CHAPTER 8

CONTROLLABILITY AND OBSERVABILITY OF MATRIX DYNAMICAL SYSTEMS AND CONTROLLABILITY OF MATRIX INTEGRO-DIFFERENTIAL EQUATIONS ON TIME SCALES

Section 8.1

Stefan Hilger [35] initiated the study of dynamic equations on time scales, which effectively manage both continuous and discrete time combination. The dynamical systems arise in a number of areas of control engineering, feedback systems. In this chapter we confine our attention to the study of controllability and observability of first order matrix dynamical systems

\[ X^A(t) = A(t)X(t) + X(\sigma(t))B(t) + F(t)U(t), \quad X(t_0) = X_0 \]  \hspace{1cm} (8.1.1)

\[ K(t) = L(t)X(t) \]  \hspace{1cm} (8.1.2)

where \( X(t) \) is an \( n \times n \) matrix, \( U(t) \) is an \( m \times n \) input piecewise rd-continuous matrix called control and \( K(t) \) is an \( r \times n \) output matrix. Here \( A(t), B(t), F(t) \) and \( L(t) \) are \( n \times n \), \( n \times n \) and \( n \times m \) and \( r \times n \) matrices respectively. Controllability, observability and realizability of dynamical systems of the type (8.1.1), (8.1.2) with \( B(t) = 0 \) (null matrix) were studied Jhon davis [21]. The results obtained in this chapter include some of the results of [[21],[56]] with \( B(t) = 0 \) and \( X \) is a vector in (8.1.1) and (8.1.2).
In section 8.2 we develop the solution to the initial value problem is expressed in terms of the two fundamental matrices of the systems \( X^\Delta(t) = A(t)X(t) \) and \( X^\Delta(t) = B^*(t)X(\sigma(t)) \) using the standard technique of variation of parameters.

In section 8.3 we prove necessary and sufficient conditions for the complete controllability and complete observability of matrix dynamical systems (8.1.1) with (8.1.2) under smoothness conditions.

Finally, in section 8.4 we develop the controllability criteria for integro-differential systems on time scales.

**Section 8.2**

Here we establish the general solutions of the initial value problem is expressed in terms of the two fundamental matrices of the systems \( X^\Delta(t) = A(t)X(t) \) and \( X^\Delta(t) = B^*(t)X(\sigma(t)) \). Throughout this chapter \( Y(t) \) stands for a fundamental matrix solution of \( X^\Delta(t) = A(t)X(t) \) and \( Z(t) \) stands for fundamental matrix solution of \( X^\Delta(t) = B^*(t)X(\sigma(t)) \).

**Theorem 8.2.1** Any solution of the homogeneous system

\[
X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t)
\]  

is of the form \( X(t) = Y(t)CZ^*(t) \) where \( C \) is a constant matrix of order \( n \)

**Proof.** Consider

\[
[Y(t)CZ^*(t)]^\Delta(t) = [Y^\Delta(t)CZ^*(t) + Y(\sigma(t))CZ^*\Delta(t)]
\]

\[
= A(t)Y(t)CZ^*(t) + Y(\sigma(t))CZ^*(\sigma(t))B(t)
\]
Hence $X(t) = Y(t)CZ^*(t)$ is a solution of (8.2.1) every solution is of the form. Let

$X(t)$ be a solution and $P(t)$ be a square matrix defined by $P(t) = Y^{-1}(t)X(t)$ implies

$X(t) = Y(t)P(t)$

Then (8.2.1) becomes

$Y^\Delta(t)P(t) + Y(\sigma(t))P^\Delta(t) = A(t)Y(t)P(t) + Y(\sigma(t))P(\sigma(t))B(t)$

if and only if $P^\Delta(t) = P(\sigma(t))B(t)$. Hence the solution of (8.2.1) is of the form

$X(t) = Y(t)CZ^*(t)$.

**Theorem 8.2.2** Any solution of (8.1.1) is of the form

$X(t) = Y(t)CZ^*(t) + \bar{X}(t)$

where $\bar{X}(t)$ is a particular solution of (8.1.1).

