Chapter 1

Introduction

Graph theory as a mathematical discipline was initiated by the renowned Swiss mathematician Leonhard Euler (1707 - 1783) in his famous discussion of the Königsberg Bridge problem entitled ‘The solution of a problem relating to the geometry of position’. It was presented at the St.Petersberg Academy on 26th August, 1735. Unfortunately, this article of Euler, published in 1736, remained an isolated contribution for nearly a hundred years. However, in the middle of the nineteenth century, there was a resurgence of interest in the area of graph theory. The natural sciences exercised their influence through investigations of
electrical networks and models for crystals and molecular structure and theoretically, the development of formal logic led to the study of binary relations in the form of graphs.

Today, graph theory is a branch of mathematics which finds applications in many areas - anthropology, architecture, biology, chemistry, computer science, economics, physics, psychology, sociology and telecommunications, to name a few. The applications of graph theory in operations research, social science, psychology and physics are detailed in C. W. Marshall [61]. J. L. Gross and J. Yellen [34] discuss a variety of graph classes with numerous illuminating examples which are of topological relevance. The development of graph theory with its applications to electrical networks, flows and connectivity are included in [11] and [22]. Ramsey theory is an interesting branch of graph theory which relates to the number theory. In [22], R. Diestel covers all major developments in the subject. More recently, the exciting notion of ‘Web graphs’ [6] has been finding applications in very many different areas. Such graphs are examples of large, dynamic, distributed graphs and share many properties with several other complex graphs [64] found in a variety of
systems ranging from social organizations to biological systems. The best barometer to indicate the growth of interest in graph theory is the explosion in the number of pages that Section 05: Combinatorics occupies in the Mathematical Reviews.

Volumes have been written on the rich theory and the very many applications of graphs. To name a few, [5], [9], [10], [32], [34], [35], [49], [71], [78]. This thesis entitled ‘Studies on some graph operators and related topics’ is a humble attempt at making a small addition to the vast ocean of results in graph theory.

‘Graph operator’ is a mapping $T : \mathcal{G} \rightarrow \mathcal{G}'$ where $\mathcal{G}$ and $\mathcal{G}'$ are families of graphs. Krausz [52] introduced the concept of the line graph and also that of ‘graph operators’. He also gave a characterization of line graphs. Whitney [79] showed that every finite connected graph except $C_3$ has at most one connected L-root.

The study of graph operators gained increasing importance
due to the study of its dynamics as detailed by E. Prisner [66].

The beginning of graph dynamics dates back to 1960s, with the publication [36] by Harary and Norman, and also the three problems of great influence posed by Ore in his monograph [65], namely

1. Determine all graphs isomorphic to their interchange graph (line graph).

2. When the interchange graph is given, is the original graph uniquely determined?

3. Investigate the repeated interchange graphs.

In the graph dynamics terminology, the first problem deals with the ‘fixedness’ and the second and third will lead to the ‘1-periodicity’ and ‘convergence’ or ‘divergence’.

The 1960s were mainly devoted to the investigation of the line graph and the line digraph operators. Several solutions to Ore’s problems for the line graph appeared in [8], [15], [72], [73]. The question of periodicity was considered only for the fixed graphs till 1970s. General periods were investigated by
Escalante [25] and it was studied for the line digraphs by Hemminger [42]. The transition number was first explicitly defined in [1].

While dealing with graph classes, a main source is the classical book by M. C. Golumbic [32]. Since then many interesting new graph classes have been studied as discussed in detail by A.Brandstädt, et.al. [13].

By applying graph operators also, we get some graph classes. The line graphs, Gallai and the anti-Gallai graphs, the cycle graphs and the edge graphs are some of the graph classes obtained by choosing appropriate graph operators. In fact, the intersection graphs form a sub collection of the graph classes obtained by using graph operators. The intersection graph is a very general notion in which objects are assigned to the vertices of a graph and two distinct vertices are adjacent if the corresponding objects have a non empty intersection. A variety of well studied graph classes such as the line graphs, the clique graphs and the block graphs are actually special types of inter-
section graphs. J. L. Szwarcfiter has made an excellent survey of the clique graphs [75]. The block graph [37], the square [38] and the complement [70] are some well studied graph operators.

