EXISTENCE OF EXPONENTIAL $\psi$-DICHOTOMY OF
MATRIX DYNAMICAL SYSTEMS ON TIME SCALES

Section 5.1

Exponential $\psi$-dichotomy generalizes the concept of hyperbolicity from autonomous to non autonomous dynamical systems, has been playing an ever more important role in the study of non autonomous dynamical systems such as ordinary differential equations, difference equations, and dynamic equations on time scales. Here we consider the first order matrix dynamical systems on time scales

$$X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t) + F(t) \quad X(v) = I_n; \quad v = \min\{T^+\}.$$  (5.1.1)

The corresponding homogeneous equation is

$$X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t) \quad X(v) = I_n; \quad v = \min\{T^+\}$$  (5.1.2)

where $A(t)$ and $B(t)$ are rd-continuous matrices of order $n \times n$, $F(t) \in C_{rd}(T, \mathbb{R}^{n \times n})$.

In this chapter first we develop some explicit necessary and sufficient criteria for the existence of exponential $\psi$-dichotomies of linear dynamic systems of the form

$$x^\Delta(t) = A(t)x(t) + f(t).$$  (5.1.3)

the corresponding homogeneous equation is

$$x^\Delta(t) = A(t)x(t)$$  (5.1.4)

on time scales. Where $A(t)$ is rd-continuous, regressive and bounded on $T^+ \cdot f$ is rd-continuous function. It is more interesting and more challenging to establish necessary
and sufficient criteria for the existence of exponential $\psi$-dichotomies of dynamical systems on general time scales.

In section 5.2, we introduce some basic preliminary results on the calculus on time scales and fundamental matrix solutions, also we establish necessary and sufficient criteria for the existence of exponential $\psi$-dichotomies for linear dynamic systems (5.1.4) on time scales.

In section 5.3 is devoted to develop relationship between the exponential $\psi$-dichotomy of the linear dynamical systems (5.1.4) and the $\psi$-bounded solutions of the non homogeneous dynamical system (5.1.3) on time scales.

In section 5.4 we extend the results obtained in section 5.2 to homogeneous matrix first order dynamical system (5.1.2) on time scales.

Finally, in section 5.5 we discuss the relationship between the exponential $\psi$-dichotomy of the linear matrix dynamical system (5.1.2) and the $\psi$-bounded solutions of the non homogeneous matrix systems (5.1.1) on time scales.

**Section 5.2**

In this section we, introduce the notion of exponential $\psi$-dichotomies on time scales and some basic results, Lemmas which are useful for later discussion.
The Euclidian norm of an \( n \times 1 \) vector \( x(t) \) is defined to be a real valued function of \( t \) and is denoted by \( \|x(t)\| = \sqrt{x^T(t)x(t)} \). The induced norm of an \( n \times n \) matrix \( A \) is defined to be

\[
|A| = \max \|Ax\| \quad \|x\| \leq 1
\]

The set of functions being both regressive and rd-continuous is denoted by

\[
\mathbb{R} = \mathbb{R}(T) = \mathbb{R}(T, R) = \mathbb{R}(T, R^{n \times n}).
\]

In this chapter, \( T \) is assumed to be unbounded above and below and

\[
\mathcal{G} = \min \{(0, \infty) \cap T\}, \quad T^+ = \{\mathcal{G} \cap T\}, \quad \mathcal{X} = \sup_{t \in T} \mu(t) \in [0, +\infty)
\]

\[
\|x\| = \sup_i x_i \quad x \in \mathbb{R}^n.
\]

Let \( \psi_i : T^+ \rightarrow (0, \infty), \ i = 1, 2, \ldots, n \), be rd continuous functions and

\[
\psi = \text{diag } [\psi_1, \psi_2, \ldots, \psi_n].
\]

The set of all functions being both regressive and rd-continuous is denoted by

\[
\mathbb{R} = \mathbb{R}(T) = \mathbb{R}(T, R^{n \times n}).
\]

**Definition 5.2.1** For \( p \in \mathbb{R} \), the inverse element is given by \( (\Theta p)(t) = \frac{-p(t)}{1 + \mu(t)p(t)} \).

**Definition 5.2.2** [63]. The dynamical system (5.1.4) is said to have an exponential \( \psi \)-dichotomy on \( T^+ \), if there exist a projection matrix \( P \) (i.e., \( P^2 = P \)) on \( \mathbb{R}^n \) and positive constants \( M_i \) and \( \alpha_i, \ i = 1, 2, \) such that

\[
| \psi(t)X(t)P^{-1}(\sigma(s))\psi^{-1}(s)| \leq M_1 e^{\Theta a_1(t, \sigma(s))} \quad t, s \in T^+ \text{ for } t \geq \sigma(s)
\]

\[
| \psi(t)X(t)(I-P)X^{-1}(\sigma(s)\psi^{-1}(s))| \leq M_2 e^{\Theta a_2(\sigma(s), t)} \quad s, t \in T^+ \text{ for } t \leq \sigma(s) \quad (5.2.1)
\]
where X is a fundamental solution matrix of (5.1.4) and I is the identity matrix. When (5.2.1) holds with $\alpha_1 = \alpha_2 = 0$, the system (5.1.4) is said to possess an ordinary $\psi$-dichotomy.

**Remark 5.2.1** We can choose an appropriate fundamental solution matrix such that the projections $P$ and $I-P$ can be written as

$$I_{ko} = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, I_{o(n-k)} = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-k} \end{bmatrix},$$

respectively, where $I_k$ is a $k \times k$ identity matrix and $I_{n-k}$ is an $(n-k) \times (n-k)$ identity matrix. In fact, there exists a non-singular matrix $B$ such that $P = BI_{ko}B^{-1}$, then (5.2.1) reduces to

$$\left| \psi(t)X(t)BI_{ko}B^{-1}X^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq M_1 e^{\Theta \alpha_1 (t, \sigma(s))}, \quad t \geq \sigma(s)$$

for all $t, s \in T^+$. Let $X_0(t) = X(t)B$, then it is easily seen that $X_0$ is also a fundamental matrix.

In addition, we also obtain the following fact in (5.2.1). If $\chi > 0$, then for any $x \in (0, \chi]$ and $\alpha > 0$, $f_1(x) := (1/x) \log(1/(1+\alpha x))$ is strictly increasing with $\lim_{x \to 0^+} f_1(x) = -\alpha$ and $f_2(x) := (1/x) \log (1 + \alpha x)$ is strictly decreasing satisfying $\lim_{x \to 0^+} f_2(x) = \alpha$.

Therefore, for $t \geq s$, we have

$$e^{\alpha(t-s)} \geq e_\alpha(t, s) \geq (1 + \alpha \chi)^{t-s/\chi}, \quad e^{-(t-s)} \leq e_\alpha(t, s) \leq (1 + \alpha \chi)^{(t-s)/\chi} \quad (5.2.3)$$

**Lemma 5.2.1** The dynamical system (5.1.4) has an exponential $\psi$-dichotomy on $T^+$ if the following conditions are satisfied:

(i) there exist positive constants $L_i$ and $\alpha_i$ $(i = 1, 2)$ such that
\[\parallel \psi(t)X(t)P\xi \parallel \leq L_1 e^{\Theta a_1(t, \sigma(s))} \parallel \psi(s)X(\sigma(s))P\xi \parallel, \ t \geq \sigma(s)\]
\[\parallel \psi(t)X(t)(I-P)\xi \parallel \leq L_2 e^{\Theta a_2(\sigma(s), t)} \parallel \psi(s)X(\sigma(s))(I-P)\xi \parallel, \ t \leq \sigma(s), \quad (5.2.4)\]

where \(\xi\) is an arbitrary n-dimensional vector;

(ii) the dynamical system (5.1.4) has bounded growth, that is, there exist \(K \geq 1\) and \(\beta > 0\) such that
\[|\psi(t)X(t)X^{-1}(\sigma(s))\psi^{-1}(s)| \leq Ke^\beta(t, \sigma(s)), \ t \geq \sigma(s). \quad (5.2.5)\]

The following Theorem is a useful property of the exponential \(\psi\)-dichotomy on time scales.

**Theorem 5.2.1** If the dynamical system (5.1.4) has an exponential \(\psi\)-dichotomy on
\[[t_0, \infty) = [t_0, \infty) \cap T^+\] for some fixed \(t_0 \geq \theta\), then it has also an exponential \(\psi\)-dichotomy on \(T^+\) with the same projection \(P\) and the same exponents \(a_1, a_2\).

