CHAPTER 3

EXISTENCE OF $\psi$-BOUNDED SOLUTIONS FOR
MATRIX DYNAMICAL SYSTEMS ON TIME SCALES

Section 3.1

Sylvester matrix and Lyapunov matrix differential equations arise in a number of areas of applied mathematics such as control systems, dynamic programming, optimal filters, quantum mechanics and systems engineering. The analytical, numerical solutions and control aspects of Sylvester matrix differential equations was studied by Fausett [30]. The aim of this chapter is to give necessary and sufficient conditions that the non-homogenous Sylvester matrix dynamical system

$$X^\Delta(t) = A(t)X(t) + X(t)B(t) + \mu(t)A(t)X(t)B(t) + F(t) \quad (3.1.1)$$

has at least one $\psi$-bounded solution for every Lebesgue $\psi$-deltaintegrable function $F$, on the time scale $T^+$. Here $\psi$ is a rd-continuous matrix function, instead of a scalar function. The calculus of time scales was initiated by Stefan Hilger(1988)[35] in order to create a theory that can unify discrete and continuous analysis. The study of dynamic equations on time scales, is an area of mathematics that has recently received a lot of attention and sheds new light on the discrepancies between continuous differential equations and discrete difference equations. It also prevents one from proving a result twice, once for differential equations and once for difference equations. The general idea, which is the main goal of Bohner and Peterson's excellent introductory text [11], is to
prove a result for a dynamic equation where the domain of the unknown function is so-called time scale. If \( T = R \), the general result obtained yields the same result concerning an ordinary differential equation. If \( T = Z \), the general result is the same result one would obtain concerning a difference equation. However, since there are infinitely many other time scales that one may work besides the real and the integers, one has a much more general result.

The present work unify the results of existence of \( \psi \)-bounded solutions of linear differential equations [23] and linear difference equations [[34], [58]] and also generalizes to Sylvester matrix dynamical systems on time scales. In section 3.2 we review the most of the results and definitions on timescales. In section 3.3 first, we present a necessary and sufficient condition for the existence of at least one \( \psi \)-bounded solution for linear non-homogenous vector dynamic equation on time scales
\[
\Delta(t) = A(t)x + f(t)
\]  
(3.1.2)
for every Lebesgue \( \psi \)-deltaintegrable function \( f \), on time scale \( T^+ \). Further, we obtain a result relating to the asymptotic behavior of solutions of (3.1.2). Furthermore, in section 3.4 we extend these results to Sylvester matrix differential equations (3.1.1), using the technique of Kronecker product of matrices on \( T^+ \).

**Section 3.2**

The purpose of this section is to review some useful results, definitions and basic properties on time scales which are needful for later discussion.

Let \( T \) be a time scale, i.e., an arbitrary non-empty closed subset of real numbers. Throughout this chapter, the time scale \( T \) is assumed to be unbounded above and below.
In this chapter, to facilitate the discussion below, we introduce some notation:

\[ T^* = [0, \infty) \cap T, \nu = \min \{0, \infty) \cap T\} \]

For \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) (* denotes transpose), let \( \|x\| = \max \{|x_1|, |x_2|, \ldots, |x_n|\} \) be the norm of \( x \). For a \( n \times n \) real matrix \( A = \{a_{ij}\} \), we define the norm \( |A| = \sup_{|A| \leq 1} \|Ax\| \).

**Theorem 3.2.1** If \( A \) is differentiable at \( t \in T^k \), then \( A(t) = A(t) + \mu(t)A^T(t) \)

**Lemma 3.2.1** The vectorization \( \text{Vec} : \mathcal{R}(T, \mathbb{R}^{n \times n}) \to \mathcal{R}(T, \mathbb{R}^n) \) is a linear and one-to-one operator. In addition, \( \text{Vec} \) and \( \text{Vec}^{-1} \) are continuous operators.

**Proof.** The fact the vectorization operator is linear and one-to-one immediate. Now, for \( A = \{a_{ij}\} \in \mathcal{R}(T, \mathbb{R}^{n \times n}) \), we have

\[
\|\text{Vec}(A)\| = \max_{1 \leq i, j \leq n} \{ |a_{ij}| \} \leq \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\} = |A|.
\]

Thus, the vectorization operator is continuous and \( \|\text{Vec}\| \leq 1 \).

In addition, for \( A = I_n \), we have \( \|\text{Vec}(I_n)\| = 1 = |I_n| \) and then

\[
\|\text{Vec}\| = 1.
\]

Obviously, the inverse of the vectorization operator, \( \text{Vec}^{-1} : \mathcal{R}(T, \mathbb{R}^n) \to \mathcal{R}(T, \mathbb{R}^{n \times n}) \), is defined by
\[
Vec^{-1} = \begin{bmatrix}
u_1 & u_{n+1} & \cdots & u_{n^2-n+1} \\
u_2 & u_{n+2} & \cdots & u_{n^2-n+2} \\
\vdots & \vdots & \ddots & \vdots \\
u_n & u_{2n} & \cdots & u_{n^2}
\end{bmatrix}
\]

where \( u = (u_1, u_2, u_3, \cdots, u_{n^2})^* \in \mathcal{R}(T, R^{n^2}) \). We have

\[
\left| Vec^{-1}(u) \right| = \max_{1 \leq i \leq n} \left\{ \sum_{j=0}^{n-1} |u_{nj+i}| \right\} \leq n \max_{1 \leq i \leq d} \|u_i\| = n\|u\|.
\]