Several graph operators and the dynamical behavior of these operators are extensively studied in [66].

It is interesting to study when the graph operators belong to some special graph classes. The inclusions between graph classes can be identified from their forbidden subgraph characterizations. The cographs, the split graphs, the threshold graphs and the line graphs are some of the interesting graph classes which admit finite forbidden subgraph characterizations and the perfect graphs, the distance hereditary graphs, the comparability graphs and the chordal graphs are some of the other interesting graph classes defined by forbidding an infinite collection of induced subgraphs.

While studying a graph operator, the study of its parameters such as clique number, independence number, chromatic num-
ber, domination number, diameter, radius, eccentricity, center etc are important. It is quite interesting to study the relationship between these parameters of $G$ and those under graph operators.

This thesis is mainly concerned with the graph operators - the ‘$P_3$ intersection graph’ and the ‘edge $C_4$ graph’.

1.1 Basic definitions

The basic notations, terminology and definitions are from ([5], [14], [32], [66] and [78]) and the basic results are from ([13], [39], [45] and [77]).

Definition 1.1.1. A graph $G = (V, E)$ consists of a collection of points, $V$ called its vertices and a set of unordered pairs of distinct vertices, $E$ called its edges. If $|V|$ is finite, then $G$ is a finite graph. The unordered pair of vertices $\{u, v\} \in E$ are called the end vertices of the edge $e = \{u, v\}$. When $u$ and $v$
are end vertices of an edge, then \( u \) and \( v \) are adjacent. If the vertex \( v \) is an end vertex of an edge \( e \), then \( e \) is incident on \( v \). Two edges which are incident with a common vertex are said to be adjacent edges. The cardinality of \( V \) is called the order of \( G \) and the cardinality of \( E \) is called the size of \( G \). A graph \( G \) of order \( n \) and size \( m \) is also denoted by \( G = (n, m) \). A graph is the null graph, denoted by \( \phi \) if it has no vertices and trivial if it has no edges.

**Definition 1.1.2.** The degree of a vertex \( v \), denoted by \( d(v) \) is the number of edges incident on \( v \). A graph \( G \) is \( k \)-regular if \( d(v) = k \) for every vertex \( v \in V \). A vertex of degree zero is an isolated vertex and of degree one is a pendant vertex. The edge incident on a pendant vertex is a pendant edge. A vertex of degree \( n - 1 \) is called a universal vertex. In a graph \( G \), the maximum degree of vertices is denoted as \( \Delta(G) \) and the minimum degree of vertices is denoted as \( \delta(G) \).

**Definition 1.1.3.** A \( v_0 - v_k \) walk in a graph \( G \) is a finite list \( v_0, e_1, v_1, e_2, v_2, \ldots, e_k, v_k \) of vertices and edges such that for \( 1 \leq i \leq k \), the edge \( e_i \) has end vertices \( v_{i-1} \) and \( v_i \). In the \( v_0 - v_k \) walk, \( v_0 \) is the origin, \( v_k \) is the terminus and \( v_1, v_2, \ldots, v_{k-1} \) are its internal vertices. If the vertices \( v_0, v_1, \ldots, v_k \) of the above
walk are distinct, then it is called a path. A path from a vertex \( u \) to a vertex \( v \) is called a \( u - v \) path. A path on \( n \) vertices is denoted by \( P_n \). If the edges \( e_1, e_2, ..., e_k \) of the walk are distinct, it is called a trail. A graph \( G \) is Eulerian if it has a closed trail containing all the edges. A nontrivial closed trail is called a cycle if its origin and internal vertices are distinct. A cycle with \( n \) vertices is denoted by \( C_n \). The length of a walk, a path or a cycle is its number of edges. A graph containing exactly one cycle is called a unicyclic graph. A graph is acyclic if it does not contain cycles. The girth of \( G \), \( g(G) \) is the length of a shortest cycle in \( G \). An acyclic graph has infinite girth. The circumference of \( G \), \( c(G) \) is the length of any longest cycle in \( G \).

