CHAPTER 7

CONTROLLABILITY, OBSERVABILITY AND REALIZABILITY OF KRONECKER PRODUCT MATRIX DYNAMICAL SYSTEMS ON TIME SCALES

Section 7.1

The importance of Kronecker product matrix dynamical systems on time scales is an interesting area of current research. There are numerous applications of Kronecker product in statistics, economics, optimization and control. A fascinating fact is that all the widely different disciplines of application depend on a common core of time scale dynamical system of the modern control system theory.

In this chapter, we shall be concerned with the first order matrix Kronecker product dynamical system on time scales represented by

\[
(X(t) \otimes Y(t))^{\Delta} = (A(t) \oplus C(t))(X(t) \otimes Y(t)) + (X(\sigma(t) \otimes Y(\sigma(t)))(B(t) \oplus D(t))
+ (F(t) \otimes G(t))(U(t) \otimes V(t)), \quad \text{with } (X(t_0) \otimes Y(t_0)) = (P_0 \otimes Q_0) \tag{7.1.1}
\]

\[
(R(t) \otimes S(t)) = (K(t) \otimes L(t))(X(t) \otimes Y(t)) \tag{7.1.2}
\]

where \(A(t), B(t), C(t), D(t), X(t)\) and \(Y(t)\) are square matrices of order \(n\) on \(J = [t_0, t_1]\), \(R(t), S(t), K(t)\) and \(L(t)\) are all matrices of order \(p \times n\) and \(F(t), G(t)\) are matrices of order \(n \times m\) and \(U(t), V(t)\) are control piecewise rd-continuous matrices of order \(n \times m\). Here we assume that the matrices \(A(t), B(t), C(t), D(t)\) are rd-continuous on closed interval \(J\). Existence of solutions of similar systems studied by Surya Narayana et al. [60].
Further Murty, Rao et al. [[51], [52]] obtained the controllability, observability criteria for continuous systems and similar results for discrete systems by S.Vladimer[61].

In this chapter we study the controllability and observability of Kronecker product system on time scales.

In section 7.2 we present the general solution of the corresponding homogeneous system

\[(X(t) \otimes Y(t))^{\Delta} = (A(t) \oplus C(t))(X(t) \otimes Y(t)) + (X(\sigma(t) \otimes Y(\sigma(t))(B(t) \oplus D(t))) \quad (7.1.3)\]

in terms of two fundamental matrix solutions of the systems

\[(X(t) \otimes Y(t))^\Lambda = (A(t) \oplus C(t))(X(t) \otimes Y(t))\] and

\[(X(t) \otimes Y(t))^\Lambda = (B(t) \oplus D(t))^\sigma(X(\sigma(t) \otimes Y(\sigma(t))).\]

We also obtain the general form of the solution of the system (7.1.1) by using the standard technique of variation of parameters.

In section 7.3 we establish the necessary and sufficient conditions for complete controllability and complete observability under certain smoothness conditions of the system (7.1.1) with (7.1.2).

In section 7.4 we present the realizability and minimal realization of (7.1.1) with (7.1.2).

Section 7.2

In this section, we present the general solution of first order matrix dynamical systems on time scales (7.1.3) in terms of the fundamental matrix solution of (7.1.4) and (7.1.5).

we then develop variation of parameters formula for the non homogeneous system(7.1.1) and present its general solution.
Theorem 7.2.1 If \((Y_1(t) \otimes Z_1(t))\) and \((Y_2^*(t) \otimes Z_2^*(t))\) are fundamental matrix solutions of 
\((X(t) \otimes Y(t))^\Delta = (A(t) \oplus C(t))(X(t) \otimes Y(t))\)

and

\((X(t) \otimes Y(t))^\Delta = (B(t) \oplus D(t))^* (X(\sigma(t)) \otimes Y(\sigma(t)))\)
respectively, then any solution of the homogeneous Kronecker product system (7.1.3)
is of the form \((X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t)) \ (\zeta_1 \otimes \zeta_2) \ (Y_2^*(t) \otimes Z_2^*(t))\),

where \(\zeta_1, \zeta_2\) are constant square matrices of order \(n\) and \(Y_1(t), Z_1(t), Y_2(t),\) and \(Z_2(t)\) are fundamental matrix solutions of

\[X^\Delta(t) = A(t)X(t), Y^\Delta(t) = C(t)Y(t), X^\Delta(t) = B^*(t)X(\sigma(t)), Y^\Delta(t) = D^*(t)Y(\sigma(t))\]

respectively.