**Proof.** Choose an \(K_1 \geq 1\) such that \(K_1 \geq e^{\beta|\lambda|(t_0, \theta)}\). Then we have
\[|\psi(t)X(t)X^{-1}(\sigma(s))\psi^{-1}(s)| \leq K_1\] for \(\theta \leq \sigma(s), \ t \leq t_0\). To obtain the conclusions, we consider the following two cases:

**Case 1:** If \(\theta \leq \sigma(s) \leq t_0 \leq t\), then
\[|\psi(t)X(t)P\psi^{-1}(\sigma(s))\psi^{-1}(s)| \leq K_1 |\psi(t)X(t)PX^{-1}(t_0)\psi^{-1}(t_0)|\]
\[\leq K_1 M_1 e^{\Theta a_1(t, t_0)} = K_1 M_1 e^{\Theta a_1(t, \sigma(s)) e^{\Theta a_1(\sigma(s), \theta)} e^{\Theta a_1(\theta, t_0)}}\]
\[\leq K_1 M_1 e^{\Theta a_1(t_0, \theta)} e^{\Theta a_1(t, \sigma(s))};\]

**Case 2:** If \(\theta \leq \sigma(s) \leq t \leq t_0\), then
\[|\psi(t)X(t)PX^{-1}(\sigma(s))\psi^{-1}(s)| \leq K^2_1 |\psi(t_0)X(t_0)PX^{-1}(t_0)\psi^{-1}(t_0)| \leq K^2_1 M_1\]
\[\leq K^2_1 M_1 e^{\Theta a_1(t_0, t_0)} = K^2_1 M_1 e^{\Theta a_1(t, \sigma(s)) e^{\Theta a_1(\sigma(s), \theta)} e^{\Theta a_1(\theta, t_0)}}\]
\[\leq K^2_1 M_1 e^{\Theta a_1(t_0, \theta)} e^{\Theta a_1(t, \sigma(s))}.\]
Therefore,
\[ | \psi(t)X(t)P\psi^{-1}(\sigma(s))\psi^{-1}(s) | \leq M_1^* e^\theta_{a1}(t, \sigma(s)), \quad \theta \leq \sigma(s) \leq t, \]
where \( M_1^* = K_1^2 M_1 e^\theta_{a1}(t_0, \theta). \)

Similarly, we have
\[ | \psi(t)X(t)(I-P)\psi^{-1}(\sigma(s))\psi^{-1}(s) | \leq M_2^* e^\theta_{a2}(\sigma(s), t), \quad \theta \leq t \leq \sigma(s) \]
where \( M_2^* = N_1^2 M_2 e^\theta_{a2}(t_0, \theta). \)

Using the above theorems we develop some necessary and sufficient criteria for the linear dynamic system (5.1.4) to have an exponential \( \psi \)-dichotomy.

**Theorem 5.2.2** Assume that \( A \in \mathbb{R} \) is bounded. The linear dynamical system (5.1.4) has an exponential \( \psi \)-dichotomy on \( T^+ \) if and only if there exist positive constants \( 0 < \theta < 1, w > 0 \) such that any solution \( x(t) \) of (5.1.4) satisfies

\[ \| \psi(t)x(t) \| \leq \theta \sup_{|t-r| \leq w} \| \psi(t)x(t) \|, \quad t \geq w \] (5.2.6)

**Proof.** Suppose the dynamical system (5.1.4) has an exponential \( \psi \)-dichotomy on \( T^+ \), then it follows from Lemma 5.2.1 that (5.2.4) holds on \( T^+ \).

Let \( x(t) \) be any solution of (5.1.4) and set
\[ x_1(t) = X(t)P\psi^{-1}(\sigma(t))x(t), \quad x_2(t) = X(t)(I-P)\psi^{-1}(\sigma(t))x(t), \]
then
\[ x(t) = X(t)P\psi^{-1}(\sigma(s))x_1(s) + X(t)(I-P)\psi^{-1}(\sigma(s))x_2(s). \]

Consider the following two cases:

**Case 1:** If \( \| \psi(s)x_2(s) \| \geq \| \psi(s)x_1(s) \| \), then, for \( t \geq s \), we have
\[ \| \psi(t)x(t) \| \geq \| \psi(t)X(t)(I-P)\psi^{-1}(\sigma(s))x_2(s) \| - \| \psi(t)X(t)P\psi^{-1}(\sigma(s))x_1(s) \|. \]

By the second inequality of (5.2.4), we have
\[ \| \psi(t)X(t)(I-P)\xi \| \geq L_2^{-1} \| \psi(s)X(s)(I-P)\xi \| e_{a2}(t, \sigma(s)) \text{ for } t \geq \sigma(s) \geq \theta. \]
Choosing $\xi = X^{-1}(\sigma(s)) x_2(s)$, for $t \geq \sigma(s) \geq \theta$, we obtain

$$|\psi(t)X(t)(I-P)X^{-1}(\sigma(s)) x_2(s)| \geq L^{-1}_2 \| \psi(s)X(s)(I-P)X^{-1}(\sigma(s)) x_2(s)\|_e a_2(t, \sigma(s))$$

$$= L^{-1}_2 \| x_2(s)\| e a_2(t, \sigma(s)).$$

For sufficiently large $t$, we have

$$\|\psi(t)x(t)\| \geq L^{-1}_2 e a_2(t, s) \| \psi(s)x_2(s)\| - L_1 e \Theta a_1(t, s) \| \psi(s)x_1(s)\|$$

$$\geq (L^{-1}_2 e a_2(t, s) - L_1 e \Theta a_1(t, s)) \| \psi(s)x_2(s)\|$$

$$\geq (1/2)(L^{-1}_2 e a_2(t, s) - L_1 e \Theta a_1(t, s)) \| \psi(s)x(s)\|.$$

**Case 2:** If $\| \psi(s)x_1(s)\| \geq \| \psi(s)x_2(s)\|$, similarly, for $s \geq t \geq \theta$, we get

$$\|\psi(t)x(t)\| \geq (1/2)(L^{-1}_1 e a_1(s, t) - L_2 e \Theta a_2(s, t)) \| \psi(s)x(s)\|.$$

This means that there exist $0 < \theta < 1$ and $w > 0$ such that

$$L^{-1}_2 e a_2(\tau + w, \tau) - L_1 e \Theta a_1(\tau + w, \tau) \geq 2\theta^{-1},$$

$$L^{-1}_1 e a_1(\tau + w, \tau) - L_2 e \Theta a_2(\tau + w, \tau) \geq 2\theta^{-1}.$$

Then $\| \psi(t)x(t)\| \leq \theta \sup_{|\tau-t| \leq w} \| \psi(\tau)x(\tau)\|$, $t \geq w$.

Conversely, assume that (5.2.6) holds. First, we show that there exists a constant $c > 1$ such that

$$\| \psi(t)x(t)\| \leq c \| \psi(s)x(s)\|$$

for $\theta \leq s \leq t \leq s + w$, where $x(t)$ is any nontrivial solution of (5.1.4). According to the condition, there exists an $N > 0$ such that $|A(t)| \leq N$ for any $t \in T^+$. It is easy to show that

$$\| \psi(t)X(t)X^{-1}(\sigma(s)) \xi \| \leq c_M(t, \sigma(s))\|\xi\|$$

for $t \geq \sigma(s)$.

Let $\xi = \psi(s)X(s)\xi^*$. For $\theta \leq \sigma(s) \leq t \leq \sigma(s) + w$, we have $\| \psi(t)X(t)\xi^* \| \leq c_N(\sigma(s) + w, \sigma(s))$, $\| \psi(s)X(s)\xi^* \| \leq c^Nw \| \psi(s)X(s)\xi^* \|$, that is,
\[ \| \psi(t)x(t) \| \leq c \| \psi(s)x(s) \| , \text{ where } c = e^{Nw}. \]

Suppose that \( x(t) \) is a nontrivial \( \psi \)-bounded solution of (5.1.4).