Definition 1.1.4. A graph \( H = (V', E') \) is called a subgraph of \( G \) if \( V' \subseteq V \) and \( E' \subseteq E \). A subgraph \( H \) is a spanning subgraph if \( V' = V \). The graph \( H \) is called an induced subgraph of \( G \) if \( E' \) is the collection of all edges in \( G \) which has both its end vertices in \( V' \). \(< V' > \) denotes the induced subgraph with vertex set \( V' \). A spanning 1-regular graph is called a 1-factor or perfect matching. A graph \( G \) is \( H \)-free if it does not contain \( H \) as an induced subgraph.
Definition 1.1.5. A graph $G$ is connected if for every $u, v \in V$, there exists a $u - v$ path. If $G$ is not connected then it is disconnected. The components of $G$ are its maximal connected subgraphs. A connected acyclic graph is called a tree.

Definition 1.1.6. The distance between two vertices $u$ and $v$ of a connected graph $G$, denoted by $d(u, v)$ or $d_G(u, v)$ is the length of a shortest $u-v$ path in $G$. The eccentricity of a vertex $u$, $e(u) = \max \{d(u,v) | v \in V(G)\}$. The radius $\text{rad}(G)$ and the diameter $\text{diam}(G)$ are respectively the minimum and the maximum of the vertex eccentricities. The center of a graph $G$, $C(G)$ is the subgraph induced by the vertices of minimum eccentricity.

Definition 1.1.7. A chord of a cycle $C$ is an edge not in $C$ whose end points lie in $C$. A graph $G$ is chordal if every cycle of length at least four in $G$ has a chord.

Definition 1.1.8. A complete graph is a graph in which each pair of distinct vertices is joined by an edge. A complete graph on $n$ vertices is denoted by $K_n$. The graph obtained by deleting any edge of $K_n$ is denoted by $K_n - \{e\}$. $K_3$ is called a triangle and a paw is a triangle with a pendant edge. A clique is a
maximal complete subgraph. The size of the largest clique in $G$ is the **clique number** $\omega(G)$. A clique of size $k$ is called a $k$-clique.

**Definition 1.1.9.** A cycle $C$ of $G$ is a $b$-cycle of $G$ if $C$ is not contained in a complete subgraph of $G$. The **bulge** of $G$, $b(G)$ is the minimum length of a $b$-cycle in $G$ if $G$ contains a $b$-cycle and is $\infty$ otherwise.

**Definition 1.1.10.** The set of all vertices adjacent to a vertex $v$ is called **open neighborhood** of $v$, denoted by $N(v)$. The **closed neighborhood** of $v$, $N[v] = N(v) \cup \{v\}$.

**Definition 1.1.11.** Assigning colors to the vertices of a graph is called a **vertex coloring**. If no two adjacent vertices receive the same color, then such a coloring is called a **proper vertex coloring**. The minimum number of colors required for a proper vertex coloring of a graph $G$ is called its **chromatic number**, denoted by $\chi(G)$.

**Definition 1.1.12.** A property $P$ of a graph $G$ is **vertex hereditary** if every induced subgraph of $G$ has the property $P$. A graph $H$ is a **forbidden subgraph** for a property $P$, if any graph $G$ which satisfies the property $P$ cannot have $H$ as an
induced subgraph.

**Definition 1.1.13.** A graph $G = (V, E)$ is isomorphic to a graph $H = (V', E')$ if there exists a bijection from $V$ to $V'$ which preserves adjacency. If $G$ is isomorphic to $H$, we write $G \cong H$.

**Definition 1.1.14.** Let $G$ be a graph. The complement of $G$, denoted by $G^c$, is the graph with vertex set same as that of $V$ and any two vertices are adjacent in $G^c$ if they are not adjacent in $G$. $K_n^c$ is called totally disconnected. A graph $G$ is self complementary if $G \cong G^c$.

