has two fixed points, namely \( 1 \) and \( -1 \).

3.3. In 1968, Malea [1] considered a metric space \( X \) with two metrics \( d \) and \( d_1 \) and established the following:

**Theorem C.** Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \) and \( T \) be a self-map of \( X \). Let \( X \) satisfying the following conditions:

\[
(3.3.1) \quad d_1(x, y) \leq d(x, y) \quad \text{for all } x, y \in X, \\
(3.3.2) \quad X \text{ is complete with respect to } d, \\
(3.3.3) \quad d(Tx, Ty) \leq \alpha d(x, y)
\]

for every \( x, y \in X \) and \( \alpha \in (0, 1) \). Then \( T \) has unique fixed point.
By weakening the hypothesis (3.3.1) - (3.3.3) Dhage and Dhoabe [2] generalized Theorem C as follows:

**Theorem D.** Let $X$ be a metric space with two metrics $d$ and $d_1$ and $T$ be a self-map of $X$. Let $X$ satisfies the following conditions:

(3.3.4) $d_1(x,y) \leq a(d(x,Tx) + d(y,Tx))$, $a \geq 1$ for every $x, y \in X$,

(3.3.5) $X$ is $T$-orbitally complete with respect to $d_1$, 

(3.3.6) $T$ is $T$-orbitally continuous with respect to $d_1$, 

(3.3.7) there exists a real number $b$ such that

$$\min \{ d(Tx,Ty), d(x,Tx), d(y,Tx) \}$$

$$+ b \min \{ d(x,Ty), d(y,Tx) \} \leq p d(x,Tx) + q d(x,y)$$

for every $x,y \in X$ and where $p$ and $q$ are non-negative constants such that $p+q = h \in (0,1)$. Then $T$ has a fixed point.

Inspired by the above results, we now generalize Theorem A and Theorem B of previous section in another direction.

**Theorem 3.** Let $X$ be a metric space with two metrics $d$ and $d_1$ and $T$ be a self-map of $X$. Let $X$ satisfies the conditions (3.3.4)-(3.3.6) and

(3.3.8) there exists a real number $b$ such that

$$\min \{ d(Tx,Ty), d(x,Tx), d(y,Ty), \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} \}$$
\[ + b \min \{ d(x, Ty), d(y, Tx) \} \cdot \frac{d(x, Ty)[1 + d(x, Tx) + d(y, Tx)]}{1 + d(x, y)} \]

\[ \leq pd(x, y) + qd(x, Tx) \]

for all \( x, y \in X \), where \( p \) and \( q \) are non-negative constants such that \( p + q = h \in (0, 1) \). Then \( T \) has a fixed point.

**Proof.** Let \( x = x_0 \in X \) be arbitrary and define a sequence \( \{ x_n \} \) by \( x_{n+1} = Tx_n \), \( n = 0, 1, \ldots \). If \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \), then assertion follows immediately. Therefore we assume that \( x_n \neq x_{n+1} \) for each \( n \in \mathbb{N} \). Now for \( x = x_{n-1} \) and \( y = x_n \) by condition (3.3.8), we obtain

\[ \min \{ d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n) \} \]

\[ \frac{d(x_n, Tx_n)[1 + d(x_{n-1}, Tx_{n-1})]}{1 + d(x_{n-1}, x_n)} \]

\[ \leq pd(x_{n-1}, x_n) + qd(x_{n-1}, Tx_{n-1}) \]

i.e.,

\[ \min \{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}) \} \cdot \frac{d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} \]
\[ + b \min \left\{ d(x_{n-1}, x_{n+1}), d(x_n, x_n) \right\} \frac{d(x_{n-1}, x_{n+1}) \left[ 1 + d(x_{n-1}, x_n) + d(x_n, x_n) \right]}{1 + d(x_{n-1}, x_n)} \]

\[ \leq pd(x_{n-1}, x_n) + q d(x_{n-1}, x_n) \]

or, \[ \min \{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} \leq (p+q)d(x_{n-1}, x_n) \]

i.e., \[ d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \]

where \( p + q = h \in (0, 1) \), since \( d(x_{n-1}, x_n) \leq hd(x_{n-1}, x_n) \).

\( h \in (0, 1) \) is impossible.