Set \( \pi(s) = \sup_{\tau \geq s} \| \psi(\tau)x(\tau) \| \) for \( s \geq 0 \), we have

\[ \| \psi(t)x(t) \| \leq \theta \sup_{|\tau-t| \leq w} \| \psi(\tau)x(\tau) \| \leq \theta \pi(s) , \ t \geq s + w. \]

Hence \( |\pi(s)| = \sup_{s \leq \tau \leq s+w} \| \psi(\tau)x(\tau) \| \), which implies that

\[ \| \psi(t)x(t) \| \leq c \| \psi(s)x(s) \| , \ 0 \leq s \leq t < \infty. \]

If \( s + nw \leq t \leq s + (n+1)w \), then

\[ \| \psi(t)x(t) \| \leq \theta^n \sup_{|\tau-t| \leq nw} \| \psi(\tau)x(\tau) \| \leq \theta^n c \| \psi(s)x(s) \| \leq \theta^{-1} c \theta^{(t-s)/w} \| x(s) \|. \]

Set \( K = \theta^{-1} c \) and \( \alpha = -(1/w) \log \theta \). Then we get

\[ \| \psi(t)x(t) \| \leq e^{-\alpha(t-s)} \| \psi(s)x(s) \| \leq e^{-\Theta(t,s)} \| \psi(s)x(s) \| , \quad 0 \leq s \leq t < \infty. \]

Carrying out arguments similar to those in Proposition 2.1 in [18], it is easily shown that there exists a \( w^* > 0 \) such that

\[ \| \psi(t)x(t) \| \leq e^{-\Theta(t,s)} \| \psi(s)x(s) \| \text{ for } w^* \leq t \leq s < \infty. \]

Since \( A \) is bounded, then (5.1.4) has \( \psi \)-bounded growth. From Lemma 5.2.1 and Theorem 5.2.1, The dynamical system (5.1.4) has an exponential \( \psi \)-dichotomy on \( T^+ \).

Section 5.3

Now we discuss the relationship between the exponential \( \psi \)-dichotomy of the linear dynamical systems (5.1.4) and the \( \psi \)-bounded solutions of the non homogeneous linear dynamical systems.
\[ x^A(t) = A(t)x(t) + f(t) \]  

(5.3.1)

where \( A \in \mathcal{R} \), \( f \in \mathcal{C}_d(T^+, \mathbb{R}^n) \), corresponding to (5.1.4). Some necessary and sufficient conditions are derived for (5.1.4) to have an exponential \( \psi \) -dichotomy.

Define

\[ C_\psi = \{ f \in \mathcal{C}_d(T^+) : = \| f \|_{C_\psi}^T = \sup_{t \in T^+} \| \psi(t)f(t) \| \}, \]

then

\[ E_\psi = \left\{ f \in C_d(T^+) : \| f \|_{E_\psi} = \sup_{t \in T^+} \int_t^{t+\omega} \| \psi(\tau)f(\tau) \| d\tau, \text{ where } T^+ \text{ is } \omega \text{-periodic with } \omega > 0 \right\} \]

clearly \( C_\psi \), \( D_\psi \) and \( E_\psi \) are Banach spaces.

**Lemma 5.3.1** If \( g \in E_\psi \) is a non-negative function with

\[ \frac{1}{\omega} \int_t^{t+\omega} g(\tau) \Delta \tau \leq K_2 \text{ for all } t \geq \vartheta, \]

Then the following conditions

(i) \[ \int_\vartheta^t e^{\Theta_2\omega}(t, \sigma(\tau))g(\tau) \Delta \tau \leq \frac{K_2\omega(1+\alpha_2\vartheta)}{1-e^{\Theta_2\omega} (\vartheta+\omega, \vartheta)} \]

(ii) \[ \int_t^\infty e^{\Theta_1\omega}(\sigma(\tau), t)g(\tau) \Delta \tau \leq \frac{K_2\omega}{1-e^{\Theta_1\omega} (\vartheta+\omega, \vartheta)} \]

hold for \( \alpha_1, \alpha_2 > 0 \) and \( t \geq \vartheta \).

The following Lemma is useful. We first assume that \( U_1 \) is the subspace of \( \mathbb{R}^n \) consisting of the initial values of all \( \psi \) -bounded solutions of (5.1.4), and
U₂ is any fixed subspace of \( \mathbb{R}^n \) supplementary to U₁ such that \( \mathbb{R}^n \) can be written as the direct sum \( \mathbb{R}^n = U₁ \oplus U₂ \).

**Lemma 5.3.2** If the system (5.3.1) has a \( \psi \) - bounded solution for \( f \in B_\psi \), where \( B_\psi \) denotes any one of the Banach spaces \( C_\psi \), \( D_\psi \) and \( E_\psi \) then there exists a positive constant \( r_{B_\psi} \) such that, for every \( f \in B_\psi \), the unique \( \psi \) -bounded solution \( z(t) \) of (5.3.1) with \( z(\theta) \in U₂ \) satisfies

\[ \| z \|_{C_\psi} \leq r_{B_\psi} \| f \|_{B_\psi}. \]

**Theorem 5.3.1** Assume that \( A \in \mathbb{R} \) is bounded. Then the system (5.1.4) has an ordinary \( \psi \) -dichotomy on \( T^+ \) if and only if (5.3.1) has at least one \( \psi \) -bounded solution for every \( f \in D_\psi \).

**Proof:** Assume that the dynamical system (5.1.4) has an ordinary \( \psi \) -dichotomy on \( T^+ \).

Then it is easy to show that

\[ x(t) = \int_{\theta}^{t} \psi(t) X(t)PX^{-1}(\sigma(s)) f(s) \Delta s - \int_{t}^{\infty} \psi(t) X(t)(I - P)X^{-1}(\sigma(s)) f(s) \Delta s \]

is a \( \psi \) -bounded solution of (5.3.1) and \( \| \psi(t) x(t) \| \leq \max \{ M₁, M₂ \} \| f \|_{B_\psi} \) for all \( t \in T^+ \).

Conversely, suppose that (5.3.1) has at least one \( \psi \) -bounded solution for every \( f \in D_\psi \).

Set

\[
H(t, \sigma(s)) = \begin{cases} 
\psi(t) X(t)PX^{-1}(\sigma(s)) & \text{for } t > \sigma(s) \geq \theta \\
-\psi(t) X(t)(I - P)X^{-1}(\sigma(s)) & \text{for } \sigma(s) > t \geq \theta
\end{cases}
\]

where \( X(t) \) is a fundamental solution matrix of (5.1.4) with \( X(\theta) = I \).
Let $z(t) = \int_0^\infty H(t, \sigma(t)) f(\tau) \Delta \tau$. For a fixed $t_1 \in T^+$, choose a function $f \in C_\psi$ which vanishes for $t \geq t_1$. Since

$$\psi(t)z(t) = \psi(t)X(t)P \int_0^{t_1} X^{-1}(\sigma(\tau)) f(\tau) \Delta \tau; t \geq t_1$$

and

$$z(\mathcal{G}) = -(I - P) \int_0^{t_1} X^{-1}(\sigma(\tau)) f(\tau) \Delta \tau \in U_2$$

then $z(t) = \int_0^{t_1} H(t, \sigma(t)) f(\tau) \Delta \tau$ is $\psi$-bounded solution of (5.3.1). By Lemma 5.3.2, we have $\|z\|_{C_\psi} \leq r_L \|f\|_{D_\psi}$. For any fixed point $s \in T^+$, we have three cases, namely (i) is right-dense (ii) is both right-scattered and left-scattered (iii) is right-scattered and left-dense. Then, it easily seen that

$$| \psi(t)X(t)P X^{-1}(\sigma(s)) \psi^{-1}(s) | \leq r_{D_\psi} (1 + \chi \|A\|_{C_\psi}) \text{ for } \sigma(s) < t$$

$$| \psi(t)X(t)(I - P) X^{-1}(\sigma(s)) \psi^{-1}(s) | \leq r_{D_\psi} (1 + \chi \|A\|_{C_\psi}) \text{ for } \sigma(s) > t. \quad (5.3.3)$$

From the rd-continuity of $\psi(t)X(t)$, it follows that (5.3.3) is also valid for $\sigma(s) = t$.

**Theorem 5.3.2** Assume that the dynamical system (5.1.4) has $\psi$-bounded growth. Then (5.1.4) has an exponential $\psi$-dichotomy on $T^+$ if and only if (5.3.1) has at least one $\psi$-bounded solution for every $f \in C_\psi$.