**Proof.** We develop a solution of the homogeneous Kronecker product matrix dynamical system (7.1.3) in the form \((X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t)) \ (K_1(t) \otimes K_2(t))\), where \(K_1(t), K_2(t)\) are square matrices of order \(n\). Then

\[(X(t) \otimes Y(t))^\Delta = (Y_1(t) \otimes Z_1(t))^\Delta \ (K_1(t) \otimes K_2(t)) + (Y_1(\sigma(t)) \otimes Z_1(\sigma(t))) \ (K_1(t) \otimes K_2(t))^\Delta \]

\[= (A(t) \oplus C(t))(Y_1(t) \otimes Z_1(t))(K_1(t) \otimes K_2(t)) + (Y_1(\sigma(t)) \otimes Z_1(\sigma(t))) \ (K_1(\sigma(t)) \otimes K_2(\sigma(t)))(B(t) \oplus D(t))\]

i.e

\[(K_1(t) \otimes K_2(t))^\Delta = (K_1(\sigma(t)) \otimes K_2(\sigma(t)))(B(t) \oplus D(t))\]

\[((K_1(t) \otimes K_2(t))^\Delta)^* = (B(t) \oplus D(t)^* (K_1(\sigma(t)) \otimes K_2(\sigma(t)))^*\]

Since \((Y_2^*(t) \otimes Z_2^*(t))\) is a fundamental matrix solution of

\[(X(t) \otimes Y(t))^\Delta = (B(t) \oplus D(t)^* (X(\sigma(t)) \otimes Y(\sigma(t)))^*\]
It follows that there exists a constant square matrix \((\zeta_3 \otimes \zeta_4)\) such that

\[
(K_1(t) \otimes K_2(t))^* = (Y_2(t) \oplus Z_2(t))^* (\zeta_3 \otimes \zeta_4)
\]

\[
\iff (K_1(t) \otimes K_2(t)) = (\zeta_3^* \otimes \zeta_4^*) (Y_2^* (t) \otimes Z_2^* (t))
\]

Hence \((X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t)) (\zeta_1^* \otimes \zeta_2^*) (Y_2^* (t) \otimes Z_2^* (t))\)

Choose \((\zeta_1 = \zeta_3^*)\) and \((\zeta_2 = \zeta_4^*)\)

**Theorem 7.2.2** Any solution \((X(t) \otimes Y(t))\) of the non-homogeneous Kronecker product matrix dynamical system (7.1.1) is of the form

\[
(X(t) \otimes Y(t)) = (Y_1(t) \otimes Z_1(t)) (\zeta_1^* \otimes \zeta_2^*) (Y_2^* (t) \otimes Z_2^* (t)) + (\Xi(t) \otimes \Upsilon(t))
\]

where \((\Xi(t) \otimes \Upsilon(t))\) is a particular solution of (7.1.1).

**Theorem 7.2.3** A particular solution of \((\Xi(t) \otimes \Upsilon(t))\) of the non-homogeneous Kronecker product matrix dynamical system (7.1.1) is given by

\[
(\Xi(t) \otimes \Upsilon(t)) = (Y_1(t) \otimes Z_1(t)) \int_{t_0}^{t} \left( (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F(s) \otimes G(s))(U(s) \otimes V(s)) \right.
\]

\[
\left. \times (Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s \right) (Y_2^*(s) \otimes Z_2^*(t))
\]