**Proof.** It can easily be verified that $X(t)$ defined by (8.2.2) is a solution (8.1.1). Now we prove every solution of (8.1.1) is this form. Let $X(t)$ be any solution of (8.1.1) and $\bar{X}(t)$ be a particular solution of (8.1.1) then $X(t) - \bar{X}(t)$ is a solution of (8.2.1) hence by Theorem 8.2.1, $X(t) - \bar{X}(t) = Y(t)CZ^*(t)$.

**Theorem 8.2.3** A particular solution of (8.1.1) is given by

$$\bar{X}(t) = Y(t)\int_s^t [Y^{-1}(\sigma(s))F(s)U(s)(Z^*)^{-1}(s)]\Delta s Z^*(t)$$

**Proof.** It can be easily verified that $\bar{X}(t)$ is a solution of (8.1.1). Using variation of parameters we can write $\bar{X}(t) = Y(t)C(t)Z^*(t)$ substituting in (8.1.1) and solving for $C(t)$ we get.
\[ [Y(t)C(t)Z^*(t)]^\Delta = A(t)Y(t)C(t)Z^*(t) + Y(\sigma(t))C(\sigma(t))Z^*(\sigma(t))B(t) + F(t)U(t) \]

\[
Y^\Delta(t)C(t)Z^*(t) + Y(\sigma(t))C^\Delta(t)Z^*(t) + Y(\sigma(t))C(\sigma(t))Z^\Delta*(t) = A(t)Y(t)C(t)Z^*(t) + Y(\sigma(t))C(\sigma(t))Z^*(\sigma(t))B(t) + F(t)U(t)
\]

implies \( C^\Delta(t) = Y^{-1}(\sigma(t))F(t)U(t)(Z^*)^{-1}(t) \)

\[
C(t) = \int_t^s [Y^{-1}(\sigma(s))F(s)U(s)(Z^*)^{-1}(s)] \Delta s.
\]

There fore \( \bar{X}(t) = Y(t) \int_t^s [Y^{-1}(\sigma(s))F(s)U(s)(Z^*)^{-1}(s)] \Delta s Z^*(t) \)

**Theorem 8.2.4** Any solution \( X(t) \) of the initial value problem (8.1.1) satisfying

\[
X(t_0) = X_0
\]

is given by

\[
X(t) = \phi(t, t_0)X_0 \xi^*(t_0, t) + \phi(t, t_0) \int_t^{t_0} \phi(t_0, \sigma(s)) F(s)U(s)(Z^*)^{-1}(s) \Delta s \xi^*(s, t_0) ds.
\]

**Proof.** Any solution \( X(t) \) of (8.1.1) is of the form

\[
X(t) = Y(t)CZ^*(t) + \int_t^{t_0} Y(t)Y^{-1}(\sigma(s))F(s)U(s)(Z^*)^{-1}(s)Z^*(t) \Delta s
\]

\[
X(t_0) = X_0 \implies X_0 = Y(t_0)CZ^*(t_0) \text{ hence } C = Y^{-1}(t_0)X_0(Z^*)^{-1}(t_0).
\]

Denote \( Y(t)Y^{-1}(t_0) = \phi(t, t_0), \)

\[
Y(t)Y^{-1}(\sigma(s)) = \phi(t, \sigma(s))
\]

\[
(Z^*)^{-1}(t_0)Z^*(t) = \xi^*(t_0, t).
\]

Then the solution of (8.1.1) becomes

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Section 8.3

In this section, we prove necessary and sufficient conditions for controllability and observability of matrix dynamical systems (8.1.1), (8.1.2) on time scales.

Definition 8.3.1 The first order matrix dynamical system (8.1.1) and (8.1.2) is said to be completely controllable on $[t_0, t_f]$, if for any given initial state $X(t_0) = X_0$, there exists an input signal $U(t)$, such that the corresponding solution of (8.1.1) and (8.1.2) satisfies

$$X(t_f) = X_f.$$ 

If the dynamical system (8.1.1) and (8.1.2) is controllable for all $X_0$ at $t = t_0$ and for all $X_f$ at $t = t_f$, then the system (8.1.1) and (8.1.2) is said to be completely controllable.