**Proof:** Assume that (5.1.4) has an exponential $\psi$-dichotomy on $T^+$. Then (5.3.2) is a solution of (5.3.1) and
\[
\|\psi(t)x(t)\| = \|f\|_{C_{\psi}} \left( \int_{0}^{t} \|\psi(t)x(t)\| \, d\tau \right) \\
\triangleq \|f\|_{C_{\psi}} \left( M_{1} \int_{0}^{t} \mathbb{E}(\alpha, \psi, \tau) \, d\tau + M_{2} \int_{0}^{t} \mathbb{E}(\psi, \tau) \, d\tau \right) \\
\triangleq \|f\|_{C_{\psi}} \left( \frac{M_{1}(1+\alpha)\mathbb{E}}{\alpha} + \frac{M_{2}\mathbb{E}}{\alpha} \right).
\]

Conversely, suppose that (5.3.1) has at least one \( \psi \)-bounded solution for every function \( f \in C_{\psi} \). For a fixed \( \mathbb{E} \in \mathcal{T}^{+} \), choose a rd-continuous function \( \eta \) such that

\[
0 \leq \eta(t) \leq 1 \quad \text{for all } t \geq \vartheta \quad \text{and} \quad \eta(t) = 0 \quad \text{for } t \geq \mathbb{E}.
\]

Set \( f(t) = \eta(t) \|\psi(t)x(t)\|^{1} \), where \( \psi(t)x(t) = \psi(t)X(t)\xi \) is any nontrivial solution of (5.3.1). Clearly \( \|f\|_{C_{\psi}} \leq 1 \),

which implies that

\[
\int_{0}^{q} H(t, \psi(t)) \|X(t)\psi(t)\|^{1} \, d\tau \leq r_{C} \quad \text{for } q \geq t_{0} \geq \vartheta.
\]

If \( \mathbb{E} = t \) for \( t = t_{1} \) then

\[
\|\psi(t)X(t)P\xi\|^{1} \leq r_{C} \quad \text{for } t \geq t_{0} \geq \vartheta \quad (5.3.4)
\]

\[
\|\psi(t)X(t)(I-P)\xi\|^{1} \leq r_{C} \quad \text{for } t \geq t_{0} \geq \vartheta.
\]

Replacing \( \xi \) by \( P\xi \) or \( (I-P)\xi \), we get

\[
(i) \int_{t_{0}}^{s} \|\psi(t)X(t)P\xi\|^{-1} \, d\tau \leq e_{\mathbb{E}}r_{C} \quad (i) \int_{t_{0}}^{t} \|\psi(t)X(t)P\xi\|^{-1} \, d\tau \quad \text{for } t \geq s \geq t_{0}
\]

\[
(5.3.5)
\]
According to the condition, the dynamic system (5.1.4) has \( \psi \)-bounded growth, then there exist a \( K \geq 1 \) and a \( \beta > 0 \) such that \( |\psi(t) X(t) X^{-1}(\sigma(s)) \psi^{-1}(s)| \leq Ke_{\beta} (t, \sigma(s)) \) for \( t \geq \sigma(s) \). Assume that \( x \) is any solution of (5.1.4) and let \( x_1(t) = X(t)PX^{-1}(\sigma(s))x(s) \), \( x_2(t) = X(t)(I - P)X^{-1}(\sigma(s))x(s) \).

Next we show that \( \|x_1(t)\| \leq e_{K}\|x(s)\| e_{\Theta rC}^{-1} (t, s) \) for \( s \leq t < \infty \)

if \( \|\psi(t) x_1(t)\| \leq K\|\psi(s) x(s)\| \) for some fixed \( s \geq \vartheta \) and \( s \leq t \leq s + rC \).

Let \( t^* = \inf \{t \in T^+ : t \geq s + rC \} \) since \( x \) is a solution of (5.1.4) then \( x(t) = \psi(t) X(t) \xi \). Replacing \( t_0 \) by \( s \) and \( s \) by \( t^* \) in the first inequality of (5.3.5), we obtain

\[
\frac{r_C}{K\|\psi(s) x(s)\|} \leq \int_{s}^{t^*} \|\psi(t) x_1(t)\|^{-1} \Delta t \leq e_{\Theta rC}^{-1} (t, s) \int_{s}^{t^*} \|\psi(t) x_1(t)\|^{-1} \Delta t \quad t \geq s + rC.
\]

By the first inequality of (5.3.4), we have

\[
\|\psi(t) x_1(t)\| \leq r_C \int_{s}^{t} \|\psi(t) x_1(t)\|^{-1} \Delta t \leq e_{K}\|\psi(s) x(s)\| e_{\Theta rC}^{-1} (t, s) \quad t \geq s + rC.
\]

Note that

\[
e_{\Theta rC}^{-1} (t, s) \geq e_{\Theta rC}^{-1}(t-s) \geq 1; \quad s \leq t \leq s + rC.
\]

This implies that

\[
\|\psi(t) x_1(t)\| \leq e_{K}\|\psi(s) x(s)\| e_{\Theta rC}^{-1} (t, s) \quad s \leq t < \infty.
\]

Similarly, if \( \|\psi(t) x_2(t)\| \leq K\|\psi(s) x(s)\| \) for some fixed \( s \geq \vartheta \) and \( \max \{ \vartheta, s-rC \} \leq t \leq s \), we have
Replacing $\xi$ by $X^{-1}(s) \psi^{-1}(s)$ $\xi$ and $w \to \infty$ in the second inequality of (5.3.5), we get

$$\left\| \psi(t)X(t)(I-P)X^{-1}(\sigma(s))\psi^{-1}(s) \right\| \leq r_C \left\{ \int_s^\infty \left( \left\| \psi(t)X(t)X^{-1}(\sigma(s))\psi^{-1}(s) \right\| \right)^{-1} \Delta t \right\} \leq r_C [K^{-1}]^{-1} \int_s^\infty e^{\beta(s,\tau)}^{-1} \Delta t, \quad t \leq \sigma(s).$$

Since $\xi$ is arbitrary, we obtain

$$\left| \psi(t)X(t)(I-P)X^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta K, \quad t \leq \sigma(s)$$

similarly

$$\left| \psi(t)X(t)PX^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta Ke^{-\beta}(t,\sigma(s)), t \geq \sigma(s).$$

Hence

$$\left| \psi(t)X(t)PX^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq (1 + r_C \beta)Ke^{-\beta}(t,\sigma(s)), t \geq \sigma(s).$$

Let $t_0 = s$, then by the first inequality of (5.3.5), we have

$$\left| \psi(t)X(t)PX^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta K[1 - e^{\beta}(t,s)]^{-1}, t \geq \sigma(s).$$

Now we have the following two cases

(1) $\chi = 0$, from Proposition 2.1[18], then

$$\left| \psi(t)X(t)PX^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq (1 + 2r_C \beta)K, t \geq \sigma(s).$$

(2) if $\chi > 0$, it follows from (5.2.3) that

$$\left| \psi(t)X(t)PX^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta K[1 - \frac{1}{(1 + \beta \chi)^{t-s/\chi}}]^{-1}, t > \sigma(s)$$

then we get
\[ |\psi(t)X(t)P^{-1}(\sigma(s))\psi^{-1}(s)| \leq r_C K(1+\beta \chi e^\beta(t,s)) \leq K(1+r_C \beta)e^{\beta(t-s)} \]

\[ \leq K(1+r_C \beta)e^{\beta \chi} . \]

Hence, we have

\[ |\psi(t)X(t)P^{-1}(\sigma(s))\psi^{-1}(s)| \leq \max \left\{ \frac{r_C K(1+\beta \chi)}{\chi}, K(1+r_C \beta)e^{\beta \chi} \right\}, t \geq \sigma(s). \]

Define

\[ N(\chi) = \begin{cases} 
(1+2r_C \beta)K & \text{if } \chi = 0, \\
\max \left\{ \frac{r_C K(1+\beta \chi)}{\chi}, K(1+r_C \beta)e^{\beta \chi} \right\}, & \text{if } \chi > 0.
\end{cases} \]

Then

\[ |\psi(t)X(t)P^{-1}(\sigma(s))\psi^{-1}(s)| \leq N(\chi) \quad \text{for} \quad t \geq s \sigma(s) \]