**Definition 1.1.15.** A graph $G$ is bipartite if the vertex set can be partitioned into two non-empty sets $U$ and $U'$ such that every edge of $G$ has one end vertex in $U$ and the other in $U'$. A bipartite graph in which each vertex of $U$ is adjacent to every vertex of $U'$ is called a complete bipartite graph. If $|U| = m$ and $|U'| = n$, then the complete bipartite graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a star. A graph $G$ is complete multipartite if the vertices can be partitioned into sets so that $\{u, v\} \in E$ if and only if $u$ and $v$ belong to different sets of the partition. A complete $k$-partite
graph with partite sets of cardinalities \( n_1, n_2, \ldots, n_k \) is denoted by \( K_{n_1,n_2,\ldots,n_k} \).

**Definition 1.1.16.** A subset \( I \subseteq V \) of vertices is **independent** if no two vertices of \( I \) are adjacent. The maximum cardinality of an independent set is called the **independence number** and is denoted by \( \alpha(G) \). A subset \( F \subseteq E \) of edges is said to be an **independent set of edges** or a **matching** if no two edges in \( F \) have a vertex in common. The maximum cardinality of a matching set of edges is the **matching number** and is denoted by \( \beta(G) \).

**Definition 1.1.17.** A subset \( K \subseteq V \) is called a **vertex cover** of \( G \) if every edge of \( G \) is incident with at least one vertex of \( K \). The minimum cardinality of a vertex cover is the **vertex covering number** \( \alpha_0(G) \).

**Definition 1.1.18.** For a graph \( G \), a subset \( V' \) of \( V(G) \) is a **k-vertex cut** of \( G \) if the number of components in \( G - V' \) is greater than that of \( G \) and \( |V'| = k \). The **vertex connectivity** of \( G \), \( \kappa(G) \) is the smallest number of vertices in \( G \) whose deletion from \( G \) increases the number of components of \( G \). A graph is **n-connected** if \( \kappa(G) \geq n \). A vertex \( v \) of \( G \) is a **cut vertex**
of $G$ if $\{v\}$ is a vertex cut of $G$. If $G$ has no cut vertices, then $G$ is a block. For $\{u, v\} \in V(G)$, the $u - v$ cut is a set $S \subseteq V(G) - \{u, v\}$ such that $G - S$ has no $u - v$ path. The edge connectivity of a graph $G$, $\kappa'(G)$ is the least number of edges whose deletion increases the number of components of $G$.

**Definition 1.1.19.** A vertex $x$ dominates a vertex $y$ if $N(y) \subseteq N[x]$. If $x$ dominates $y$ or $y$ dominates $x$, then $x$ and $y$ are comparable. Otherwise, they are incomparable. The Dilworth number of a graph $G$, $dilw(G)$ is the largest number of pairwise incomparable vertices of $G$.

As an example, $dilw(C_4) = 2$.

**Definition 1.1.20.** A subset $S \subseteq V$ of vertices is a dominating set if each vertex of $G$ that is not in $S$ is adjacent to at least one vertex of $S$. If $S$ is a dominating set then $N[S] = V$.

A dominating set of minimum cardinality in $G$ is called a minimum dominating set, its cardinality is called the domination number of $G$ and it is denoted by $\gamma(G)$.

**Definition 1.1.21.** A dominating set $S$ is an independent dominating set if $S$ is an independent set. The independent domination number of a graph $G$, $\gamma_i(G)$ is the minimum cardinality
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of an independent dominating set in $G$. The minimum cardinality of a maximal independent set of vertices in $G$ is also the same as $\gamma_i(G)$. A subset $S \subseteq V$ is a total (open) dominating set if $N(S) = V$. The total (open) domination number of a graph $G$, $\gamma_t(G)$ is the minimum cardinality of a total dominating set in $G$. A dominating set $S$ is a connected dominating set if $< S >$ is a connected subgraph of $G$ and the corresponding domination number is the connected domination number $\gamma_c(G)$. A dominating set $S$ is a paired dominating set if $< S >$ has a perfect matching and the corresponding domination number is the paired domination number $\gamma_{pr}(G)$. The paired domination number exists for all graphs with out isolated vertices. A dominating set $S$ is a clique dominating set if $< S >$ is a complete graph. The minimum cardinality of a clique dominating set, if it exists is the clique domination number $\gamma_{cl}(G)$. A clique dominated graph is a graph that contains a dominating clique.