Thus \( Vec^{-1} \) is a continuous operator. Also, if we take \( u = VecA \) in the above inequality, then the following inequality holds

\[
|A| \leq n\|VecA\| \text{ for every } A \in \mathcal{R}(T, R^{n \times n}).
\]

Now by applying the Vec operator to the matrix Sylvester dynamical system (3.1.1) and using Kronecker product properties, we have

\[
W^\Delta(t) = G(t)W(t) + \hat{F}(t), \quad W(t_0) = W_0. \tag{3.2.1}
\]

The corresponding homogeneous equation is

\[
W^\Delta(t) = G(t)W(t), \quad W(t_0) = W_0. \tag{3.2.2}
\]

where \( W(t) = Vec X(t), \hat{U}(t) = Vec U(t), \hat{F}(t) = Vec F(t) \) and

\[
G(t) = [B^* \otimes I_n + I_n \otimes A + \mu(t)(B^* \otimes A)], \text{ is an } n^3 \times n^2 \text{ rd-continuous, regressive matrix.}
\]

**Lemma 3.2.2** The matrix valued function \( X(t) \) is a solution (3.1.1) on \( T^+ \) if and only if the vector valued function \( \hat{X}(t) = Vec(X(t)) \) is a solution of the equation (3.2.1) on \( T^+ \).
Lemma 3.2.3 Let $Y(t)$ and $Z(t)$ be the fundamental matrices for the matrix dynamic equations

$$X^\Lambda(t) = A(t)X(t), \ t \in T^+ \tag{3.2.3}$$

and

$$X^\Lambda(t) = B^*(t)X(t), \ t \in T^+ \tag{3.2.4}$$

respectively. Then matrix $W(t) = (Z(t) \otimes Y(t))$ is a fundamental matrix of (3.2.2).

If, in addition, $Y(v) = I_n, Z(v) = I_n$, then $W(v) = I_{n^2}$.

**Proof.** From Remark 2.5.1 and Theorem 3.2.1, we have

$$(Z(t) \otimes Y(t))^\Lambda = Z^\Lambda (t) \otimes Y(t) + Z(\sigma(t)) \otimes Y^\Lambda (t)$$

$$= B^* (t)Z(t) \otimes Y(t) + Z(\sigma(t)) \otimes A(t)Y(t)$$

$$= (B^* \otimes I_n)(Z(t) \otimes Y(t)) + (Z(t) + \mu(t)B^*(t)Z(t)) \otimes A(t)Y(t)$$

$$= (B^* \otimes I_n)(Z(t) \otimes Y(t)) + [(I_n \otimes A(t)) + \mu(t)(B^* (t) \otimes A(t))](Z(t) \otimes Y(t))$$

$$= [(B^* \otimes I_n) + (I_n \otimes A(t)) + \mu(t)(B^* (t) \otimes A(t))](Z(t) \otimes Y(t))$$

$$= G(t)(Z(t) \otimes Y(t)), \text{ for all } t \in T^+.$$ 

On the other hand, the matrix $(Z(t) \otimes Y(t))$ is invertible matrix for all $t \in T^+$

(because $Z(t)$ and $Y(t)$ are invertible matrices for all $t \in T^+$). Thus $(Z(t) \otimes Y(t))$ is the fundamental matrix of (3.2.2). Clearly, $W(v) = (Z(v) \otimes Y(v)) = I_n \otimes I_n = I_{n^2}$.

This completes the proof.

Let $A$ be a regressive and rd-continuous $n \times n$ matrix on $T^+$ and the associated linear dynamic equation of (3.1.2) is
\[ x^\Delta = A(t)x. \]  

(3.2.5)

Also let \( Y \) be the fundamental matrix of (3.2.5) for which \( Y(v) = I_n \). Let \( X_1 \) denote the subspace of \( \mathbb{R}^{n \times n} \) consisting of all matrices which are values of \( \psi \)-bounded solution of (3.2.5) on \( T^+ \) at \( t = v \) and let \( X_2 \) be an arbitrary fixed subspace of \( \mathbb{R}^{n \times n} \), supplementary to \( X_1 \), let \( P_1, P_2 \) denote the corresponding projections of \( \mathbb{R}^{n \times n} \) on to \( X_1, X_2 \) respectively.

Then \( \overline{X}_1 \) denote the subspace of \( \mathbb{R}^{n^2} \) consisting of all vectors which are values of \( I_n \otimes \psi \)-bounded solution of (3.2.2) on \( Z^+ \) at \( n = 1 \) and \( \overline{X}_2 \) a fixed subspace of \( \mathbb{R}^{n^2} \), supplementary to \( \overline{X}_1 \). Let \( Q_1, Q_2 \) denote the corresponding projections of \( \mathbb{R}^{n^2} \), on to \( \overline{X}_1, \overline{X}_2 \) respectively.

**Theorem 3.2.2** Let \( A(t), B(t) \in \mathcal{R}(T^+, \mathbb{R}^{n \times n}) \). and \( F(t) \) be rd-continuous matrix function on \( T^+ \). If \( Y(t) \) and \( Z(t) \) be the fundamental matrices for the matrix dynamic equations (3.2.3) and (3.2.4), then

\[
\hat{X}(t) = \int_{v}^{t} \left( (Z(t) \otimes Y(t))Q_1(Z^{-1}(\sigma(s) \otimes Y^{-1}(\sigma(s))) \right) \hat{F}(s) \Delta s \\
- \int_{v}^{t} \left( (Z(t) \otimes Y(t))Q_2(Z^{-1}(\sigma(s) \otimes Y^{-1}(\sigma(s))) \right) \hat{F}(s) \Delta s
\]

(3.2.6)

is a solution of (3.2.1) on \( T^+ \).