It follows from (5.3.6) and (5.3.7) that

\[ |\psi(t)X(t)P^{-1}(\sigma(s))\psi^{-1}(s)| \leq e_{N(\chi)}e_{\Theta^{-1}_e}(t,\sigma(s)) \quad \text{for} \quad t \geq \sigma(s) \geq \mathcal{D} \]

\[ |\psi(t)X(t)(1-P)X^{-1}(\sigma(s))\psi^{-1}(s)| \leq e_{r_C \beta K e_{\Theta^{-1}_e}(\sigma(s),t)} \quad \text{for} \quad \sigma(s) \geq t \geq \mathcal{D}. \]

This implies that (5.1.4) has an exponential \( \psi \)-dichotomy on \( T^+ \).
Section 5.4

In this section we extend the results obtained in sections 5.2 and 5.3 to matrix first order dynamical systems on time scales. Here we develop some necessary and sufficient criteria for the existence of exponential \( \psi \)-dichotomy for the matrix differential system

\[
X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t) \quad X(v) = I_n \quad v = \min\{T^+\}
\]

(5.4.1)

where \( A(t) \) and \( B(t) \) are regressive and rd-continuous matrices of order \( n \times n \). The solution of (5.4.1) can be expressed in terms of two fundamental matrix solutions of the system \( \phi(t,t_0) \) and \( X^\Delta(t) = B(t)^*X(\sigma(t)) \).

Lemma 5.4.1 Let \( Y(t) \) and \( Z(t) \) are fundamental matrices of the systems \( X^\Delta(t) = A(t)X(t) \) with \( Y(v) = I_n \) and \( X^\Delta(t) = B(t)^*X(\sigma(t)) \) with \( Z(v) = I_n \), then the matrix \( Y_CZ^* \) is the fundamental matrix of (5.4.1) (where \( C \) is constant matrix of order \( n \)).

Definition 5.4.1 The dynamical system (5.4.1) is said to have an exponential \( \psi \)-dichotomy on \( T^+ \), if there exist a projection matrix \( P_1 \) and \( P_2 \) (i.e., \( P_1 + P_2 = I \) and \( P_i^2 = P_i \), \( i = 1, 2 \)) on \( \mathbb{R}^{n \times n} \) and positive constants \( M_i \) and \( \alpha_i \), \( i = 1, 2 \), such that

\[
|\psi(t)Y(t)Z(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq M_1 e^{\alpha_1}(t, \sigma(s)), t \geq \sigma(s),
\]

\[
|\psi(t)Y(t)Z(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq M_2 e^{\alpha_2}(\sigma(s), t), t \leq \sigma(s), t, s \in T^+
\]

(5.4.2)

where \( YZ^* \) is a fundamental solution matrix of (5.4.1). When (5.4.2) holds with \( \alpha_1 = \alpha_2 = 0 \), then (5.4.1) is said to possess an ordinary \( \psi \)-dichotomy.

In fact, there exists a non-singular matrix \( B \) such that \( P_1 = B I_{n_0} B^{-1} \) and \( P_2 = B I_{(n-k)} B^{-1} \) then (5.4.2) reduces to

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\[|\psi(t)Y(t)Z^*(t) B I_{k_0} B^{-1}(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq M_1 e^{\Theta s_1}(t, \sigma(s)), t \geq \sigma(s)\]

\[|\psi(t)Y(t)Z^*(t) B I_{(n-k)} B^{-1}(Z^*)^{-1}\sigma(s)Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq M_2 e^{\Theta s_2}(\sigma(s), t), t \leq \sigma(s).\]

Then it is easily seen that \((YZ^*)_0(t)=(YZ^* B)\) is also a fundamental matrix.

**Lemma 5.4.2** The dynamical system (5.4.1) has an exponential \(\psi\) -dichotomy on \(T^+\) if the following conditions are satisfied:

(i) there exist positive constants \(L_i\) and \(\alpha_i\) (\(i = 1, 2\)) such that

\[|\psi(t)Z^*(t)Y(t)P_1 \xi| \leq L_1 e^{\Theta \alpha_1}(t, \sigma(s))|\psi(s)Y(s)(Z(s)^*P_1 \xi)|, t \geq \sigma(s)\]

\[|\psi(t)Y(t)Z^*(t)P_2 \xi| \leq L_2 e^{\Theta \alpha_2}(\sigma(s), t)|\psi(s)Y(s)Z^*(s))P_2 \xi|, t \leq \sigma(s), \quad (5.4.3)\]

where \(\xi\) is an arbitrary \(n\)-dimensional vector;

(ii) the dynamical system (5.4.1) has \(\psi\) -bounded growth, that is, there exist \(K \geq 1\) and \(\beta > 0\) such that

\[|\psi(t)Y(t)Z^*(t) (Z^{-1})^*(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq K e^{\beta}(t, \sigma(s)), t \geq \sigma(s). \quad (5.4.4)\]

**Theorem 5.4.1** If the dynamical system (5.4.1) has an exponential \(\psi\) -dichotomy on \([t_0, \infty)=[t_0, \infty) \cap T\) for some fixed \(t_0 \geq \theta\), then it has also an exponential \(\psi\) -dichotomy on \(T^+\) with the same projections \(P_1\) and \(P_2\) and the same exponents \(\alpha_1, \alpha_2\).

**Proof.** Choose an \(K_1 \geq 1\) such that \(K_1 \geq e_{|A|}(t_0, \theta)\). Then we have

\[|\psi(t)Y(t)Z^*(t) (Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq K_1 \text{ for } \theta \leq \sigma(s), t \leq t_0.\]

To obtain the conclusions, we consider the following two cases:

**Case 1:** If \(\theta \leq \sigma(s) \leq t_0 \leq t\), then

\[|\psi(t)Y(t)Z^*(t) P_1 (Z^{-1})^*(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq K_1|\psi(t_0)Y(t_0)Z^*(t_0) P_1 (Z^{-1})(t_0)Y^{-1}(t_0)\psi^{-1}(t_0)|\]

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\[ \leq K_1 M_1 e_{\Theta_{al}}(t, t_0) = K_1 M_1 e_{\Theta_{al}}(t, \quad \sigma(s)) e_{\Theta_{al}}(\sigma(s), \vartheta) e_{\Theta_{al}}(\vartheta, t_0) \]

\[ \leq K_1 M_1 e_{al}(t_0, \vartheta) e_{\Theta_{al}}(t, \quad \sigma(s)); \]

**Case 2:** If \( \vartheta \leq \sigma(s) \leq t \leq t_0 \), then

\[ | \psi(t) Y(t) Z^*(t) P_1 (Z^*)^{-1} \big( \sigma(s) \big) Y^{-1}(\sigma(s)) \psi^{-1}(s) | \]

\[ \leq K^2 \quad \left| \psi(t_0) Y(t_0) Z(t_0) P_1 (Z^*)^{-1}(t_0) Y^{-1}(t_0) \psi^{-1}(t_0) \right| \]

\[ \leq K^2 M_1 e_{\Theta_{al}}(t_0, t) = K^2 M_1 e_{\Theta_{al}}(t, \sigma(s)) e_{\Theta_{al}}(\sigma(s), \vartheta) e_{\Theta_{al}}(\vartheta, t_0) \]

\[ \leq K^2 M_1 e_{\Theta_{al}}(t_0, \vartheta) e_{\Theta_{al}}(t, \sigma(s)). \]

Therefore,

\[ | \psi(t) Y(t) Z(t) P_1 (Z^*)^{-1} \big( \sigma(s) \big) Y^{-1}(\sigma(s)) \psi^{-1}(s) | \leq M^*_1 \quad e_{\Theta_{al}}(t, \quad \sigma(s)), \quad \vartheta \leq \sigma(s) \leq t, \]

where \( M^*_1 = K^2 M_1 e_{\Theta_{al}}(t_0, \vartheta). \)

Similarly, we have

\[ | \psi(t) Y(t) Z^*(t) P_2 (Z^*)^{-1} \big( \sigma(s) \big) Y^{-1}(\sigma(s)) \psi^{-1}(s) | \leq M^*_2 \quad e_{\Theta_{al}}(\sigma(s), t), \quad \vartheta \leq t \leq \sigma(s), \]

where \( M^*_2 = N^2 M_2 e_{\Theta_{al}}(t_0, \vartheta). \)

**Theorem 5.4.2** Assume that \( A, B \in \mathbb{R} \) are bounded. The dynamical system (5.4.1) has an exponential \( \psi \) -dichotomy on \( T^+ \) if and only if there exist positive constants \( 0 < \theta < 1, \)

\( w > 0 \) such that any solution \( X(t) \) of (5.4.1) satisfies

\[ \| \psi(t) x(t) \| \leq \theta \quad \sup_{|t-t'| \leq w} \| \psi(t) x(t) \|, \quad t \geq w. \quad (5.4.5) \]

**Proof.** Suppose the system (5.4.1) has an exponential \( \psi \) -dichotomy on \( T^+ \), then it follows from Lemma 5.4.1 that (5.4.3) holds on \( T^+ \).