Definition 1.1.22. The subgraph weakly induced by a set $S$ of vertices is the graph $< S >_w$ whose vertex set is $N[S]$ and whose edge set consists of those edges in $E(G)$ with at least one vertex, and possibly both, in $S$. A dominating set $S$ is called a weakly connected dominating set if $< S >_w$ is connected. The corre-
sponding domination number is **weakly connected domination number**, \( \gamma_w(G) \). The cardinality of the weakly connected independent dominating set is the **weakly connected independent domination number**, denoted by \( i_w(G) \).

For example:-

\[ G: \]

\[ \begin{array}{c}
  h \\
  j \\
  l \\
  n \\
  p
\end{array} \]

\[ \begin{array}{c}
  a \\
  b \\
  c \\
  d \\
  e \\
  f \\
  g
\end{array} \]

\[ \begin{array}{c}
  i \\
  k \\
  m \\
  o \\
  q
\end{array} \]

![Figure 1.1.1](image)

Then, \( \gamma(G) = 5 \) (\( \{a, b, d, f, g\} \) is a dominating set of minimum cardinality),

\( \gamma_i(G) = 7 \) (\( \{h, i, b, d, f, p, q\} \) is an independent dominating set of minimum cardinality),

\( \gamma_t(G) = 6 \) (\( \{a, b, d, e, f, g\} \) is a total dominating set of minimum cardinality),

\( \gamma_c(G) = 7 \) (\( \{a, b, c, d, e, f, g\} \) is a connected dominating set of
minimum cardinality),
\[ \gamma_{pr}(G) = 6 \] (\( \{a, b, c, d, f, g\} \) is a paired dominating set of minimum cardinality),
\[ \gamma_w(G) = 5 \] (\( \{a, b, d, f, g\} \) is a weakly connected dominating set of minimum cardinality),
\[ i_w(G) = 7 \] (\( \{h, i, b, d, f, p, q\} \) is a dominating set of minimum cardinality).

The graph \( G \) is not a clique dominated graph.

**Definition 1.1.23.** A subset \( S' \subseteq E \) is an edge dominating set if every edge not in \( S' \) is adjacent to some edge in \( S' \). The **edge domination number** \( \gamma'(G) \) of \( G \) is the minimum cardinality of all edge dominating sets of \( G \). A subset \( S' \subseteq E \) is an efficient edge dominating set for \( G \) if each edge in \( E \) is dominated by exactly one edge in \( S' \). The **efficient edge domination number** of \( G \) is denoted by \( \gamma'_e(G) \).

**Definition 1.1.24.** The **intersection graph** is a graph whose vertex set is a collection of objects and any two vertices are adjacent if the corresponding objects intersect. The intersection graph of all the edges of \( G \) is the **line graph** of \( G \) denoted by \( L(G) \). Thus, the line graph \( L(G) \) of a graph \( G \) is a graph that
has a vertex for every edge of $G$, and two vertices of $L(G)$ are adjacent if and only if they correspond to two edges of $G$ with a common end vertex.

**Illustration:**

![Diagram of graphs $G$ and $L(G)$](image)

**Figure 1.1.2**

**Definition 1.1.25.** The $k$-path graph corresponding to a graph $G$ has the set of all paths of length $k$ as vertices and two vertices in the $k$-path graph are adjacent whenever the intersection of the corresponding paths form a path of length $k - 1$ in $G$ and their union forms either a cycle or a path of length $k + 1$ in $G$. 
Definition 1.1.26. For any graph $G$, the $n^{th}$ iterated graph under the operator $\Phi$ is iteratively defined as $\Phi^1(G) = \Phi(G)$ and $\Phi^n(G) = \Phi(\Phi^{n-1}(G))$ for $n > 1$. A graph $G$ is $\Phi^n$-complete if $\Phi^n(G)$ is a complete graph. We say that $G$ is convergent under $\Phi$ if $\{\Phi^n(G), n \in N\}$ is finite. If $G$ is not convergent under $\Phi$, then $G$ is divergent under $\Phi$. A graph $G$ is periodic if there is some natural number $n$ with $G = \Phi^n(G)$. The smallest such number $n$ is called the period of $G$. A graph $G$ is $\Phi$-fixed if the period of $G$ is one. The transition number $t(G)$ of a convergent graph $G$ is zero if $G$ is periodic and is the smallest number $n$ such that $\Phi^n(G)$ is periodic otherwise. A graph $G$ is mortal if for some $n \in N$, $\Phi^n(G) = \phi$, the null graph. A semibasin is any subset $B$ of the class of graphs $G$ with $\Phi(B) \subseteq B$. A basin is a semibasin $B$ if its compliment is also a semibasin.