Proceeding in this manner we obtain

\[ d(x_n, x_{n+1}) \leq hd(x_{n-1}, x_n) \leq h^2d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^nd(x_0, x_1). \]

We show that \( \{ T^n x \} \) is a Cauchy sequence with respect to \( d_1 \). We have

\[ d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + \cdots + d(x_{n+p-1}, x_{n+p}) \]

\[ \leq h^n d(x_0, x_1) + \cdots + h^{n+p} d(x_0, x_1) \]

\[ \leq \frac{h^n}{1-h} d(x_0, x_1). \]

Since \( d_1(x, y) \leq a (d(x, Tx) + d(y, Tx)) \), we get

\[ d_1(x_n, x_{n+p}) \leq a(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p})) \]
\[ d_1(x_n, x_{n+p}) \leq a \left( n^2 d(x_0, x_1) + \frac{n^2}{1-h} \cdot d(x_0, x_1) \right) \]
\[ = a h^n \left( 1 + \frac{h}{1-h} \right) d(x_0, x_1) \]
\[ = \frac{ah^n}{1-h} \cdot d(x_0, x_1) \xrightarrow{\ n \to \infty \ } 0 \]

This shows that \( \{T^n x\} \) is a Cauchy sequence with respect to the metric \( d_1 \). The metric space \( X \) being \( T \)-orbitally complete with respect to \( d_1 \), there is some \( u \in X \) such that \( \lim_{n \to \infty} x_n = u \).

By \( T \)-orbital continuity of \( T \), we get \( u = \lim_{n \to \infty} x_{n+1} = \lim_{n \to \infty} T x_n \)
\[ = T \lim_{n \to \infty} x_n = Tu \]. Thus \( T \) has a fixed point and this completes the proof of the theorem.

**Remark 5.** If we take \( b = -1 \) and \( q = 0 \) in (3.3.8) of Theorem 3, then we get Theorem A due to Pachpatte [5] as a corollary with hypothesis (3.3.4) replaced by hypothesis (3.3.1).

**Theorem 4.** Let \( X \) be a metric space with two metrics \( d \) and \( d_1 \) and \( T \) be a self-map of \( X \). Let \( X \) satisfies the conditions (3.3.4) - (3.3.6) and

\[ \text{(3.3.9) } \quad \text{there exists a real number } b \text{ such that} \]
\[ \min \{d(Tx, Ty), d(x, Tx), d(y, Ty), d(Tx, T^2x), d(y, T^2x)\} + b \min \{d(x, Ty), d(y, Tx), d(x, T^2x), d(Ty, T^2x)\} \]
\[ \leq pd(x, y) + qd(x, Tx) \]
for all \( x, y \in X \), where \( p \) and \( q \) are non-negative constants such that \( p + q = h \) and \( h \in (0, 1) \). Then \( T \) has a fixed point in \( X \).

**Proof.** Let \( x = x_0 \in X \) be arbitrary and define a sequence \( \{x_n\} \) by \( x_{n+1} = Tx_n \), \( n = 0, 1, \ldots \). If \( x_n = x_{n+1} \) for some \( n \in \mathbb{N} \), then assertion follows immediately. Therefore we assume that \( x_n \neq x_{n+1} \) for each \( n \in \mathbb{N} \). Now for \( x = x_{n-1} \) and \( y = x_n \), by condition (3.3.9), we have

\[
\min \left\{ d(Tx_{n-1}, Tx_n), d(x_{n-1}, Tx_{n-1}), d(x_n, Tx_n) \right\} + \frac{1}{p} \min \left\{ d(x_{n-1}, Tx_{n-1}), d(x_{n-1}, Tx_n), d(x_n, Tx_{n-1}), d(Tx_n, Tx_{n-1}) \right\} \\
\leq p d(x_{n-1}, x_n) + q d(x_{n-1}, Tx_{n-1})
\]

i.e.,

\[
\min \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_n, x_{n+1}) \right\} + \frac{1}{p} \min \left\{ d(x_{n-1}, x_{n+1}), d(x_n, x_n), d(x_{n-1}, x_{n+1}), d(x_{n+1}, x_{n+1}) \right\} \\
\leq p d(x_{n-1}, x_n) + q d(x_{n-1}, x_n)
\]

or,

\[
\min \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n) \right\} \leq (p+q) d(x_{n-1}, x_n)
\]

i.e.,

\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n)
\]
where \( p + q = h \in (0, 1) \), since \( d(x_{n-1}, x_n) \leq h d(x_{n-1}, x_n) \),
\( h \in (0, 1) \) is impossible.

Proceeding in this manner, we obtain

\[
d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \leq h^2 d(x_{n-2}, x_{n-1}) \leq \cdots \leq h^n d(x_0, x_1)
\]

The remaining proof of this theorem is exactly the same as that of Theorem 3.