Let \( X(t) \) be any solution of (5.4.1) and set

\[ X_1(t) = Y(t) Z^*(t) P_1 (Z^*)^{-1}(t) Y^{-1}(t) X(t), \quad X_2(t) = Z^*(t) Y(t) P_2 (Z^*)^{-1}(t) Y^{-1}(t) X(t), \]

then
\[ X(t) = Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_1(s) \]
\[ + Y(t)Z^*(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_2(s) \]

Consider the following two cases:

**Case 1:** If \(|\psi(s)X_2(s)| \geq |\psi(s)X_1(s)|\), then, for \(t \geq s\), we have
\[ |\psi(t)X(t)| \geq |\psi(t)Y(t)Z^*(t)P_2Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_2(s)| \]
\[ - |\psi(t)Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_1(s)|. \]

By the second inequality of (5.2.4), we have
\[ \|\psi(t)Y(t)Z^*(t)P_2\xi\| \geq L^{-1/2} \|\psi(S)Y(s)Z^*(s)P_2\xi\| e_a(t, s) \text{ for } t \geq s \geq \theta. \]

Choosing \(\xi = (Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_2(s)\), for \(t \geq \sigma(s) \geq \theta\), we obtain
\[ |\psi(t)Y(t)Z^*(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_2(s)| \]
\[ \geq L^{-1/2} |\psi(t)Y(t)Z^*(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X_2(s)| e_a(t, \sigma(s)) \]
\[ = L^{-1/2} |X_2(s)| e_a(t, s). \]

For sufficiently large \(t\), it is easy to show that
\[ |\psi(t)X(t)| \geq L^{-1/2} e_a(t, s) |\psi(s)X_2(s)| - L_1e \Theta a_1(t, s) |\psi(s)X_1(s)| \]
\[ \geq (L^{-1/2} e_a(t, s) - L_1e \Theta a_1(t, s)) |\psi(s)X_2(s)| \]
\[ \geq (1/2)(L^{-1/2} e_a(t, s) - L_1e \Theta a_1(t, s)) |\psi(s)X(s)|. \]

**Case 2:** If \(|\psi(s)X_1(s)| \geq |\psi(s)X_2(s)|\), similarly, for \(s \geq t \geq \theta\), we get
\[ |\psi(t)X(t)| \geq (1/2)(L^{-1/2} e_a(t, s) - L_2e \Theta a_2(s, t)) |\psi(s)X(s)|. \]

This means that there exist \(0 < \theta < 1\) and \(w > 0\) such that
\[ L^{-1/2} e_a(t + w, s) - L_1e \Theta a_1(t + w, s) \geq 2\theta^{-1}, \]
\[ L^{-1} e_a(t + w, s) - L_2e \Theta a_2(t + w, s) \geq 2\theta^{-1}. \]
Then \( \psi(t)X(t) \leq \theta \) \( \sup_{|\tau-t| \leq w} \psi(\tau)X(\tau) \), \( t \geq w \).

Conversely assume that (5.4.5) holds. We first show that there exists a constant \( c > 1 \) such that
\[
|\psi(t)X(t)| \leq c|\psi(s)X(s)| \quad \text{for} \quad \theta \leq s \leq t \leq s + w,
\]
where \( X(t) \) is any nontrivial solution of (5.4.1).

According to the condition, there exists \( N_1, N_2 > 0 \) such that
\[
|A(t)| \leq N_1, \quad |B(t)| \leq N_2, \quad \text{for any} \quad t \in T^+.
\]
It is easy to show that
\[
\|\psi(t)Y(t)Z^*(t)(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)\xi\| \leq e_m(t, \sigma(s))\|\xi\| \quad \text{for} \quad t \geq \sigma(s).
\]
Let \( \xi = \psi(s)Y(s)Z^*(t)\xi^s \). For \( \theta \leq s \leq t \leq s + w \),
we have
\[
\|\psi(t)Y(t)Z^*(s)\xi^s\| \leq e_N(s + w, s)\|\psi(s)Y(s)Z^*(s)\xi^s\|
\]
\[
\leq e^Nw\|\psi(s)Y(s)Z^*(s)\xi^s\|,
\]
that is,
\[
|\psi(t)X(t)| \leq c|\psi(s)X(s)|, \quad \text{where} \quad c = e^Nw.
\]

Suppose that \( X(t) \) is a nontrivial \( \psi \)-bounded solution of (5.4.1).

Set \( \pi(s) = \sup_{t \geq s} |\psi(\tau)X(\tau)| \) for \( s \geq \theta \), we have
\[
|\psi(t)X(t)| \leq \theta \sup_{|\tau-t| \leq w} |\psi(\tau)X(\tau)| \leq \theta \pi(s), \quad t \geq s + w.
\]
Hence \( |\pi(s)| = \sup_{s \leq \tau \leq s + w} |\psi(\tau)X(\tau)| \), which implies that
\[
|\psi(t)X(t)| \leq c|\psi(s)X(s)|, \quad \theta \leq s \leq t < \infty.
\]

If \( s + nw \leq t \leq s + (n + 1)w \), then
\[
|\psi(t)X(t)| \leq \theta^n \sup_{s \leq \tau \leq s + w} |\psi(\tau)X(\tau)| \leq \theta^n c|\psi(s)X(s)|.
\]
\[ |\tau - t| \leq nw \leq \theta - 1 c \theta^{(s)w} | \psi(s)X(s)|. \]

Set \( K = \theta^{-1} c \) and \( \alpha = -(1/w) \log \theta \). Then we get

\[ |\psi(t)X(t)| \leq Ne^{-\alpha(t-s)} |\psi(s)X(s)| \leq Ne_{\alpha}(t, s) |\psi(s)X(s)|, \quad \theta \leq s \leq t < \infty. \]

Carrying out arguments and extended to matrix systems similar to those in Proposition 2.1 in [18], it is easily shown that there exists a \( w^* > \theta \) such that

\[ |\psi(t)X(t)| \leq Ne^{-\alpha(s,t)} |\psi(s)X(s)| \text{ for } w^* \leq t \leq s < \infty. \]

Since \( A \) and \( B \) are bounded, then (5.4.1) has \( \psi \)-bounded growth. From Lemma 5.4.1 and Theorem 5.4.1, The matrix system (5.4.1) has an exponential \( \psi \)-dichotomy on \( T^+ \).

**Section 5.5**

The purpose of this section is to develop the relationship between the exponential \( \psi \)-dichotomy of the linear matrix dynamical system (5.4.1) and the \( \psi \)-bounded solutions of the non homogeneous matrix dynamical systems

\[ X^\Delta(t) = A(t)X(t) + X(\sigma(t))B(t) + F(t) \quad (5.5.1) \]

where \( A, B \in \mathbb{R} \) and \( F(t) \in C_{rd}(T) \).

Define

\[ C_\psi = \{ F \in C_{rd}(T^+) : \left| F \right|_{C_\psi} = \sup_{t \in T^+} |\psi(t) F(t)| \}, \]

\[ D_\psi = \left\{ F \in C_{rd}(T^+) : \left| F \right|_{D_\psi} = \sup_{\theta} \left| \int \Psi(\tau) F(\tau) \Delta \tau \right| \right\}, \]

\[ E_\psi = \left\{ F \in C_{rd}(T) : \left| F \right|_{E_\psi} = \sup_{t \in T^+} \frac{1}{\omega} \int_{t}^{t+\omega} |\psi(\tau) F(\tau) \Delta \tau|, \text{ where } T^+ \text{ is } \omega \text{- periodic with } \omega > 0 \right\}. \]
clearly $C_\psi$, $D_\psi$ and $E_\psi$ are Banach spaces.