Theorem 8.3.1 If the dynamical system (8.1.1) and (8.1.2) $(S_1)$ is completely controllable on the closed interval $J = [t_0, t_f]$ if and only if the $n \times n$ symmetric controllability matrix

$$V(t_0, t_f) = \int_{t_0}^{t_f} \phi(t_0, \sigma(s))F(s)F^*(s)\phi^*(t_0, \sigma(s))\Delta s \text{ is non-singular}$$

is non-singular

and in this case the control

$$U(t) = -(F(t))^* \phi^*(t_0, \sigma(t))V^{-1}(t_0, t_f)[X_0 - \phi(t_0, t_f)X_f] \xi^*(t_0, t)$$

defined on $t_0 \leq t \leq t_f$, transfers $X(t_0) = X_0$ to $X(t_f) = X_f$. 

$$X(t) = \phi(t, t_0)X_0 \xi^*(t_0, t) + \phi(t, t_0)[\int_{t_0}^{t} \phi(t_0, \sigma(s))F(s) \xi^*(s, t_0)\Delta s] \xi^*(t, t_0).$$
**Proof.** Suppose that $V(t_0,t_f)$ is non-singular, then the control $U(t)$ defined by (8.3.2) exists. Now substituting (8.3.1) in (8.3.2) with $t = t_f$, we have

$$X(t_f) = \phi(t_0, t_f)[X_0 - \int_{t_0}^{t_f} \phi(t_0, \sigma(s)) F(s) F^*(s) \phi^*(t_0, \sigma(s)) V^{-1}(t_0, t_1) [\xi_0 - \phi(t_0, t_f) \xi_f] \Delta s]$$

$$= X_f.$$ 

Hence the dynamical system (8.3.1) and (8.3.2) is completely controllable on J.

Conversely, suppose that the dynamical system (8.1.1) and (8.1.2) is completely controllable on J, then we have to show that $V(t_0,t_f)$ is non-singular. Suppose that $V(t_0,t_f)$ is singular. Since $V(t_0,t_f)$ is not invertible, then there exists a non zero $n \times 1$ vector $\alpha$ such that

$$\alpha^* V(t_0,t_f) \alpha = \int_{t_0}^{t_f} \alpha^* \phi(t_0, \sigma(s)) F(s) F^*(s) \phi^*(t_0, \sigma(s)) \alpha \Delta s$$

$$= \int_{t_0}^{t_f} \theta^*(t_0, \sigma(s)) \theta(t_0, \sigma(s)) \Delta s$$

where $\theta(\sigma(s), t_0) = F^*(t_0, \sigma(s)) \alpha$

$$= \int_{t_0}^{t_f} \|\phi\|_\infty^2 \Delta s \geq 0 . \quad (8.3.3)$$

From (8.3.3) $V(t_0,t_f)$ is positive semi definite. Suppose that there exists some $\beta \neq 0$ such that $\beta^* V(t_0,t_f) \beta = 0$ then from (8.3.3) $\theta = \eta$ when $\alpha = \beta$ implies

$$\int_{t_0}^{t_f} \|\eta\|_c^2 \Delta s = 0 .$$

Using the property of norm, we have
\( \eta(\sigma(s),t_0) = 0 \quad t_0 \leq t \leq t_f \)

since \( S_1 \) is completely controllable, then there exists a control \( U(t) \) making \( X(t_f) = 0 \).

If \( X(t_0) = \beta \), then from (8.3.3) we have

\[
\beta = -\int_{t_0}^{t_f} \theta^T(t_0,\sigma(s)) F(s) U(s) \Delta s
\]

consider \( \|\beta\|^2 = \beta^* \beta \)

\[
= -\int_{t_0}^{t_f} U(s) F^*(s) \theta^* (t_0,\sigma(s)) \beta \Delta s
\]

\[
= -\int_{t_0}^{t_f} U^*(s) \eta(s, t_0) \Delta s
\]

\[
= 0.
\]

Thus \( \beta = 0 \), which is a contradiction to our assumption. Hence \( V(t_0, t_1) \) is positive definite and is therefore non-singular.

Now we turn our attention to the concept of observability for dynamical system on a time scales.

**Definition 8.3.2** The time scale dynamical system \( S_1 \) is completely observable on \( J = [t_0,t_1] \) if for any time \( t_0 \) and any initial state \( X(t_0) = X_0 \) there exists a finite time \( t_1 > t_0 \) such that the knowledge of \( U(t) \) and \( L(t) \) for \( t_0 \leq t \leq t_1 \) suffices to determine \( X_0 \) uniquely.