Definition 1.1.27. The touching number of a cycle is the cardinality of the set of all edges having exactly one of its end vertices on the cycle. For every integer $n \geq 3$, the $n$-touching number $t_n(G)$ of a graph $G$ is the supremum of all touching numbers of $C_n$, provided $G$ contains some $C_n$. If $G$ contains no $C_n$ then $t_n(G)$ is undefined. The vertex touching number of an induced $C_k$ is the cardinality of the set of all vertices which
are adjacent to exactly one vertex of the $C_k$. The vertex touching number of a graph $vt_k(G)$ is the supremum of all vertex touching numbers of induced $C_k$, provided $G$ contains some induced $C_k$.

For example, for the graph $G$ in Figure 1.1.3, $t_5(G) = 7$, $t_3(G) = 5$, $vt_5(G) = 1$, $vt_3(G) = 5$.

![Figure 1.1.3](image)

**Definition 1.1.28.** A graph $G$ whose vertex set can be partitioned into an independent set and a clique is a **split graph**.

**Definition 1.1.29.** A graph $G$ is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$.

**Definition 1.1.30.** A graph $G$ is a **threshold graph** if it can be obtained from $K_1$ by recursively adding isolated vertices and universal vertices.
Definition 1.1.31. A connected graph is a block graph if every maximal 2-connected subgraph (block) is complete. A graph is a geodetic graph if for every pair of vertices there is a unique path of minimum length between them and a graph is weakly geodetic if for every pair of vertices of distance two, there is a unique common neighbor.

Definition 1.1.32. A graph that can be reduced to the trivial graph by taking complements within components is called a co-graph.

Definition 1.1.33. For every integer $w$: $1 \leq w \leq \delta(G)$, a $w$-container between any two distinct vertices $u$ and $v$ of $G$ is a set of $w$ internally vertex disjoint paths between them. Let $C_w(u,v)$ denote a $w$-container between $u$ and $v$. In $C_w(u,v)$, the parameter $w$ is the width of the container. The length of the container is the longest path in $C_w(u,v)$. The $w$-wide diameter of $G$, $D_w(G)$ is the minimum number $l$ such that there is a $C_w(u,v)$ of length $l$ between any pair of distinct vertices $u$ and $v$.

For example:-
For this graph $G$, $C_3(a, b) = \{a - b, a - c - b, a - e - b\}$. Length of this $C_3(a, b) = 2$. $D_3(G) = 3$.

**Definition 1.1.34.** For any $k$, the **diameter variability** arising from the change of edges of a graph $G$ are as follows.

$D^{-k}(G)$: The least number of edges whose addition to $G$ decreases the diameter by (at least) $k$.

$D^{+0}(G)$: The maximum number of edges whose deletion from $G$ does not change the diameter.

$D^{+k}(G)$: The least number of edges whose deletion from $G$ increases the diameter by (at least) $k$.

**Definition 1.1.35.** The graph obtained from $G$ by subdividing...
each edge of $G$ exactly once is called the subdivision of $G$ and is denoted by $S(G)$.

**Definition 1.1.36.** The union of two vertex disjoint graphs $G$ and $H$ denoted by $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

**Definition 1.1.37.** The join of two graphs $G$ and $H$ denoted by $G \vee H$ is the graph obtained from the union $G \cup H$ by adding the edges \{u - v : u \in V(G) and v \in V(H)\}. The graph $K_1 \vee 2K_2$ is called a bow. The moth [58] graph is $K_1 \vee \{P_3 \cup 2K_1\}$.