**Proof.** It is easily seen that \( \hat{X}(t) \) is the solution of (3.2.1) on \( T^+ \).
Let $\psi_i : T^+ \to (0, \infty)$, i=1, 2,..,n, be regressive and rd- continuous functions, and define
\[ \psi = \text{diag}[\psi_1, \psi_2, \ldots, \psi_n]. \]

Then $\psi$ - is regressive and rd- continuous and invertible on $T^+$

**Definition 3.2.1** A function $\phi : T^+ \to R^n$ said to be $\psi$ - bounded on $T^+$ if $\psi(t)\phi(t)$ is bounded on $T^+$ (i.e., there exists $L_1 > 0$ such that $\|\psi(t)\phi(t)\| \leq L_1$, for all $t \in T^+$)

Extend this definition for matrix functions.

**Definition 3.2.2** A matrix function $F : T^+ \to R^{n \times n}$ is said to be $\psi$ - bounded on $T^+$ if the matrix function $\psi F$ is bounded on $T^+$ (i.e., there exists $L_2 > 0$ such that $|\psi(t)F(t)| \leq L_2$, for all $n \in T^+$)

**Definition 3.2.3** A function $f : T^+ \to R^n$ is said to be Lebesgue $\psi$ – integrable on $T^+$ if $f$ is $\Delta$-measurable and $f$ is Lebesgue $\Delta$(delta) integrable on $T^+$

(i.e., $\int_{T^+} \|\psi(s)f(s)\|\Delta s < \infty$).

Extend this definition for matrix functions.

**Definition 3.2.4** A matrix function $F : T^+ \to R^{n \times n}$ is said to be Lebesgue $\psi$ – delta integrable on $T^+$ if $F$ is $\Delta$-measurable and $F$ is Lebesgue delta integrable on $T^+$

(i.e., $\int_{T^+} |\psi(s)F(s)|\Delta s < \infty$).

The following Lemmas play a vital role in the proof of main result.
Lemma 3.2.4 The matrix function $F : T^+ \to \mathbb{R}^{n \times n}$ is $\psi$-deltaintegrable on the time scale $T^+$ if and only if the vector function $\text{Vec}F(t)$ is $I_n \otimes \psi$-deltaintegrable on $T^+$.

**Proof.** From the proof of Lemma 3.2.2, it follows that

$$\frac{1}{n} |A| \leq \|\text{Vec}A\|_\mathcal{R}(T^+, \mathbb{R}^{n^2}) \leq A \quad \text{for every} \quad A \in \mathcal{R}(T^+, \mathbb{R}^{n \times n}).$$

Put $A(t) = \psi(t)F(t)$ in the above inequality, we have

$$\frac{1}{n} |\psi(t)F(t)| \leq \|(I_n \otimes \psi(t))\text{Vec}F(t)\|_\mathcal{R}(T^+, \mathbb{R}^{n^2}) \leq |\psi(t)F(t)|, \quad t \in T^+ \quad (3.2.7)$$

for all matrix functions $F(t)$.

Suppose that $F(t)$ is $\psi$-bounded on $T^+$. From (3.2.7)

$$\|(I_n \otimes \psi(t))\text{Vec}F(t)\|_\mathcal{R}(T^+, \mathbb{R}^{n^2}) \leq |\psi(t)F(t)|,$$

From definition 3.2.4 and 3.2.3, $\hat{F}(t)$ is Lebesgue $I_n \otimes \psi$-deltaintegrable on $T^+$.

Conversely, suppose that $\hat{F}(t)$ is $I_n \otimes \psi$-bounded on $T^+$. Again from (3.2.7), we have

$$|\psi(t)F(t)| \leq n \|(I_n \otimes \psi(t))\text{Vec}F(t)\|_{\mathbb{R}^{n^2}}$$

hence from, definitions 3.2.3 and 3.2.4, $F(t)$ is $\psi$-bounded on $T^+$.

Lemma 3.2.5 The matrix function $F : T^+ \to \mathbb{R}^{n \times n}$ is $\psi$-bounded on the time scale $T^+$, if and only if the vector function $\text{Vec}F(t)$ is $I_n \otimes \psi$-bounded on $T^+$.

**Proof.** The proof easily follows from the inequality (3.2.7).
Section 3.3

In this section, we prove a necessary and sufficient conditions for the existence of at least one \( \psi \)-bounded solution for the vector dynamical system (3.1.2), via Lebesgue \( \psi \)-deltaintegrable function on \( T^+ \). And also obtain a result relating to asymptotic behavior of \( \psi \)-bounded solution of (3.1.2).

Theorem 3.3.1 If \( A \in \mathbb{R} \) is a \( n \times n \) real matrix, then (3.1.2) has at least one \( \psi \)-bounded solutions on \( T^+ \) for every Lebesgue \( \psi \)-deltaintegrable function \( f \) on \( T^+ \) if and only if there is a positive constant \( M \) such that

\[
\psi(t)Y(t)P_1Y^{-1}(\sigma(s))\psi^{-1}(s) \leq M, \quad \text{for } v \leq \sigma(s) \leq t, \\
\psi(t)Y(t)P_2Y^{-1}(\sigma(s))\psi^{-1}(s) \leq M, \quad \text{for } v \leq t \leq \sigma(s), \quad \forall t, s \in T^+. 
\]