Now we present a necessary and sufficient condition for the system \( S_1 \) to be completely observable.
**Theorem 8.3.2** The system $S_1$ is completely observable on $J$ if and only if the $n \times n$ symmetric observability matrix

$$W(t_0, t_1) = \int_{t_0}^{t_1} \phi^*(s, t_0) L^*(s) L(s) \phi(s, t_0) \Delta s$$

is non-singular.

**Proof.** Suppose $W(t_0, t_1)$ that is non-singular. Without loss of generality we assume zero input i.e., $U(t) = 0$.

The output becomes

$$K(t) = L(t) \xi(t),$$

where $\xi(t) = \phi(t, t_0) \xi_0$

multiplying (8.3.5) on the left by $\phi^*(t, t_0) L^* (t)$ and integrating from $t_0$ to $t_1$ we obtain

$$\int_{t_0}^{t_1} \phi^*(s, t_0) L^* (s) K(s) \Delta s = W(t_0, t_1) \xi_0$$

since $W(t_0, t_1)$ is non singular, $\xi_0$ can be determined uniquely. Hence the dynamical system $S_1$ is completely observable (c.o).

Conversely, suppose that $S_1$ is c.o., then we prove that $W(t_0, t_1)$ is non-singular. Since $W(t_0, t_1)$ is symmetric, we can construct the quadratic form

$$\alpha^* W(t_0, t_1) \alpha = \int_{t_0}^{t_1} \alpha^* \phi^*(s, t_0) L^* (s) L(s) \phi(s, t_0) \Delta s$$

$$= \int_{t_0}^{t_1} \| \phi \|^2 \Delta s \geq 0$$

(8.3.6)
where $\alpha$ is an arbitrary $n \times 1$ vector and $\eta(s,t_0) = L(s)\phi(s,t_0)\alpha$ from (8.3.6) $W(t_0,t_1)$ is positive semi definite. Suppose that there exists some $\beta \neq 0$ such that $\beta^*W(t_0,t_1)\beta = 0$ then from (8.3.6) with $\eta = \theta$ when $\alpha = \beta$ implies

$$\int_{t_0}^{t_1} \| \theta \|^2_e = 0.$$  

Using the property of norm, we have $\theta(s,t_0) = 0; t_0 \leq s \leq t_1$ implies $L(s)\phi(s,t_0)\beta = 0, t_0 \leq s \leq t_1$. From (8.3.5) this implies that when $\xi_0 = \beta$, the output is identically zero throughout the interval, so that $\xi_0$ cannot be determined from a knowledge of $K(t)$. This contradicts the supposition that $S_1$ is completely observable.

Hence $W(t_0,t_1)$ is positive definite, and therefore non-singular.

Section 8.4

In this section, we study the controllability of the following first order matrix dynamic integro-differential equation on time scales

$$X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t) + \int_{t_0}^{t} [K_1(t,s)X(s) + X(\sigma(s))K_2(t,s)]Ds + C(t)U(t)D^*(t)$$  

(8.4.1)

satisfying $X(t_0) = X_0$, where $K_1(t,s), K_2(t,s)$ are rd-continuous square matrices of the order $(n \times n), C(t), D(t) \in C_{rd}[T^+,R^{mn}]$ and the input $U(t) \in C_{rd}[T^+,R^{mm}]$.

First, we develop the variation of parameters formula for the dynamical system

$$X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t) + \int_{t_0}^{t} [K_1(t,s)X(s) + X(\sigma(s))K_2(t,s)]Ds + F(t)$$  

(8.4.2)

satisfying $X(t_0) = X_0$ and $F \in C_{rd}[T^+,R^{n \times n}]$.  

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Further, we obtain sufficient conditions for controllability of the first order matrix integro-differential system on time scales (8.4.1).

Here we develop the method of variation of parameters formula for integro-differential system on time scales in terms of resolvent kernels and then offer sufficient conditions for the controllability of the first order matrix integro-differential system on time scales.