**Proof.** We develop a particular solution of the Kronecker product matrix dynamical system (7.1.1) in the form

\[
(\Xi(t) \otimes \Upsilon(t)) = (Y_1(t) \otimes Z_1(t)) (K_1(t) \otimes K_2(t)) (Y_2^* (t) \otimes Z_2^* (t))
\]

where \(K_1(t), K_2(t)\) are square matrices of order \(n\). Then substituting \((\Xi(t) \otimes \Upsilon(t))\) in (7.1.1), we get

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\[
[ (Y_1(t) \otimes Z_1(t))(K_1(t) \otimes K_2(t))(Y_2^*(t) \otimes Z_2^*(t))]^A
\]

\[\begin{align*}
= (A(t) \otimes C(t))(Y_1(t) \otimes Z_1(t))(K_1(t) \otimes K_2(t))(Y_2^*(t) \otimes Z_2^*(t)) \\
+ (Y_1(\sigma(t)) \otimes Z_1(\sigma(t)))(K_1(\sigma(t)) \otimes K_2(\sigma(t)))(Y_2^*(\sigma(t)) \otimes Z_2^*(\sigma(t)))(B(t) \otimes D(t)) \\
+ (F(t) \otimes G(t))(U(t) \otimes V(t))
\end{align*}\]

\[
(Y_1(t) \otimes Z_1(t))^A (K_1(t) \otimes K_2(t))(Y_2^*(t) \otimes Z_2^*(t)) + \\
(Y_1(\sigma(t)) \otimes Z_1(\sigma(t)))(K_1(\sigma(t)) \otimes K_2(\sigma(t)))(Y_2^*(t) \otimes Z_2^*(t))
\]

\[\begin{align*}
= (A(t) \otimes C(t))(Y_1(t) \otimes Z_1(t))(K_1(t) \otimes K_2(t))(Y_2^*(t) \otimes Z_2^*(t)) \\
+ (Y_1(\sigma(t)) \otimes Z_1(\sigma(t)))(K_1(\sigma(t)) \otimes K_2(\sigma(t)))(Y_2^*(\sigma(t)) \otimes Z_2^*(\sigma(t)))(B(t) \otimes D(t)) \\
+ (F(t) \otimes G(t))(U(t) \otimes V(t))
\end{align*}\]

imply

\[
(Y_1(\sigma(t)) \otimes Z_1(\sigma(t)))(K_1(t) \otimes K_2(t))^A (Y_2^*(t) \otimes Z_2^*(t)) = (F(t) \otimes G(t))(U(t) \otimes V(t))
\]

\[
(K_1(t) \otimes K_2(t))^A = (Y_1(\sigma(t)) \otimes Z_1(\sigma(t)))^{-1} (F(t) \otimes G(t))(U(t) \otimes V(t))(Y_2^*(t) \otimes Z_2^*(t))^{-1}
\]

\[
(K_1(t) \otimes K_2(t)) = \int_{t_0}^t (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F(s) \otimes G(s))(U(s) \otimes V(s))(Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s
\]

hence a particular solution is given by

\[
(\overline{X}(t) \otimes \overline{Y}(t) = (Y_1(t) \otimes Z_1(t))[\int_{t_0}^t (Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1} (F(s) \otimes G(s))(U(s) \otimes V(s))(Y_2^*(s) \otimes Z_2^*(s))^{-1} \Delta s]
\]

\[\times (Y_2^*(t) \otimes Z_2^*(t))\]
Theorem 7.2.4 Any solution \((X(t) \otimes Y(t))\) of the initial value problem (7.1.1) satisfying
\[ X(t_0) \otimes Y(t_0) = P_0 \otimes Q_0 \]
is given by
\[
(X(t) \otimes Y(t) = \phi(t,t_0)(P_0 \otimes Q_0) \xi^*(t,t_0) \\
+ \phi(t,t_0) \int_{t_0}^t (\phi(t_0,\sigma(s))(F(s) \otimes G(s))(U(s) \otimes V(s)) \xi^*(s,t_0) \Delta s) \xi^*(t_0,t)
\]
(7.2.1)
where \(\phi(t,\sigma(s)) = (Y_1(t) \otimes Z_1(t))(Y_1(\sigma(s)) \otimes Z_1(\sigma(s)))^{-1}\)
(7.2.2)
and
\[
\xi^*(s,t) = (Y_2^*(s) \otimes Z_1^*(s))^{-1}(Y_2^*(t) \otimes Z_1^*(t)).
\]
(7.2.3)

Section 7.3

In this section, we prove necessary and sufficient conditions for controllability and observability of the Kronecker product dynamical system (7.1.1) and (7.1.2) on time scales.