Proof. First, we suppose that the non-homogeneous dynamical system (3.1.2) has at least one \( \psi \)-bounded solutions on \( T^+ \) for every Lebesgue \( \psi \)-deltaintegrable function \( f \) on \( T^+ \). We define the sets:

\( H_\psi = \{ x : T^+ \to \mathbb{R}^n : x \text{ is } \psi \text{-bounded and rd- continuous on } T^+ \} \)

\( J_\psi = \{ x : T^+ \to \mathbb{R}^n : x \text{ is lebesgue } \psi \text{-integrable on } T^+ \} \)

\( K_\psi = \{ x : T^+ \to \mathbb{R}^n : x \text{ is rd-continuous on all intervals } J \subset T^+, \psi \text{-bounded solution on } T^+ \}, \quad x(v) \in P_2, x^\Delta(t) - A(t)x(t) \in J_\psi \)

It is well know that \( H_\psi \) and \( J_\psi \) are real Banach spaces with the norms

\[
\|x\|_{H_\psi} = \sup_{t \in \mathcal{T}} \|\psi(t)x(t)\|, \\
\|x\|_{J_\psi} = \int_{\mathcal{T}} \|\psi(t)x(t)\| \Delta t
\]
respectively. Clearly the set $K_\psi$ is a real linear space with the following norm, defined by

$$\|x\|_{K_\psi} = \sup_{t \geq v} \|\psi(t)x(t)\| + \|x^\Delta(t) - A(t)x(t)\|_{H_\psi}.$$  

**Claim:** $(K_\psi, \|\cdot\|_{K_\psi})$ is a real Banach space.

Let $\{x_n\}_{n=1}^\infty$ be any sequence in $K_\psi$. Then $\{x_n\}_{n=1}^\infty$ is in $H_\psi$.

And $x^\Delta_n(t) - A(t)x_n(t) \in J_\psi$.

Since $H_\psi$ is a real Banach space, there exists a rd-continuous and $\psi$-bounded function $x: T^+ \to R^n$ such that

$$\lim_{n \to \infty} \psi(t)x_n(t) = x(t) \text{ on } T^+.$$  

Denote $\bar{x}(t) = \psi^{-1}(t)x(t) \in H_\psi$. From

$$\|x_n(t) - x(t)\| \leq \|\psi^{-1}(t)\| \|\psi(t)x_n(t) - x(t)\|,$$  

It follows that $\lim_{n \to \infty} x_n(t) = x(t)$ on $T^+$. Thus, $\bar{x}(v) \in X_2$. Since $x^\Delta_n(t) - A(t)x_n(t) \in J_\psi$, then $f_n(t) = \psi(t)(x^\Delta_n(t) - A(t)x_n(t))$, is a fundamental sequence in $L$ (the Banach space of all vector functions which are Lebesgue delta integrable on $T^+$ with the norm defined in $J_\psi$). Thus there exists a function $f$ in $L$ such that

$$\lim_{n \to \infty} \int_v^\infty \|f_n(t) - f(t)\|\,\Delta t = 0.$$  

Let $\bar{f}(t) = \psi^{-1}(t)f(t)$, then $\bar{f}(t) \in J_\psi$. For a fixed, but arbitrary, $t \geq v$, we have

$$\bar{x}(t) - \bar{x}(v) = \lim_{n \to \infty} \left( x_n(t) - x_n(v) \right)$$
It follows that \( \lim_{t \to +\infty} \| \|\Delta(t) - \bar{x}(t) \| \| = 0 \). Thus \( K_v \) is a real Banach space on \( T^+ \).

Now we define the operator \( S: K_v \to J_v \). Let \( Sx = x^\Delta - A(x_v)_0 \). Clearly \( S \) is linear and bounded, with \( |S| \leq 1 \). Let \( Sx = 0 \). Then \( x^\Delta = A(x_v)_0 \). Thus \( x^\Delta \) is a \( \psi \)-bounded solution of \( (3.2.5) \). Then \( x^\Delta \in K_v \). It follows that \( \lim_{t \to +\infty} \| x^\Delta(t) - A(x_v)_0 \| \| = 0 \). From \( (3.1.2) \) it follows that \( \lim_{t \to +\infty} \| x^\Delta(t) - x(t) \| \| = 0 \). Thus \( K_v \) is a real Banach space on \( T^+ \).

It follows that \( \lim_{t \to +\infty} \| x^\Delta(t) - \bar{x}(t) \| \| = 0 \). Thus \( K_v \) is a real Banach space on \( T^+ \).
one linear operator from a Banach space $K_{\psi}$ onto a Banach space $J_{\psi}$, then the inverse operator $S^{-1}$ is also bounded. Therefore, there exists a positive constant $M = \|S^{-1}\|^{-1}$ such that, for $f \in J_{\psi}$ and for the solution $x \in K_{\psi}$ of (3.1.2), we have

$$\|x\|_{K_{\psi}} = \sup_{t \geq v} \| \psi(t) x(t) \| + \| x^\Delta - A(t)x \|_{J_{\psi}},$$

$$\|S^{-1}f\| = \sup_{t \geq v} \| \psi(t) x(t) \| + \| F \|_{J_{\psi}},$$

$$\sup_{t \geq v} \|\psi(t)x(t)\| \leq \left(\|S^{-1}\| - 1\right) \int_{v}^{\infty} \|\psi(t) f(t)\| \Delta t \leq M \int_{v}^{\infty} \|\psi(t) f(t)\| \Delta t.$$ 

It is easily seen that the solution of (3.1.2) on $T^+$ is

$$x(t) = \int_{v}^{\infty} K(t, \sigma(s)) f(s) ds,$$

where

$$K(t, s) = \begin{cases} Y(t)P_1 Y^{-1}(s), & \text{for } v \leq s \leq t \\ -Y(t)P_2 Y^{-1}(s), & \text{for } v \leq t \leq s. \end{cases}$$

For any fixed point $s \in T^+$, we have the following three cases.