**Remark 6.** If we choose \( b = -1 \) and \( q = e \) in (3.3.9) of Theorem 4, then we get Theorem B due to Mishra [2] as a corollary with hypothesis (3.3.4) replaced by hypothesis (3.3.1).

3.4. In this section we discuss sequence of orbitally continuous mappings on a compact metric space which satisfy (3.1.3) with \( u_1 = u_3 = x \) and \( u_2 = u_4 = y \). Our result is motivated by a result of Achari [4].

**Theorem 5.** Let \((X, d)\) be a compact metric space, \( \{T_n\} \) be a sequence of orbitally continuous functions of \(X\) into itself satisfying

\[
(3.4.1) \quad \min\{d(T_n x, T_{n+1} y), d(x, T_n x), d(y, T_{n+1} y) \} \leq \frac{d(y, T_{n+1} y) [1 + d(x, T_n x)]}{1 + d(x, y)}
\]

\[
+ \min\{d(x, T_{n+1} y), d(y, T_n x) \frac{d(x, T_{n+1} y) [1 + d(x, T_n x) + d(y, T_n x)]}{1 + d(x, y)} \}
\]
\[ p \cdot d(x, y) + q \cdot d(x, T_n x) \]

for all \( x, y \in X \) and each \( n = 1, 2, \ldots \), where \( a, p \) and \( q \) are real numbers satisfying \( 0 < p + q < 1 \). If \( \{ T_n \} \) converges pointwise to the function \( T \), then \( T \) has a fixed point. Indeed, every cluster point of sequence \( \{ x_n \} \) of fixed points \( x_n \) of \( T_n \) is a fixed point of \( T \).

**Proof.** Let \( x_0 \in X \) be arbitrary and construct the sequence \( x_0 = x_0 \),
\[ x_1 = T_1 x_0, x_2 = T_2 x_1, \ldots, x_n = T_n x_{n-1}, \ldots \]
By (3.4.1), for \( x = x_{n-1} \) and \( y = x_n \), we have

\[
\min \left\{ d(T_n x_{n-1}, T_{n+1} x_n), d(x_{n-1}, T_n x_{n-1}), d(x_n, T_{n+1} x_n), \frac{d(x_n, T_{n+1} x_n) [1 + d(x_{n-1}, T_n x_{n-1})]}{1 + d(x_{n-1}, x_n)} \right\}
\]

\[
+ a \min \left\{ d(x_{n-1}, T_{n+1} x_n), d(x_n, T_{n+1} x_n), \frac{d(x_{n-1}, T_{n+1} x_n) [1 + d(x_{n-1}, T_n x_{n-1}) + d(x_n, T_n x_{n-1})]}{1 + d(x_{n-1}, x_n)} \right\}
\]

\[
\leq p \cdot d(x_{n-1}, x_n) + q \cdot d(x_{n-1}, T_n x_{n-1})
\]

i.e.,

\[
\min \left\{ d(x_n, x_{n+1}), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) [1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} \right\}
\]
\[ + a \min \{ d(x_{n-1}, x_{n+1}) \cdot d(x_n, x_n) \cdot \frac{d(x_{n-1}, x_{n+1})[1 + d(x_{n-1}, x_n) + d(x_n, x_n)]}{1 + d(x_{n-1}, x_n)} \} \]

\[ \leq pd(x_{n-1}, x_n) + qd(x_{n-1}, x_n) \]

or \( \min \{ d(x_n, x_{n+1}), d(x_{n-1}, x_n) \} \leq (p+q) d(x_{n-1}, x_n) \)

i.e., \( d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \)

where \( p+q = h \in (0, 1) \), since \( d(x_{n-1}, x_n) \leq h d(x_{n-1}, x_n) \), \( h \in (0, 1) \) is impossible.