Definition 7.3.1 The matrix dynamical systems \(S_1\) given by (7.1.1) and (7.1.2) is said to be completely controllable if for \(t_0\), any initial state \((X(t_0) \otimes Y(t_0)) = (P_0 \otimes Q_0)\) and any given final state \((P_f \otimes Q_f)\) there exists a finite time \(t_1 > t_0\) and a control
\[(U(t) \otimes V(t)), t_0 \leq t \leq t_1\] such that \((X(t_1) \otimes Y(t_1)) = (P_f \otimes Q_f)\).

Theorem 7.3.1 The time scale dynamical system \(S_1\) is completely controllable on the closed interval \(J = [t_0, t_1]\) if and only if the \(n^2 \times n^2\) symmetric controllability matrix
\[
M(t_0,t_1) = \int_{t_0}^{t_1} \phi(t_0,\sigma(s))(F \otimes G)(s)(F \otimes G)^*(s)\phi^*(t_0,\sigma(s)) \Delta s
\]
(7.3.1)
where \(\phi(t,\sigma(s))\) is defined in (7.2.3), is non-singular. In this case the control
\begin{equation}
(U(t) \otimes V(t)) = -(F \otimes G)^* (t) \phi^* (t_0, \sigma(s)) M^{-1} (t_0, t_1) \{(P_0 \otimes Q_0) - \phi(t_0, t_1)(P_f \otimes Q_f )\}
\end{equation}

(7.3.2)

defined on \( t_0 \leq t \leq t_1 \),

transfers \( (X(t_0) \otimes Y(t_0)) = (P_0 \otimes Q_0) \) to \( (X(t_1) \otimes Y(t_1)) = (P_f \otimes Q_f ) \).

**Proof.** Suppose that \( M(t_0, t_1) \) is non-singular, then the control defined by (7.3.2) exists.

Now substituting (7.3.2) in (7.2.2) with \( t = t_1 \), we have

\[
(X(t_1) \otimes Y(t_1)) = \phi(t_1, t_0) \left[ P_0 - \int_{t_0}^{t_1} \phi(t_0, \sigma(s))(F \otimes G)(s)(F \otimes G)^* (s) \phi^* (t_0, \sigma(s)) M^{-1} (t_0, t_1) \{(P_0 \otimes Q_0) - \phi(t_0, t_1)(P_f \otimes Q_f )\} \Delta s \right]
\]

\[
= \phi(t_1, t_0) \phi(t_0, t_1) = (P_1 \otimes Q_f ).
\]

Hence the dynamical system \( S_1 \) is completely controllable.

Conversely, suppose that the dynamical system \( S_1 \) is completely controllable on \( J \), then we have to show that \( M(t_0, t_1) \) is non-singular. Then there exists a non zero \( n^2 \times 1 \) vector \( \alpha \) such that

\[
\alpha^* M(t_0, t_1) \alpha = \int_{t_0}^{t_1} \alpha^* \phi(t_0, \sigma(s))(F \otimes G)(s)(F \otimes G)^* (s) \phi^* (t_0, \sigma(s)) \alpha \Delta s
\]

\[
= \int_{t_0}^{t_1} \theta^* (\sigma(s), t_0) \theta(\sigma(s), t_0) \Delta s = \int_{t_0}^{t_1} \| \theta \|^2 \Delta s \geq 0 \quad (7.3.3)
\]

where \( \theta = (F \otimes G)^* (s) \phi^* (t_0, \sigma(s)) \alpha \). From (7.3.3) \( M(t_0, t_1) \) is positive semi definite.