**Case (1):** If $s$ is a right-dense. Then there exist a sequence of time scale points $s_k \in T^+ (s_k > s), k \in \mathbb{N}$, such that $\lim_{k \to \infty} s_k = s$ from the right. Let $\delta_k = s_k - s$ and define $f$ as

$$f(t) = \begin{cases} \psi^{-1}(t) \xi, & \text{for } s \leq t \leq s + \delta_k \\ 0, & \text{otherwise} \end{cases}$$

where $\xi$ is any fixed constant vector, $s \geq v$. Then $f(t) \in J_{\psi}$ and $\|f\|_{J_{\psi}} = \delta_k \xi$. Therefore,
\[ \|\psi(t)x(t)\| = \left\| \int_s^{\psi(t)K(t,\sigma(\tau))\psi^{-1}(\tau)} \xi \Delta \tau \right\| \leq M\delta \|\xi\| . \]

Dividing by \( \delta_k \) and letting \( k \to \infty \), for \( t \neq s \), we have

\[ \left| \psi(t)K(t,\sigma(s))\psi^{-1}(s)\xi \right| \leq M\|\xi\| . \]

Since \( \xi \) is arbitrary, then

\[ \left| \psi(t)K(t,\sigma(s))\psi^{-1}(s) \right| \leq M . \]

**Case(2):** If \( s \) is both right-scattered and left-scattered, define \( f \) as

\[
 f(t) = \begin{cases} 
 \psi^{-1}(t)\xi, & \text{for } \rho(s) \leq t \leq s \\
 0, & \text{otherwise.}
\end{cases}
\]

Note that \( \|f\|_{L^\infty} = \mu(\rho(s))\|\xi\| \); we have

\[ \left\| \int_{\rho(s)}^{\psi(t)K(t,\sigma(\tau))\psi^{-1}(\tau)} \xi \Delta \tau \right\| \leq M\mu(\rho(s))\|\xi\| . \]

Dividing by \( \mu(\rho(s)) \) in the above inequality, we get

\[ \left| \psi(t)K(t,\sigma(s))\psi^{-1}(s)\xi \right| \leq M\|\xi\| . \]

This implies

\[ \left| \psi(t)K(t,\sigma(s))\psi^{-1}(s) \right| \leq M . \]

**Case(3):** If \( s \) is right scattered and left dense, define \( f \) as

\[
 f(t) = \begin{cases} 
 \psi^{-1}(t)\xi, & \text{for } s \leq t \leq \sigma(s) \\
 0, & \text{otherwise.}
\end{cases}
\]

Clearly \( \|f\|_{L^\infty} = \mu(s)\|\xi\| \), we have
\[ \|\psi(t)x(t)\| = \left\| \int_{s}^{t} \psi(t)K(t,\sigma(\tau))\psi^{-1}(\tau)\xi(\tau)\Delta \tau \right\| \leq M\mu(s)\|\xi\|. \]

Dividing by \( \mu(s) \) in the above inequality, we get

\[ \left\| \psi(t)K(t,\sigma(s))\psi^{-1}(s)\xi(s) \right\| \leq M\|\xi\|. \]

This implies \( \left| \psi(t)K(t,\sigma(s))\psi^{-1}(s) \right| \leq M \)

which is equivalent with (3.3.1).

Conversely, suppose that the condition (3.3.1) holds. We have to prove that
dynamical system (3.1.2) has at least one \( \psi \)-bounded solution on \( T^+ \). Consider the function

\[ x(t) = \int_{s}^{t} Y(t)P_{1}\psi^{-1}(\sigma(s))f(s)\Delta s - \int_{s}^{\infty} Y(t)P_{2}\psi^{-1}(\sigma(s))f(s)\Delta s, \]

for \( t \geq v \)

where \( f \) is Lebesgue \( \psi \)-deltaintegrable function on \( T^+ \). It is easy to see that \( x(t) \) is a \( \psi \)-bounded solution on \( T^+ \) of (3.1.2). This completes the proof.

**Remark 3.3.1** If the time scale \( T = R \), then \( T^+ = R_+ = [0,\infty) \) and Theorem 3.3.1 becomes Theorem 3.3.1 of [23]. And also if the time scale \( T = Z \), the \( T^+ = N = \{0,1,2,...\} \)
and Theorem 3.3.1 becomes Theorem 3.2.1 of [34]. Therefore, Theorem 3.3.1 unifies Theorem 3.3.1 of [23] and Theorem 3.2.1 of [34].

**Theorem 3.3.2** Suppose that

(1) the fundamental matrix \( Y(t) \) of (3.2.5) satisfies the conditions:

(a) \( \lim_{t \to \infty} \psi(t)Y(t)P_1 = 0 \),
\[(b) \quad \left| \psi(t) Y(t) P_1 Y^{-1}(\sigma(s)) \psi^{-1}(s) \right| \leq M, \quad \text{for } v \leq s \leq t, \]
\[
\left| \psi(t) Y(t) P_2 Y^{-1}(\sigma(s)) \psi^{-1}(s) \right| \leq M, \quad \text{for } v \leq t \leq s
\]

where \( M \) is a positive constant.