**Definition 1.1.38.** The corona of two graphs $G_1 = (n_1, m_1)$ and $G_2 = (n_2, m_2)$, denoted by $G_1 \circ G_2$, is the graph obtained by taking one copy of $G_1$ and $n_1$ copies of $G_2$, and then joining the $i^{th}$ vertex of $G_1$ to every vertex in the $i^{th}$ copy of $G_2$.

**Definition 1.1.39.** The cartesian product of two graphs $G$ and $H$ denoted by $G \times H$ is the graph with $V(G \times H) = \{(u, v) : u \in V(G) and v \in V(H)\}$ and any two vertices $(u_1, v_1), (u_2, v_2) \in G \times H$ are adjacent if one of the following holds.

(i) $u_1 = u_2$ and $v_1 - v_2 \in E(H)$

(ii) $u_1 - u_2 \in E(G)$ and $v_1 = v_2$. 
1.2 Basic lemmas and theorems

Lemma 1.2.1. [8] *The line graph* $L(G)$ *has nine forbidden subgraphs.*

Figure 1.2.1 gives the nine forbidden subgraphs of $L(G)$.

Lemma 1.2.2. [19] *$G$ is a cograph if and only if* $G$ *is* $P_4$-*free.*
Lemma 1.2.3. [68] If $G$ is a cograph, then the domination number of $G$ is at most two.

Lemma 1.2.4. [29] If $G$ is a graph without isolated vertices, then $\gamma(G) \leq \text{dilw}(G)$.

Lemma 1.2.5. [16] A graph $G$ is a threshold graph if and only if $\text{dilw}(G) = 1$.

Lemma 1.2.6. [16] A graph $G$ is a threshold graph if and only if $G$ contains no induced $\{2K_2, C_4\}$ and no $P_4$.

Lemma 1.2.7. [44], [48] A graph $G$ is a block graph if and only if $b(G) = \infty$.

Lemma 1.2.8. [44], [48] A graph $G$ is weakly geodetic if and only if $b(G) \geq 5$

Lemma 1.2.9. [30] A graph $G$ is a split graph if and only if $G$ contains no induced $2K_2, C_4$ and no $C_5$.

Lemma 1.2.10. [77] For a connected graph $G$, $D^{+i}(G) \leq \kappa'(G)$.

Lemma 1.2.11. [26], [31] A connected graph $G$ is Eulerian if and only if the degree of each vertex of $G$ is even.

Theorem 1.2.12. [15] For any graph $G = (n, m)$, $\gamma'(G) \leq \lfloor n/2 \rfloor$. 
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Theorem 1.2.13. [4] For any connected graph $G$ of even order $n$, $\gamma'(G) = n/2$ if and only if $G$ is isomorphic to $K_n$ or $K_{n/2,n/2}$.

Theorem 1.2.14. [4] For any tree $T$ of order $n \neq 2$, $\gamma'(T) \leq (n - 1)/2$, equality holds if and only if $T$ is isomorphic to the subdivision of a star.

Theorem 1.2.15. [4] Let $G = (n, m)$ be a connected unicyclic graph. Then $\gamma'(G) = \lfloor n/2 \rfloor$ if and only if $G$ is isomorphic to either $C_4$, $C_5$, $C_7$, $C_{3,k}$ or $C_{4,k}$ for some $k \geq 0$.

Theorem 1.2.16. [41] For a connected graph $G$, $\text{diam}(G) - 1 \leq \gamma_c(G) \leq 2\beta(G)$.

Theorem 1.2.17. [23] For a connected graph $G$, $\gamma_c(G) \leq 2\alpha(G) - 1$.

Theorem 1.2.18. (Whitney’s theorem) [79] Let $G$ be a simple graph with at least three vertices. Then $G$ is 2-connected if and only if for each pair of distinct vertices $u$ and $v$ of $G$ there are two internally disjoint $u - v$ paths in $G$.