Proceeding in this way, one obtains

\[ d(x_n, x_{n+1}) \leq h d(x_{n-1}, x_n) \leq \ldots \leq h^n d(x_0, x_1) \]

By routine calculation one can easily show that the following inequalities hold:

\[ d(x_1, x_j) \leq \sum_{i=1}^{j-1} d(x_i, x_{i+1}) \leq h^j \frac{d(x_0, x_1)}{1-h}, \quad j > 1. \]

Thus the sequence \( \{ x_n \} \) is Cauchy. Since \( X \) is complete and \( \{ T_n \} \) are orbitally continuous, \( T_n \) has a fixed point for infinitely many of \( n \)'s. So there is a subsequence \( \{ T_n(x) \} \) of \( \{ T_n \} \) such that each \( \{ T_n(x) \} \) has a fixed point, say \( x_1 \). By compactness we may (by taking a subsequence) assume that \( \{ x_1 \} \) converges to some \( x \) in \( X \). We shall show that \( x \) is a fixed point of \( T \). If \( x = f(x) \) for only finitely many of \( r \)'s, then
\[ T^x = \lim_{r \to x} = T^n_r(x) \]
\[ = \lim_{r \to x} T^n_r(x) \cdot x_r \]
\[ = \lim_{r \to x} x_r \]
\[ = x. \]

Therefore we may assume that \( x_r \neq x \) for infinitely many of \( x \)'s. This completes the proof.

Finally, we prove a result similar to Theorem 5, but for different kind of mappings.

**Theorem 6.** Let \((X,d)\) be a compact metric space, \(\{T_n\}\) be a sequence of orbitally continuous functions of \(X\) into itself satisfying

\[
(3.4.2) \quad \min \left[ \left\{ d(T_n x, T_{n+1} y) \right\}^2 + d(x, y) \cdot d(T_n x, T_{n+1} y) \cdot d(y, T_{n+1} y) \right] \leq a \cdot \min \left[ d(x, T_n x) \cdot d(y, T_{n+1} y), \quad d(x, T_{n+1} y) \cdot d(y, T_n x) \right] \\
\leq p \cdot d(x, T_n x) \cdot d(y, T_{n+1} y) + q \cdot d(x, y) \cdot d(T_n x, T_{n+1} y) \]

for all \( x, y \in X \) and each \( n = 1, 2, \ldots \), where \( a, p, \) and \( q \)
are real numbers satisfying $0 < p + q < 1$. If $\{T_n\}$ converges pointwise to the function $T$, then $T$ has a fixed point. Indeed, every cluster point of sequence $\{x_n\}$ of fixed points $x_n$ of $T_n$ is a fixed point of $T$.

**Proof.** Let $x_0 \in X$ be arbitrary and construct the sequence $x_0 = x_1 = T_1 x_0, x_2 = T_2 x_1, \ldots, x_n = T_n x_{n-1}, \ldots$.

By (3.4.2), for $x = x_{n-1}$ and $y = x_n$, we have

$$\min \left\{ \begin{array}{c} \{d(T_n x_{n-1}, T_{n+1} x_n) \}^2, \ d(x_{n-1}, x_n), \ d(T_n x_{n-1}, T_{n+1} x_n), \\ \{d(x_n, T_{n+1} x_n) \}^2 \end{array} \right\}$$

$$+ a \ min \left\{ d(x_{n-1}, T_n x_{n-1}), \ d(x_n, T_{n+1} x_n), \ d(x_n, T_{n+1} x_n) \right\}$$

$$\leq p d(x_{n-1}, T_n x_{n-1}) d(x_n, T_{n+1} x_n) + q d(x_{n-1}, x_n) d(T_n x_{n-1}, T_{n+1} x_n)$$

i.e.,

$$\min \left\{ \{d(x_n, x_{n+1}) \}^2, \ d(x_{n-1}, x_n), \ d(x_n, x_{n+1}), \ d(x_n, x_{n+1}) \right\}$$

$$+ a \ min \left\{ d(x_{n-1}, x_n), \ d(x_n, x_{n+1}), d(x_{n-1}, x_{n+1}), d(x_n, x_n) \right\}$$

$$\leq p d(x_{n-1}, x_n) d(x_n, x_{n+1}) + q d(x_{n-1}, x_n) d(x_n, x_{n+1})$$

or,

$$\min \left\{ \{d(x_n, x_{n+1}) \}^2, \ d(x_{n-1}, x_n), \ d(x_n, x_{n+1}) \right\} \leq (p+q) d(x_{n-1}, x_n) d(x_n, x_{n+1})$$
i.e., \[ d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n), \]

where \( p + q = h \in (0,1) \), since \( d(x_{n-1}, x_n) \cdot d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n) \cdot d(x_n, x_{n+1}), h \in (0,1) \) is impossible.

Proceeding in this way one obtains

\[ d(x_n, x_{n+1}) \leq h \cdot d(x_{n-1}, x_n) \leq \ldots \leq h^n \cdot d(x_0, x_1). \]

The rest of the proof is exactly the same as that of Theorem 5.