Suppose that there exists some \( \beta \neq 0 \) such that \( \beta^* M(t_0, t_1) \beta = 0 \) then from (7.3.3) with \( \theta = \eta \) when \( \alpha = \beta \), implies

\[
\int_{t_0}^{t_1} \| \eta \|^2 \Delta s = 0
\]
using the properties of norms, we have

$$\eta(\sigma(s), t_0) = 0, t_0 \leq t \leq t_1.$$  \hfill (7.3.4)

Since $S_1$ is completely controllable, so there exists a control $(U(t) \otimes V(t))$ making

$$(X(t_1) \otimes Y(t_1)) = 0 \quad \text{if} \quad (X(t_0) \otimes Y(t_0)) = \beta.$$ Hence from (7.2.2) we have

$$\beta = -\int_{t_0}^{t_1} \phi(t_0, \sigma(s))(F \otimes G)(s)((U(s) \otimes V(s))\Delta s)] .$$

Consider

$$\|\beta\|^2 = \beta^* \beta = -\int_{t_0}^{t_1}((U(s) \otimes V(s))^*(F \otimes G)^*(s)\phi^*(t_0, \sigma(s))\beta \Delta s$$

$$= -\int_{t_0}^{t_1}((U(s) \otimes V(s))^* \eta(\sigma(s), t_0)\Delta s = 0$$

hence $\beta = 0$, which is a contradiction to our assumption. Thus $M(t_0, t_1)$ is positive definite and is therefore non-singular.

Now we confine our attention to the concept of observability on a timescale dynamical system.

**Definition 7.3.2** The dynamical systems (7.1.1) is completely observable on

$J = [t_0, t_1]$ if for any time $t_0$ and any initial state $(X(t_0) \otimes Y(t_0)) = (P_0 \otimes Q_0)$ there exists a finite time $t_1 > t_0$ such that the knowledge of $(U(t) \otimes V(t))$ and $(R(t) \otimes S(t))$ for $t_0 \leq t \leq t_1$ suffices to determine $(P_0 \otimes Q_0)$ uniquely.

Now we present a necessary and sufficient condition for the system (7.1.1), (7.1.2) to be completely observable.

**Theorem 7.3.2** The dynamical system $S_1$ on time scales is completely observable on $J$ if and only if the $n^2 \times n^2$ symmetric observability matrix
\[
W(t_0, t_1) = \int_{t_0}^{t_1} \phi^*(s, t_0) (K \otimes L)^*(s)(K \otimes L)(s)\phi(s, t_0)\Delta s
\]

is non-singular.

**Proof.** Suppose that \(W(t_0, t_1)\) is non-singular. It is simpler to consider the case of zero input and it does not entail any loss of generality. Since the concept is not altered in the presence of a known input signal. Implies \((R(t) \otimes S(t)) = (K \otimes L)(t)(X(t) \otimes Y(t))\) since from

\[
(X(t) \otimes Y(t)) = \phi(t, t_0)(P_0 \otimes Q_0)\xi^*(t, t_0)
\]

we have

\[
(R(t) \otimes S(t)) = (K \otimes L)(t)\phi(t, t_0)(P_0 \otimes Q_0)\xi^*(t, t_0).
\]  \hspace{1cm} (7.3.5)

Pre multiplying (7.3.5) with \(\phi^*(t, t_0)(K \otimes L)^*(t)\), post multiply with \(\xi^*(t_0, t)\) and integrating from \(t_0\) to \(t_1\), we get

\[
\int_{t_0}^{t_1} \phi^*(s, t_0) (K \otimes L)^*(s)(R(s) \otimes S(s))\Delta s = W(t_0, t_1)(P_0 \otimes Q_0).
\]

Since \(W(t_0, t_1)\) is non-singular, \((P_0 \otimes Q_0)\) can be determined uniquely. Hence the dynamical system \(S_1\) is completely observable.