(2) the function \( f : T^+ \to \mathbb{R}^n \) is Lebesgue \( \psi \)-deltaintegrable on \( T^+ \). Then every \( \psi \)-bounded solution \( x(t) \) of (3.1.2) satisfies

\[
\lim_{t \to \infty} \| \psi(t)x(t) \| = 0.
\]

**Proof.** Let \( x(t) \) be a \( \psi \)-bounded solution of (3.1.2). There is a positive constant \( M \) such that

\[
\| \psi(t)x(t) \| \leq M, \quad \text{for all } t \geq v. \quad \text{We consider the function}
\]

\[
y(t) = x(t) - Y(t) P_1 x(v) - \int_{v}^{t} Y(t) P_1 Y^{-1}(\sigma(s)) f(s) \Delta s + \int_{t}^{\infty} Y(t) P_2 Y^{-1}(\sigma(s)) f(s) \Delta s, \quad \forall t \geq v.
\]

From the hypothesis, it follows that the function \( y(t) \) is a \( \psi \)-bounded solution of (3.2.5).

Then \( y(v) \in X_1 \). On the other hand, \( P_1 y(v) = 0 \). Therefore, \( y(v) = P_2 y(0) \in X_2 \). Thus, \( y(v) = 0 \) and then \( y(t) = 0 \) for \( t \geq 0 \). Thus, for \( t \geq v \) we have

\[
x(t) = Y(t) P_1 x(v) + \int_{v}^{t} Y(t) P_1 Y^{-1}(\sigma(s)) f(s) \Delta s - \int_{t}^{\infty} Y(t) P_2 Y^{-1}(\sigma(s)) f(s) \Delta s, \quad \forall t \geq v.
\]

Since \( f \) is a Lebesgue \( \psi \)-deltaintegrable on \( T^+ \), for a given \( \varepsilon > 0 \), there exists \( t_1 \geq v \) such that

\[
\int_{t_1}^{\infty} \| \psi(s) f(s) \| \Delta s < \frac{\varepsilon}{2M}, \quad \text{for } t \geq t_1.
\]

Moreover, there exists \( t_2 > t_1 \) such that, for \( t \geq t_2 \).
\[
|\psi(t)Y(t)P_1| \leq \frac{\varepsilon}{2} \left[ \|x(v)\| + \int_v^t \|Y^{-1}(\sigma(s))f(s)\| \Delta s \right]^{-1}
\]

Then, for \( t \geq t_2 \) we have
\[
\|\psi(t)x(t)\| \leq \|\psi(t)Y(t)P_1\|\|x(v)\| + \int_v^t \|\psi(t)Y(t)P_1\|\|Y^{-1}(\sigma(s))f(s)\| \Delta s
\]
\[
+ \int_{t_1}^t \|\psi(t)Y(t)P_2Y^{-1}(\sigma(s))\psi^{-1}(s)\|\|\psi(s)f(s)\| \Delta s
\]
\[
+ \int_t^\infty \|\psi(t)Y(t)P_2\|\|Y^{-1}(\sigma(s))f(s)\| \Delta s
\]
\[
\leq \|\psi(t)Y(t)P_1\|\|x(v)\| + \int_v^t \|Y^{-1}(\sigma(s))f(s)\| \Delta s
\]
\[
+ M \int_{t_1}^\infty \|\psi(s)f(s)\| \Delta s < \varepsilon.
\]

This shows that \( \lim_{t \to \infty} \|\psi(t)x(t)\| = 0 \).

**Remark 3.3.2** Theorem 3.3.2 is no longer true if we require that the function \( f \) be \( \psi \)-bounded on \( T^+ \), instead of condition (2) of the Theorem 3.3.2. Even if the function \( f \) is such that
\[
\lim_{t \to \infty} \|\psi(t)f(t)\| = 0.
\]

This illustrate with the following example.

**Example 3.3.1** Consider the linear dynamical system (3.2.5) with \( A(t) = O_2 \)

then \( Y(t) = I_2 \) is the fundamental matrix of (3.2.5).

Consider
\[
\psi(t) = \begin{bmatrix}
\frac{1}{t+1} & 0 \\
0 & t+1
\end{bmatrix}
\]
then there exist projections
\[ P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \]

satisfies (3.3.1) with \( M = 1 \).

If we take \( f(t) = \begin{bmatrix} \sqrt{t+1} \\ \frac{1}{(t+1)^2} \end{bmatrix} \),

then \( \lim_{t \to \infty} \| \psi(t) f(t) \| = 0. \)

On the other hand, the solutions of the system (3.1.2) are

\[
x(t) = \begin{cases} 
\left( \int \sqrt{(t+1) \Delta t + c_1} \right) \\
\left( \int \frac{1}{t+1} \Delta t + c_2 \right)
\end{cases}
\]

\[
= \begin{cases} 
\left( \frac{2}{3} (t+1)^{3/2} + c_1 \right) \\
\left( -\frac{1}{t+1} + c_2 \right)
\end{cases}
\]

when \( T = R \)

\[
\left( \sum_{k=1}^{t} \sqrt{k} + c_1 \right)
\]

\[
\left( \sum_{k=1}^{t} \frac{1}{k^2} + c_2 \right)
\]

when \( T = Z. \)

It follows that the solutions of the dynamical system (3.1.2) are \( \psi \)-unbounded on \( T' \).