We consider the matrix linear integro-differential system on time scales

\[
X^\Delta(t) = A(t)X(t) + \int_{t_0}^{t} K_1(t,s)X(s)\Delta s + F(t) \quad X(t_0) = X_0
\]

(8.4.3)

where \( A(t), K_1(t,s) \) are \((n \times n)\) rd-continuous matrices for \( t, s \in T^+ \) and \( F \in c_{rd}[T^+, R^{n \times n}] \) and \( X(t) \) is a square matrix of order \( n \). For \( t_0 < s < t < \infty \), we get

\[
\phi(t,s) = A(t) + \int_{s}^{t} K_1(t,\tau)\Delta \tau
\]

(8.4.4)

\[
R_1(t,s) = I_n + \int_{s}^{t} R_1(t,\xi)\phi(\xi,s)\Delta \xi
\]

(8.4.5)

where \( K_1(t,s) = \phi(t,s) = R_1(t,s) = 0 \) for \( t_0 \leq t < s \).

**Theorem 8.4.1** Any solution of (8.4.2) with initial condition \( X(t_0) = X_0 \), is of the form

\[
X(t) = R_1(t,t_0)X_0 + \int_{t_0}^{t} R_1(t,s)F(s)\Delta s
\]

where \( R_1(t,s) \) is the unique solution of the partial differential equation on time scales.
\[
\frac{\partial R_1}{\partial s}(t, s) + R_1(t, \sigma(s))A(s) + \int_0^t R_1(t, \sigma(\xi))K_1(\xi, s)\Delta \xi = 0 \quad (8.4.6)
\]

with \( R_1(t, t) = I_\ast \).

**Proof.** Since \( \phi \) is rd-continuous, it follows that \( R_1 \) in (8.4.5) exists and subsequently

\[
\frac{\partial R_1}{\partial s}
\]

exists and satisfies (8.4.6). Let \( X(t) \) be a solution of (8.4.2) for \( t \geq t_0 \) then if we set

\[ P(s) = R_1(t, s)X(s), \]

we have

\[
P^\Delta(s) = \frac{\partial R_1}{\partial s}X(s) + R_1(t, \sigma(s))X^\Delta(s)
\]

\[
= \frac{\partial R_1}{\partial s}X(s) + R_1(t, \sigma(s))[A(s)X(s) + \int_{t_0}^s K_1(s, u)X(u)\Delta u + F(s)]
\]

integrating between the limits \( t_0 \) to \( t \)

\[
P(t) - P(t_0) = \int_{t_0}^t \left[ \frac{\partial R_1}{\partial s}X(s) + R_1(t, \sigma(s))A(s)X(s) + R_1(t, \sigma(s))F(s) \right] \Delta s
\]

\[ + \int_{t_0}^t R_1(t, \sigma(s)) \int_{t_0}^s K_1(s, u)Z(u)\Delta u] \Delta s. \]

It implies that

\[
P(t) - R_1(t_0, t)X_0 = \int_{t_0}^t \left[ \frac{\partial R_1}{\partial s} + R_1(t, \sigma(s))A(s) + \int_{t_0}^s R_1(t, \sigma(s))K_1(s, u)\Delta u \right] X(s) \Delta s
\]

\[ + \int_{t_0}^t R_1(t, \sigma(s))F(s) \Delta s. \]

Using (8.4.6), we get

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\[ P(t) = R_1(t, t_0)X_0 + \int_{t_0}^{t} R_1(t, \sigma(s))F(s)\Delta s. \]

Since \( P(t) = R_1(t, t)X(t) \) and \( R_1(t, t) = I_n \)

\[ X(t) = R_1(t, t_0)X_0 + \int_{t_0}^{t} R_1(t, \sigma(s))F(s)\Delta s \]

Let \( X(t) \) be the solution of (8.4.6) satisfying \( X(t_0) = X_0 \) existing for \( t_0 \leq t < \infty \). Then

\[ \int_{t_0}^{t} R_1(t, \sigma(s))X^\Delta(s)\Delta s = R_1(t, t)X(t) - R_1(t, t_0)X_0 - \int_{t_0}^{t} \frac{\partial R_1(t, s)}{\Delta s}X(s)\Delta s \]

\[ = \int_{t_0}^{t} R_1(t, \sigma(s))F(s)\Delta s - \int_{t_0}^{t} \frac{\partial R_1(t, s)}{\Delta s}X(s)\Delta s. \]