Conversely, suppose that the dynamical system \(S_1\) is completely observable. Then we prove that \(W(t_0, t_1)\) is non singular. Since \(W(t_0, t_1)\) is symmetric, we can construct the quadratic form

\[
\alpha^* W(t_0, t_1)\alpha = \int_{t_0}^{t_1} \alpha^* \phi^*(s, t_0) (K \otimes L)^*(s)(K \otimes L)(s)\phi(t_0, s)\alpha \Delta s
\]  \hspace{1cm} (7.3.6)
\[
\int_{t_0}^{t_1} \| \eta(s, t_0) \|^2 \Delta s \geq 0
\]

where \( \alpha \) is an arbitrary column \( n^2 \)-vector and \( \eta(s, t_0) = (K \otimes L)(s) \phi(t_0, s) \alpha \). From (7.3.6) \( W(t_0, t_1) \) is positive semi definite. Suppose that there exists some \( \beta \neq 0 \) such that 
\[
\beta^T W(t_0, t_1) \beta = 0
\]
then from (7.3.6) with \( \eta=0 \) when \( \alpha=\beta \), implies
\[
\int_{t_0}^{t_1} \| \theta(s, t_0) \|^2 \Delta s = 0 \Rightarrow \theta(s, t_0) = 0, t_0 \leq s \leq t_1.
\]

\[
\Rightarrow (K \otimes L)(s) \phi(t_0, s) \beta = 0, t_0 \leq s \leq t_1
\]

from (7.3.5), this implies that when \( (P_0 \otimes Q_0) = \beta \), the output is identically zero throughout the interval, so that \( (P_0 \otimes Q_0) \) cannot be determined in this case from knowledge of \( (R(t) \otimes S(t)) \).

This contradicts the supposition that \( S_1 \) is completely observable.

Hence \( W(t_0, t_1) \) positive definite, therefore non-singular.

Section 7.4

The linear dynamical system theory, the concept of realizability refers to the ability to characterize a known output, in terms of linear dynamical system with some input.

In this section, we present equivalent conditions for realizability.

**Definition 7.4.1** [21] The dynamical system (7.1.1), (7.1.2) of dimension \( n^2 \) is a realization of the weighting patterns \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \) if, for all \( t \) and \( s \).
If a realization of the system exists, then the weighting patterns are called realizable. The system is a minimal realization if no realization of \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \) if the dimension less than \( n^2 \) exists.

For the dynamical system (7.1.1), (7.1.2), the output signal \( (R(t) \otimes S(t)) \) corresponding to a given input \( (U(t) \otimes V(t)) \), the weighting patterns

\[
G_1(t, \sigma(s)) = (K(t) \otimes L(t))\phi(t, \sigma(s))(F(s) \otimes G(s)) \quad \text{and} \quad G_2(t, s) = \xi(t, s)
\]

is given by

\[
(R(t) \otimes S(t)) = \int_{t_0}^{t} G_1(t, \sigma(s))(U(s) \otimes V(s))G_2^*(t, s) \Delta s. \tag{7.4.1}
\]

**Theorem 7.4.1** The weighting patterns \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \) are realizable if and only if there exists rd-continuous matrix functions \( H_1(t) \) of order \( p^2 \times m^2 \) and \( F_1(t) \) of order \( m^2 \times n^2 \) such that

\[
G_1(t, \sigma(s)) = H_1(t) \ F_1(\sigma(s))
\]

and rd-continuous matrix functions \( H_2(t) \) of order \( m^2 \times p^2 \) and \( F_2(t) \) of order \( p^2 \times n^2 \)

\[
G_2(t, s) = H_2(t)F_2(t).
\]

**Proof.** Suppose that \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \) are realizable. We may assume that the system (7.1.1) and (7.1.2) is one such realization.