If we consider

\[
f(t) = \begin{bmatrix} \frac{1}{t+1} \\ \frac{1}{(t+1)^3} \end{bmatrix},
\]

then we have
\[
\int_T \|\psi(t) f(t)\| \Delta t = \begin{cases}
\int_0^\infty \|\psi(t) f(t)\| dt = 1 & \text{when } T = \mathbb{R} \\
\sum_{k=0}^\infty \|\psi(t) f(t)\| = \frac{\pi^2}{6} & \text{when } T = \mathbb{Z}.
\end{cases}
\]

Therefore \( f(t) \) is \( \psi \)-delta integrable function on \( T^+ \). On the other hand, the solutions of the system (3.1.2) are

\[
x(t) = \begin{cases}
\left( \log(t+1) + c_1 \right) \\
-\frac{1}{2(t+1)^2} + c_2
\end{cases}, \quad \text{when } T = \mathbb{R}
\]

\[
\left( \sum_{k=1}^t \frac{1}{k} + c_1 \right) \\
\sum_{k=1}^t \frac{1}{t^3} + c_2
\end{cases}, \quad \text{when } T = \mathbb{Z}.
\]

It is easily seen that the above solutions are \( \psi \)-bounded on \( T^+ \) if and only if \( c_2 = 0 \). In this case, \( \lim_{t \to \infty} \|\psi(t) x(t)\| = 0 \).

Note that the asymptotic properties of the components of the solutions are not the same. This is obtained by using a matrix function \( \psi \) rather than a scalar function.

**Section 3.4**

In this section we obtain a necessary and sufficient condition for the existence of \( \psi \)-bounded solution for Sylvester first order matrix dynamical system (3.1.1), via \( \psi \)-delta integrable matrix function \( F \), on \( T^+ \). And also obtain a result relating to asymptotic behavior of \( \psi \)-bounded solution of (3.1.1).
**Theorem 3.4.1** If $A(t)$ and $B(t)$ are regressive and rd-continuous $n \times n$ real matrices on $T^+$, then (3.1.1) has at least one $\psi$-bounded solution on $T^+$ for every Lebesgue $\psi$-delta integrable matrix function $F : T^+ \rightarrow R^{n \times n}$ on $T^+$, if and only if there exists a positive constant $M$ such that

\[
\left| \begin{bmatrix} Z(t) \otimes \psi(t)Y(t) \end{bmatrix}Q_1 \begin{bmatrix} Z^{-1}(\sigma(s)) \otimes Y^{-1}(\sigma(s))\psi^{-1}(s) \end{bmatrix} \right| \leq M \quad \text{for } \nu \leq \sigma(s) \leq t
\]

\[
\left| \begin{bmatrix} Z(t) \otimes \psi(t)Y(t) \end{bmatrix}Q_2 \begin{bmatrix} Z^{-1}(\sigma(s)) \otimes Y^{-1}(\sigma(s))\psi^{-1}(s) \end{bmatrix} \right| \leq M \quad \text{for } \nu \leq t \leq \sigma(s) \quad (3.4.1)
\]

**Proof.** Suppose that the equation (3.1.1) has at least one $\psi$-bounded solution on $T^+$.

For every Lebesgue $\psi$-delta integrable matrix function $F : T^+ \rightarrow R^{n \times n}$. Let $\hat{F} : \mathbb{R} \rightarrow R^{n^2}$ be a Lebesgue $I_n \otimes \psi$-delta integrable function on $T^+$.

From Lemma 3.2.4, it follows that the matrix function $F(t) = \text{Vec}^{-1} \hat{F}(t)$ is Lebesgue $\psi$-delta integrable matrix function on $T^+$. From the hypothesis, the system (3.1.1) has at least one $\psi$-bounded solution $X(t)$ on $T^+$. From Lemma 3.2.2, it follows that the vector valued function $\hat{X}(t) = \text{Vec}X(t)$ is a $I_n \otimes \psi$-bounded solution of (3.2.1) on $T^+$.

Thus, the system (3.2.1) has at least one $I_n \otimes \psi$-bounded solution on $T^+$ for every Lebesgue $I_n \otimes \psi$-delta integrable function $\hat{F}$ on $T^+$.

From Theorem 3.3.1 and Lemma 3.2.3, there is a positive constant $M$ such that the fundamental matrix $W(t)$ of the system (3.2.2) satisfies the condition

\[
\left| \begin{bmatrix} I_n \otimes \psi(t) \end{bmatrix}W(t)Q_1W^{-1}(\sigma(s))(I_n \otimes \psi^{-1}(s)) \right| \leq M \quad \text{for } \nu \leq \sigma(s) \leq t
\]

\[
\left| \begin{bmatrix} I_n \otimes \psi(t) \end{bmatrix}W(t)Q_2W^{-1}(\sigma(s))(I_n \otimes \psi^{-1}(s)) \right| \leq M \quad \text{for } \nu \leq t \leq \sigma(s)
\]
put \( W(t) = Z(t) \otimes Y(t) \) and using Kronecker product properties, the condition (3.4.1) holds.

Conversely, suppose that (3.4.1) holds for some \( K > 0 \).

Let \( F : T^+ \rightarrow R^{n \times n} \) be a Lebesgue \( \psi \) -deltaintegrable matrix function on \( T^+ \).

From Lemma 3.2.4, it follows that the vector valued function \( \hat{F}(t) = \text{Vec} F(t) \) is a Lebesgue \( I_n \otimes \psi \) -deltaintegrable function on \( T^+ \).

From Theorem 3.3.1, it follows that the vector dynamical system (3.2.1) has at least one \( I_n \otimes \psi \) -bounded solution on \( T^+ \). Let \( u(t) \) be this solution.