Using (8.4.6), we get

\[ \int_{t_0}^{t} R_1(t, \sigma(s))[X^\Delta(s) - A(s)X(s)] - \int_{t_0}^{s} K_1(s, \xi)X(\xi)\Delta \xi - F(s)\Delta s = 0 . \]

Since \( R_1(t, \sigma(s)) \) is non-zero rd-continuous for \( t_0 \leq s \leq t < \infty \). Then

\[ X^\Delta(s) - A(s)X(s) - \int_{t_0}^{s} K_1(s, \xi)X(\xi)\Delta \xi - F(s) = 0 \]

Therefore \( X(s) \) is a solution of

\[ X^\Delta(s) = A(s)X(s) + \int_{t_0}^{s} K_1(s, \xi)F(\xi)\Delta \xi. \]

**Theorem 8.4.2** The first order matrix integro-differential system on time scales

\[ X^\Delta(t) = A(t)X(t) + \int_{t_0}^{t} K_1(t, s)X(s)\Delta s + C(t)U(t) \]

\[ X(t_0) = X_0 \]
is completely controllable, if and only if the controllability matrix

\[ \phi(t_0, t_1) = \int_{t_0}^{t_1} R_1(t_0, \sigma(s))C^*(s)C(s)R_1^*(t_0, \sigma(s)) \Delta s \]

is non-singular, where \( R_1(t, \sigma(s)) \) is the resolvent matrix. The control function

\[ U(t) = -C(t)R_1^*(t_0, t)\phi^{-1}(t_0, t_1)[X_0 - R(t_0, t_1)X_1] \]

defined for \( t_0 < t < t_1 \) transfers \( X(t_0) = X_0 \) to \( X(t_1) = X_1 \)

**Proof.** Any solution \( X(t) \) of (8.4.7) is given by

\[ X(t) = R(t_1, t_0)X_1 + \int_{t_0}^{t} R_1(t, \sigma(s))C^*(s)U(s) \Delta s \]

and

\[ X(t_1) = R(t_1, t_0)[X_0 + \int_{t_0}^{t_1} R_1(t_0, \sigma(s))C^*(s)U(s) \Delta s] \]

\[ = R_1(t_1, t_0)[X_0 + \int_{t_0}^{t_1} R(t_0, \sigma(s))C(s)(-C^*(s)R^*(t_0, \sigma(s))\phi^{-1}(t_0, t_1))(X_0 - R(t_0, t_1)X_1) \Delta s] \]

\[ = R_1(t_1, t_0)[X_0 - \phi(t_0, t_1)\phi^{-1}(t_0, t_1)(X_0 - R(t_0, t_1)X_1)] = X_1. \]

Similarly we obtain the converse as in the previous theorem.

**Theorem 8.4.3** Assume that \( B(t) \) and \( K_2(t, s) \) are rd-continuous \((n \times n)\) matrices for \( t \in T^+, (t, s) \in T \) and \( F \in C[R_+, R^{\text{aux}}] \). Then the solution of

\[ X^\Delta(t) = X(\sigma(t))B(t) + \int_{t_0}^{t} X(\sigma(s))K_2(t, s) \Delta s + D(t)U(t) \]

\[ X(t_0) = X_0 \]

is given by
\[ X(t) = X_0 R_2^*(t, t_0) + \int_{t_0}^{t} D(s) U^*(s) R_2^*(t, s) \Delta s \]

where \( R_2(t, s) \) is the resolvent kernel and is the unique solution of

\[ \frac{\partial}{\Delta s} (R_2^*(t, s)) + B(s) R_2^*(t, s) + \int_{s}^{t} K_2(\xi, s) R_2^*(t, \sigma(\xi)) \Delta \xi = 0 \quad (8.4.9) \]

with \( R_2(t, t) = I_n \).