**Lemma 5.5.1.** If $G \in E_\psi$ is a non-negative function with

$$
\frac{1}{\omega} \int_0^{t+\omega} \psi(\tau)G(\tau)\Delta \tau \leq K_2 \quad \text{for all } t \geq \theta, \text{then}
$$

$$
\int_0^t e^{\Theta \alpha_1 (t, \sigma(\tau))} \psi(\tau)G(\tau)\Delta \tau \leq \frac{K_2 \omega (1 + \alpha_1 \chi)}{1 - e^{\Theta \alpha_1 (\theta + \omega, \theta)}}
$$

$$
\int_t^\infty e^{\Theta \alpha_2 (t, \sigma(\tau))} \psi(\tau)G(\tau)\Delta \tau \leq \frac{K_2 \omega}{1 - e^{\Theta \alpha_2 (\theta + \omega, \theta)}}
$$

hold for $\alpha_1, \alpha_2 > 0$ and $t \geq \theta$.

The following Lemma will be very useful. We first assume that $U_1$ is the subspace of $\mathbb{R}^n$ consisting of the initial values of all $\psi$-bounded solutions of (5.4.1), and $U_2$ is any fixed subspace of $\mathbb{R}^n$ supplementary to $U_1$ such that $\mathbb{R}^n$ can be written as the direct sum

$$
\mathbb{R}^n = U_1 \oplus U_2.
$$

**Lemma 5.5.2** If (5.5.1) has a $\psi$-bounded solution for $F \in B_\psi$, where $B_\psi$, denotes any one of the Banach spaces $C_\psi$, $D_\psi$ and $E_\psi$ then there exists a positive constant $r_{B_\psi}$ such that, for every $F \in B_\psi$, the unique $\psi$-bounded solution $Q(t)$ of (5.5.1) with $Q(\theta) \in U_2$ satisfies $\|Q\|_{C_\psi} \leq r_{B_\psi} |F|_{B_\psi}$.

**Theorem 5.5.1** Assume that $A, B \in \mathcal{R}$ are bounded. Then (5.5.1) has an ordinary $\psi$-dichotomy on $\mathbb{T}^+$ if and only if (5.5.1) has at least one $\psi$-bounded solution for every $F \in D_\psi$.

**Proof:** Assume that (5.5.1) has an ordinary $\psi$-dichotomy on $\mathbb{T}^+$. Then it is easy to show
\[ X(t) = \int_{\varphi}^t \psi(t)Y(t)Z^*(t)P_1(Z^*(\sigma(t)))^{-1}Y(\sigma(t))^{-1}\psi^{-1}(\tau)F(\tau)\Delta \tau - \int_{t}^{\infty} \psi(t)Y(t)Z^*(t)P_2(Z^*(\sigma(t)))^{-1}Y(\sigma(t))^{-1}\psi^{-1}(\tau)F(\tau)\Delta \tau \quad (5.5.2) \]

is a \( \psi \)-bounded solution of (5.5.1) and \( |\psi(t)X(t)| \leq \max\{M_1, M_2\}|F|_{D_\psi} \) for all \( t \in T^+ \).

Conversely suppose that (5.5.1) has at least one \( \psi \)-bounded solution for every \( F \in D_\psi \).

Set
\[
W(t, \sigma(s)) = \begin{cases} 
\psi(t)Y(t)Z^*(t)P_1(Z(\sigma(s))^*)^{-1}Y(\sigma(s))^{-1}\psi^{-1}(s) & \text{for } t > \sigma(s) \geq \vartheta \\
-\psi(t)Y(t)Z^*(t)P_2(Z(\sigma(s))^*)^{-1}Y(\sigma(s))^{-1}\psi^{-1}(s) & \text{for } \sigma(s) > t \geq \vartheta
\end{cases}
\]

where \( Y(t)Z^*(t) \) is a fundamental solution matrix of (5.4.1) with \( YZ^*(\vartheta) = I_n \).

Let \( Q(t) = \int_{\vartheta}^{\infty} W(t, \sigma(\tau))F(\tau)\Delta \tau \). For a fixed \( t_1 \in T^+ \),

choose a function \( F \in D_\psi \), which vanishes for \( t \geq t_1 \). Since

\[
\psi(t)Q(t) = \psi(t)Y(t)Z(t)P_1 \int_{\vartheta}^{t_1} (Z(\sigma(\tau))^*)^{-1}Y(\sigma(\tau))^{-1}\psi^{-1}(\tau))F(\tau)\Delta \tau; t \geq t_1
\]

and

\[
Q(\vartheta) = -P_2 \int_{\vartheta}^{t_1} (Z(\sigma(\tau))^*)^{-1}Y(\sigma(\tau))^{-1}\psi^{-1}(\tau))F(\tau)\Delta \tau \in U_2 \text{ then}
\]

\[
Q(t) = \int_{\vartheta}^{t_1} W(t, \sigma(\tau))F(\tau)\Delta \tau
\]

is \( \psi \)-bounded solution of (5.5.1). By Lemma 5.5.2, we have \( \|Q\|_C \leq r_L \|F\|_{D_\psi} \). For any fixed point \( s \in T^+ \).
we have three cases as in the following: (1) is right-dense; (2) is both right-scattered and left-scattered; (3) is right-scattered and left-dense.

Then, we get

\[
|\psi(t)Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)|
\]

\[
\leq r_{D\nu}(1 + \chi X[A|_{C_{\psi}} + |B|_{C_{\psi}}]) \text{ for } t > \sigma(s)
\]

\[
|\psi(t)Y(t)Z^*(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)|
\]

\[
\leq r_{D\nu}(1 + \chi[A|_{C_{\psi}} + |B|_{C_{\psi}}]) \text{ for } \sigma(s) < t. \quad (5.5.3)
\]

From the rd-continuity of \(\psi(t)Y(t)Z^*(t)\), it follows that (5.5.3) is also valid for \(\sigma(s) = t\).

**Theorem 5.5.2** Assume that (5.4.1) has \(\psi\) -bounded growth. Then (5.4.1) has an exponential \(\psi\) -dichotomy on \(T^+\) if and only if (5.5.1) has at least one \(\psi\) -bounded solution for every \(F \in C_{\psi}\).

**Proof:** Assume that (5.4.1) has an exponential \(\psi\) -dichotomy on \(T^+\). Then (5.5.2) is a solution of (5.5.1) and

\[
|\psi(t))X(t)| = \|F\|_{C_{\psi}} \left( \int_t^\infty |\psi(t)Y(t)Z^*(t)P_1(Z^*(\sigma(s)))^{-1}Y^{-1}(\sigma(s))\psi^{-1}(\tau)| \Delta \tau \\
+ \int_t^\infty |\psi(t)Y(t)Z^*(t)P_2Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(\tau)| \Delta \tau \right)
\]

\[
\leq \|F\|_{C_{\psi}} \left( M_1 \int_t^\infty e_{\Theta\alpha_1}(\sigma(\tau), t) \Delta \tau + M_2 \int_t^\infty e_{\Theta\alpha_2}(\sigma(\tau), t) \Delta \tau \right)
\]

\[
\leq \|F\|_{C_{\psi}} \left( \frac{M_1(1 + \alpha_1 \zeta)}{\alpha_1} + \frac{M_2}{\alpha_2} \right).
\]

Conversely suppose that (5.5.1) has at least one \(\psi\) -bounded solution for every function
\( F \in C_\nu \) For a fixed \( w \in T^+ \), choose a rd-continuous function \( \eta \) such that \( 0 \leq \eta(t) \leq 1 \) for all \( t \geq 0 \) and \( \eta(t) = 0 \) for \( t \geq w \). Set \( F(t) = \eta(t)X(t)|\psi(t)|X(t)|^{-1} \), where \( \psi(t)X(t) = \psi(t)Y(t)Z(t) \xi \) is any nontrivial solution of (5.5.1). Clearly \( |F|_{C_\nu} \leq 1 \). (denote \( r_C = r_{C,\psi} \))