From (3.2.7), it follows that the matrix function \( u(t) = \text{Vec}^{-1} v(t) \) is a \( \psi \) -bounded solution of the equation (3.1.1) on \( T^+ \) (because \( F(t) = \text{Vec}^{-1} \hat{F}(t) \)).

Thus the Sylvester matrix dynamical system (3.1.1) has at least one \( \psi \) -bounded solution on \( T^+ \) for every Lebesgue \( \psi \) -deltaintegrable matrix function \( F \) on \( T^+ \).

This completes the proof.

In the following theorem, we obtain sufficient conditions for the asymptotic behavior of \( \psi \) -bounded solution of the dynamical system (3.1.1).

**Theorem 3.4.2** Suppose that:

1. let \( Y(t) \) and \( Z(t) \) be fundamental matrices of (3.2.3),(3.2.4) respectively and satisfies the conditions:

   (a) \( \lim_{t \to \infty} (Z(t) \otimes \psi(t) Y(t)) Q_i = 0 \),

   (b) condition (3.4.1) of Theorem 3.4.1,
(2) the function $F : T^+ \to R^{n \times n}$ is Lebesgue $\psi$-deltaintegrable matrix function on $T^+$.

Then every $\psi$-bounded solution $X(t)$ of (3.1.1) satisfies

$$\lim_{t \to \infty} \|\psi(t)X(t)\| = 0.$$  

**Proof.** Let $X(t)$ be a $\psi$-bounded solution of (3.1.1). From Lemma 3.2.5, it follows that the function $\hat{X}(t) = \text{Vec} X(t)$ is a $I_n \otimes \psi$-bounded solution on $T^+$ of the vector dynamical system (3.2.1). Also from Lemma 3.2.4, the function $\hat{F}(t)$ is Lebesgue $I_n \otimes \psi$-delta integrable on $T^+$, from the Theorem 3.3.2, it follows that

$$\lim_{t \to \infty} \left\| (I_n \otimes \psi(t)) \hat{X}(t) \right\| = 0.$$  

Now, from the inequality (3.2.7), we have

$$\|\psi(t)X(t)\| \leq n \| (I_n \otimes \psi(t)) \|, \quad t \in T^+$$

and then

$$\lim_{t \to \infty} \|\psi(t)X(t)\| = 0$$

The proof is now complete.

**Remark 3.4.1** Note that Theorem 3.4.2 is no longer true if we require that the function $F$ be $\psi$-bounded on $T^+$ (more, even $\lim_{t \to \infty} \|\psi(t)F(t)\| = 0$), instead of the condition (2).

This illustrate with the following example.

**Example 3.4.1** Consider the dynamical system (3.1.1) with $A(t) = O_2$, $B(t) = O_2$.

Then $Y(t) = I_2$, $Z(t) = I_2$ are fundamental matrices of (3.2.3) and (3.2.4).
Consider \( \psi(t) = \begin{bmatrix} 1 & 0 \\ \frac{t}{t+1} & \frac{1}{t+1} \end{bmatrix} \) then there exist projections

\[
P_1 = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}
\]

satisfies the condition (3.4.1) with \( M=1 \).

First we take

\[
F(t) = \begin{bmatrix} \sqrt{t+1} & \sqrt{t+1} \\ \frac{1}{(t+1)^2} & \frac{1}{(t+1)^2} \end{bmatrix}
\]

Then \( \lim_{t \to \infty} |\psi(t)F(t)| = 0 \).

On the other hand the solutions of the dynamical system (3.1.1) are

\[
X(t) = \begin{cases} 
\begin{bmatrix} \frac{2}{3} \sqrt{(t+1)^3} + c_1 & \frac{2}{3} \sqrt{(t+1)^3} + c_2 \\ -\frac{1}{t+1} + c_3 & -\frac{1}{t+1} + c_4 \end{bmatrix} & \text{when } T = \mathbb{R} \\
\sum_{k=1}^{t} \sqrt{k} + c_1 & \sum_{k=1}^{t} \sqrt{k} + c_2 \\
\sum_{k=1}^{t} \frac{1}{k^2} + c_3 & \sum_{k=1}^{t} \frac{1}{k^2} + c_4 & \text{when } T = \mathbb{Z}.
\end{cases}
\]

It follows that the solutions of the system (3.1.1) is \( \psi \)-bounded.

Now, we take
\[ F(t) = \begin{bmatrix} \frac{1}{t+1} & \frac{1}{(t+1)^3} \\ \frac{1}{(t+1)^3} & \frac{1}{(t+1)^3} \end{bmatrix}, \] then

\[ \int_0^\infty |\psi(t)F(t)|dt = 1 \]

\[ \sum_{k=1}^\infty |\psi(t)F(t)| = \frac{\pi^2}{6}. \]

On the other hand the solution of the system

\[ X(t) = \begin{cases} \log(t+1) + c_1 & \log(t+1) + c_2 \\ -\frac{1}{2(t+1)^2} + c_3 & -\frac{1}{2(t+1)^2} + c_4 \end{cases}, \text{ when } T = \mathbb{R} \]

\[ \begin{cases} \sum_{k=1}^t \frac{1}{k} + c_1 & \sum_{k=1}^t \frac{1}{k} + c_2 \\ \sum_{k=1}^t \frac{1}{k^3} + c_3 & \sum_{k=1}^t \frac{1}{k^3} + c_4 \end{cases}, \text{ when } T = \mathbb{Z} \]

It is easily seen that the solutions are \( \psi \)-bounded on \( T^+ \) if and only if \( c_3 = c_4 = 0 \). In this case \( \lim_{t \to \infty} |\psi(t)X(t)| = 0. \)