**Theorem 8.4.4** The matrix integro-differential system (8.4.8) is completely controllable, if and only if, the controllability matrix

\[ \xi(t_0, t_1) = \int_{t_0}^{t_1} R_2(t_0, \sigma(s)) D^*(s) D(s) R_2^*(t_0 \sigma(s)) \Delta s \]

is non-singular, where \( R_2(t, s) \) is the resolvent matrix. The control function \( U(t) \) given by

\[ U(t) = -D(t) R_2^*(t_0, t) \xi^{-1}(t_0, t) [X_0^* - R_2(t_0, t_1) X_1^*] \]

defined for \( t_0 < t < t_1 \), transfers

\[ X(t_0) = X_0 \text{ to } X(t_1) = X_1. \]

We shall now consider the superposition of these systems and present a set of sufficient conditions for the complete controllability of matrix integro-differential system (8.4.1) on time scales.

**Theorem 8.4.5** The matrix integro-differential system (8.4.1) satisfying \( X(t_0) = X_0 \) has a unique solution given by

\[ X^\Delta(t) = R_1(t, t_0) X_0 R_2^*(t, t_0) + \int_{t_0}^{t} R_1(t, \sigma(s)) C(s) U(s) D^*(s) R_2^*(t, \sigma(s)) \Delta s \]
where $R_1$ and $R_2$ are the solutions of partial differential equations on time scales (8.4.6) and (8.4.9) respectively.

**Theorem 8.4.6** The matrix integro-differential system (8.4.1) is completely controllable if and only if the

$$\phi(t_0, t_1) = \int_{t_0}^t R_1(t_0, \sigma(s))C^*(s)C(s)R_1^*(t_0, \sigma(s))\Delta s$$

and

$$\xi(t_0, t_1) = \int_{t_0}^t R_2^*(t_0, \sigma(s))D(s)D^*(s)R_2(t_0, \sigma(s))\Delta s$$

are non-singular, where $R_1(t, s)$ and $R_2(t, s)$ are resolvent matrices. The control function $U(t)$ is given by

$$U(t) = -C(t)R_1^*(t_0, t)\phi^{-1}(t_0, t_1)[X_0 - R_1(t_0, t_1)X_1R_1^*(t_1, t_0)]\xi^{-1}(t_0, t_1)R_2^*(t_0, t_1)D(t)$$

(8.4.10)

for $t_0 < t < t_1$ transfers $X(t_0) = X_0$ to $X(t_1) = X_1$.

**Proof.** Any solution $X(t)$ of the matrix integro-differential equation on time scales is given by

$$X(t) = R_1(t, t_0)X_0R_2^*(t, t_0) + \int_{t_0}^t R_1(t, \sigma(s))C(s)U(s)D^*(s)R_2^*(t, \sigma(s))\Delta s. \quad (8.4.11)$$

Substituting the general form of the control $U(t)$ defined (8.4.10) in (8.4.11), we get

$$X(t_1) = R_1(t_1, t_0)X_0R_2^*(t_1, t_0) + R_1(t_1, t_0)\int_{t_0}^{t_1} R_1(t, \sigma(s))C(s)[-C^*(s)R_1^*(t_0, \sigma(s))\phi^{-1}(t_0, t_1)]$$

$$[X_0 - R_1(t_0, t_1)X_1R_2^*(t_1, t_0)]\xi^{-1}(t_0, t_1)R_2(t_1, t_0)D(s)]D^*(s)R_2^*(t_0, \sigma(s))\Delta s$$
\[ = R_1(t_1, t_0)X_0R_2^*(t_1, t_0) - R_1(t_1, t_0)\phi(t_0, t_1)\phi^{-1}(t_0, t_1)[X_0 - R_1(t_0, t_1)X_1R_2(t_1, t_0)] \]
\[ \times \xi(t_0, t_1)\xi^{-1}(t_0, t_1)R_2^*(t_0, t_1) \]
\[ = R_1(t_1, t_0)X_0R_2^*(t_1, t_0) - R_1(t_1, t_0)X_0R_2^*(t_0, t_1) + R_1(t_1, t_0)R(t_0, t_1)X_1R_2^*(t_1, t_0)R_2^*(t_0, t_1) \]
\[ = R_1(t_1, t_1)X_1R_2^*(t_1, t_1) = X_1. \]

Conversely, suppose that the system (8.4.1) is completely controllable. Then it can be easily seen as in Theorem 8.3.1, \( \phi(t_0, t_1) \) and \( \xi(t_0, t_1) \) are non-singular.

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