Implies

\[
\begin{align*}
&\int_{t_0}^{\infty} W(t, \sigma(\tau)[\psi(\tau)X(\sigma(\tau))]^{-1} \Delta \tau \leq r_C \quad \text{w} \geq t_0 \geq 0 \quad \text{and} \quad t \geq r \\
&\left\| \psi(t)Y(t)Z^*(t)P_1 \xi \right\| \int_{t_0}^{\infty} \left\| \psi(\tau)Y(\sigma(\tau))Z^*(\sigma(\tau))P_1 \xi \right\|^{-1} \Delta \tau \leq r_C \quad \text{for} \quad t \geq t_0 \geq 0 \\
&\left\| \psi(t)Y(t)Z^*(t)P_2 \xi \right\| \int_{s}^{\infty} \left\| \psi(\tau)Y(\sigma(\tau))Z^*(\sigma(\tau))P_2 \xi \right\|^{-1} \Delta \tau \leq r_C \quad \text{for} \quad t \geq t_0 \geq 0 \quad (5.5.4)
\end{align*}
\]

Replacing \( \xi \) by \( P_1 \xi \) or \( P_2 \xi \), we get

\[
\begin{align*}
&\int_{t_0}^{\infty} \left\| \psi(t)Y(t)Z^*(t)P_1 \xi \right\|^{-1} \Delta \tau \leq e^\rho r_C^{-1} (t, s) \int_{t_0}^{\infty} \left\| \psi(\tau)Y(\tau)Z^*(\tau)P_1 \xi \right\|^{-1} \Delta \tau \quad \text{for} \quad t \geq s \geq t_0 \\
&\int_{s}^{\infty} \left\| \psi(t)Y(t)Z^*(t)P_2 \xi \right\|^{-1} \Delta \tau \leq e^\rho r_C^{-1} (s, t) \int_{t_0}^{\infty} \left\| \psi(\tau)Y(\tau)Z^*(\tau)P_2 \xi \right\|^{-1} \Delta \tau \quad \text{for} \quad t \leq s \leq w.
\end{align*}
\]

According to the condition, the dynamical system (5.5.1) has \( \psi \)-bounded growth, then there exist a \( K \geq 1 \) and a \( \beta > 0 \) such that \( |\psi(t)Y(t)Z^*(t)(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s)| \leq Ke^\beta(t, s) \) for \( t \geq \sigma(s) \). Assume that \( X(t) \) is any solution of (5.5.1) and let

\[
\begin{align*}
X_1(t) &= Y(t)Z(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X(s), \\
X_2(t) &= Y(t)Z^*(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))X(s).
\end{align*}
\]

Next we show that \( |X_1(t)| \leq e_{K}|X(s)| e^{\rho r_C^{-1}}(t, s) \) for \( s \leq t < \infty \).
if $|\psi(t) X_1(t)| \leq K|\psi(s) X(s)|$ for some fixed $s \geq \vartheta$ and $s \leq t \leq s + r_C$.

Let $t^* = \inf\{t \in T^*|t \geq s + r_C\}$ since $X(t)$ is a solution of (5.5.1) then $X(t) = \psi(t)Y(t)Z(t) \xi$.

Replacing $t_0$ by $s$ and $s$ by $t^*$ in the first inequality of (5.5.5), we obtain

$$\frac{r_C}{K|\psi(s) X(s)|} \leq \int_s^t \frac{|(\psi(\tau))X_1(\sigma(\tau))|^{-1} \Delta \tau}{s} \leq e_{\Theta r_C^{-1}}(t, s) \frac{t}{s} \frac{|(\psi(\tau))X_1(\sigma(\tau))|^{-1} \Delta \tau}{s} \quad t \geq s + r_C.$$

By the first inequality of (5.5.4), we have

$$|\psi(t) X_1(t)| \leq r_C \int_s^t \frac{|(\psi(\tau))X(\sigma(\tau))|^{-1} \Delta \tau}{s} \leq e_K \frac{|\psi(s) X(s)|}{s} e_{\Theta r_C^{-1}}(t, s) \quad t \geq s + r_C.$$

Note that

$$e_{\Theta r_C^{-1}}(t, s) \geq e_{\Theta r_C^{-1}(t-s)} \geq 1; s \leq t \leq s + r_C.$$  

This implies that

$$|\psi(t) X_1(t)| \leq e_K \frac{|\psi(s) X(s)|}{s} e_{\Theta r_C^{-1}}(t, s) \quad s \leq t < \infty \quad (5.5.6)$$

Similarly, if $|\psi(t) X_2(t)| \leq K|\psi(s) X(s)|$ for some fixed $s \geq \vartheta$ and

$$\max\{\vartheta, s-r_C\} \leq t \leq s.$$

we have

$$|X_2(t)| \leq e_K |X(s)| e_{\Theta r_C^{-1}}(s, t), \quad \vartheta \leq t < s \quad (5.5.7)$$

Replacing $\xi$ by $(Z^*)^{-1}(s)Y^{-1}(s)\psi^{-1}(s) \xi$ and putting $w \rightarrow \infty$ in the second inequality of (5.5.5), we get
\[
\left| \psi(t)Y(t)Z^*(t)P_1Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right|
\leq r_C \int_0^\infty \left| \psi(t)Y(t)Z^*(\tau)(Z^{-1}(\sigma(\tau))Y^{-1}(\sigma(\tau))\psi^{-1}(\tau)\xi \right| \Delta \tau
\leq r_C \left[ K^{-1} \epsilon^{-1} \int_0^\infty [e_\beta(s,\tau)]^{-1} \Delta \tau, \quad t \leq s \right.
\]

Since \( \xi \) is an arbitrary, we obtain
\[
\left| \psi(t)Y(t)Z^*(t)P_1Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta K, \quad t \leq \sigma(s)
\]
similarly
\[
\left| \psi(t)Y(t)Z^*(t)P_2(Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta K e_\beta(t,\sigma(s)), t \geq \sigma(s).
\]
then
\[
\left| \psi(t)Y(t)Z^*(t)P_1(Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq (1 + r_C \beta) K e_\beta(t,\sigma(s)), t \geq \sigma(s)
\]

Let \( t_0=s \), then by the first inequality of (5.5.5), we have
\[
\left| \psi(t)Y(t)Z^*(t)P_1(Z^{-1}(s))Y^{-1}(s)\psi^{-1}(s) \right| \leq r_C \beta K [1 - e_\phi(t,s)]^{-1}, t \geq s.
\]

Now we consider the two cases

(i) If \( \chi = 0 \), it follows from Proposition 2.1 in [18] that
\[
\left| \psi(t)Y(t)Z^*(t)P_1(Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq (1 + 2r_C \beta) K, t \geq \sigma(s).
\]

(ii) If \( \chi > 0 \), it follows from (5.2.3) that
\[
\left| \psi(t)Y(t)Z^*(t)P_1(Z^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_C \beta K [1 - \left( \frac{1}{1 + \beta \chi} \right)^{t-s/\chi}]^{-1}, t > \sigma(s).
\]

Then we get
\[ \left| \psi(t)Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq r_CK(1+\beta\chi)e_\beta(t,\sigma(s)) \leq K(1+r_C\beta)e^{\beta(t-s)} \]

\[ \leq K(1+r_C\beta)e^{\beta\chi}. \]

Hence we have

\[ \left| \psi(t)Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq \max \left\{ \frac{r_CK(1+\beta\chi)}{\chi}, K(1+r_C\beta)e^{\beta\chi} \right\}, t \geq \sigma(s). \]

Define

\[ N(\chi) = \begin{cases} 
(1+2r_C\beta)K & \text{if } \chi = 0, \\
\max \left\{ \frac{r_CK(1+\beta\chi)}{\chi}, K(1+r_C\beta)e^{\beta\chi} \right\}, & \text{if } \chi > 0. 
\end{cases} \]

Then

\[ \left| \psi(t)Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq N(\chi) \text{ for } t \geq \sigma(s). \]

It follows from (5.5.6) and (5.5.7) that

\[ \left| \psi(t)Y(t)Z^*(t)P_1(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq e_{N(\chi)e_{\Theta_c^{-1}}}(t,\sigma(s)) \text{ for } t \geq \sigma(s) \geq \vartheta. \]

\[ \left| \psi(t)Y(t)Z^*(t)P_2(Z^*)^{-1}(\sigma(s))Y^{-1}(\sigma(s))\psi^{-1}(s) \right| \leq e_{rc\beta K e_{\Theta_c^{-1}}}(\sigma(s),t) \text{ for } \sigma(s) \geq t \geq \vartheta. \]

Hence the dynamical system (5.5.1) has an exponential \( \psi \)-dichotomy on \( T^+ \).