Now we write
\( G_1(t, \sigma(s)) = (K(t) \otimes L(t)) \varphi(t, \sigma(s))(F(s) \oplus G(s)) \)

\[ = (K(t) \otimes L(t)) \varphi(t, 0) \varphi(0, \sigma(s))(F(s) \oplus G(s)) \]

\[ = H_1(t) F_1(\sigma(s)) \]

where \( H_1(t) = (K(t) \otimes L(t)) \varphi(t, 0) \),

\[ F_1(\sigma(s)) = \varphi(0, \sigma(s))(F(s) \oplus G(s)) \]

and

\[ G_2(t, s) = \psi(t, s) = \psi(s, 0) \psi(0, t) \]

\[ = H_2(s) F_2(t) \]

Conversely, suppose that there exist matrices \( H_1, H_2, F_1 \) and \( F_2 \) with \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \), then the system

\[ (X(t) \otimes Y(t))^\Delta = (I \otimes F_1)(X(t) \otimes Y(t))(F_2 \otimes I) \]

and

\[ (R(t) \otimes S(t)) = (I \otimes H_1)(X(t) \otimes Y(t))(H_2 \otimes I) \]

is a realization of \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \). Since the transition matrices of the zero system (state equation (7.1.1) and with \( (A(t) \oplus C(t)) = (B(t) \oplus D(t)) = 0 \) are identity matrices.

Hence the Theorem.

The following Theorem gives a necessary and sufficient condition for minimal realization using controllability and observability.

**Theorem 7.4.2** The linear system (7.1.1) and (7.1.2) on time scales is a realization of the weighting patterns \( G_1(t, \sigma(s)) \) and \( G_2(t, s) \). Then this realization is minimal if and only if for some \( t_0 \) and \( t_1 > t_0 \), then the state equation is both controllable and observable on \( [t_0, t_1] \).

**Proof.** Writing the weighting pattern in terms of both realization products
(K(t) ⊗ L(t))(F(s) ⊗ G(s)) = H_1(t)F_1(s)
and
(F(s) ⊗ G(s))^* (K(t) ⊗ L(t))^* = F_1^*(s)H_1^*(t)
this implies

(K(t) ⊗ L(t))^* (K(t) ⊗ L(t))(F(s) ⊗ G(s))(F(s) ⊗ G(s))^*

= (K(t) ⊗ L(t))^* H_1(t)F_1(s)(F(s) ⊗ G(s))^* \forall t, s
for any t_0 and t_1 > t_0, it is possible to integrate this expression with respect to t and then
with respect to s to obtain

\[ \int_{t_0}^{t_1} (K(t) \otimes L(t))^* (K(t) \otimes L(t))(F(s) \otimes G(s))(F(s) \otimes G(s))^* \Delta t \]

= \[ \int_{t_0}^{t_1} (K(t) \otimes L(t))^* H_1(t)F_1(s)(F(s) \otimes G(s))^* \Delta t \]

W(t_0, t_1)M(t_0, t_1) = \[ \int_{t_0}^{t_1} (K(t) \otimes L(t))^* H_1(t) \Delta t \int_{t_0}^{t_1} F_1(s)(F(s) \otimes G(s))^* \Delta s \]
since the RHS is the product of n^2×m^2 and m^2×n^2 matrix and as such it cannot have full rank. Therefore M(t_0, t_1) W(t_0, t_1) cannot be simultaneously invertible. Further more this argument holds regardless as t_0 and t_1 > t_0, so that the state equation (7.1.1) and with

\[ (A(t) \oplus C(t)) = (B(t) \oplus D(t)) = 0 \]
cannot both controllable and observable on any interval. This argument is independent of the t_0 and t_1 chosen. A similar argument given above demonstrates with G_2(t, s) and so sufficiency is established.

Conversely, suppose that the given state equation (7.1.1) is a minimal realization of the weighting patterns G_1(t, \omega(s)) and G_2(t, s) with (A(t) \oplus C(t)) = (B(t) \oplus D(t)) = 0.

We begin by showing that if either

\[ W(t_0, t_1) = \int_{t_0}^{t_1} (K \otimes L)^*(s)(K \otimes L)(s) \Delta s \]
or

\[ M(t_0, t_1) = \int_{t_0}^{t_1} (F \otimes G)(s)(F \otimes G)^* (s) \Delta s \]

are in singular for all \( t_0, t_1 \). Then the minimality violated. By positive definiteness of the observability and controllability matrices \( M(t_0, t_1) \) and \( W(t_0, t_1) \) are invertible on \([t_0, t_1]\).

This shows that the minimal realization of state equation is both controllable and observable on \([t_0, t_1]\).