Theorem 1.2.19. (Menger’s theorem) [62], [21] Let $u$ and $v$ be two non adjacent vertices of a graph $G$. Then the maximum number of internally disjoint $u - v$ paths in $G$ is the minimum number of vertices in a $u - v$ separating set.
Theorem 1.2.20. (Generalized Whitney's theorem) [17]

A simple graph $G$ is $n$-connected if and only if, given any pair of distinct vertices $u$ and $v$ of $G$, there are at least $n$ internally disjoint $u - v$ paths in $G$.

1.3 New definitions

Definition 1.3.1. [59] The $P_3$ intersection graph of a graph $G$, $P_3(G)$ is the intersection graph of all induced 3-paths in $G$. That is, $P_3(G)$ has the induced paths on three vertices in $G$ as its vertices and two distinct vertices in $P_3(G)$ are adjacent if the corresponding induced 3-paths in $G$ intersect. If $a_1 - a_2 - a_3$ is an induced 3-path in $G$ then the corresponding vertex in $P_3(G)$ is denoted by $a_1a_2a_3$.

Definition 1.3.2. A graph $G$ is a $P_3$ intersection graph if there exists a graph $H$ such that $G \cong P_3(H)$.

In Figure 1.3.1 a graph $G$ and its $P_3(G)$ are shown.
Definition 1.3.3. [57] The edge $C_4$ graph of a graph, $E_4(G)$ is a graph whose vertices are the edges of $G$ and two vertices in $E_4(G)$ are adjacent if the corresponding edges in $G$ are either incident or are opposite edges of some $C_4$ in $G$. This graph class is also known by the name edge graph in [66].

In $E_4(G)$ any two vertices are adjacent if the union of the corresponding edges in $G$ induce any one of the graphs $P_3$, $C_3$, $C_4$, $K_4 - \{e\}$, $K_4$. If $a_1 - a_2$ is an edge in $G$, the corresponding vertex in $E_4(G)$ is denoted by $a_1a_2$.

Definition 1.3.4. A graph $G$ is an edge $C_4$ graph if there exists a graph $H$ such that $G \cong E_4(H)$.

In Figure 1.3.2 a graph $G$ and its $E_4(G)$ are shown.
This section is a survey of results related to that of ours.

The $H$ - intersection graph $Int_H(G)$ [66] is the intersection graph of all subgraphs of $G$ that are isomorphic to $H$. If $H$ is $K_2$ then $Int_H(G)$ is the line graph. Trotter [76] characterized the graphs for which $Int_{K_2}(H)$ is perfect. The 3-edge graph is the intersection graph of the set of all 3-edges of $G$ [67]. The $K_3$ intersection graph is the 3-edge graph provided every edge
lies in some triangle [66]. In [2], Akiyama and Chvátal have characterized the graphs for which $IntP_3(G)$ is perfect.

In [27], Daniela derived some properties of the girth and connectivity of the path graphs. In [51], Knor and Niepel characterized the graphs isomorphic to their path graphs.

In [7], Bandelt and others proved that a bipartite graph is dismantlable if and only if its edge $C_4$ graph is dismantlable and a bipartite graph is neighborhood-Helly if and only if its edge $C_4$ graph is neighborhood-Helly. For any given graph $G$, the edge graph is a supergraph of $L(G)$. In [50] it has been shown that for any graph $G$ without isolated vertices, there is a graph $H$ such that $C(H) = G$ and $C(L(H)) = L(G)$.

Many types of dominations and their characteristics are discussed in [24], [39], [40]. In [18], efficient algorithms are developed for finding a minimum cardinality of connected dominating set and a minimum cardinality Steiner tree in permutation graphs. In [20], forbidden subgraph conditions sufficient to im-
ply the existence of a dominating clique are given.

The concept of edge domination was introduced by Mitchell and Hedetniemi in [63]. The edge dominating sets and the bounds for the edge domination number $\gamma'$ are studied in [47]. In [15], an upper bound for $\gamma'(G)$ is obtained. Again, these bounds are modified in [4] for a connected graph $G$ of even order, tree and a connected unicyclic graph. In [23], [41] a bound for the connected domination number of a graph $G$ with regard to the diameter of the graph, the vertex independence number and the matching number of a graph are obtained.

In [29], it is observed that for graphs $G$ without isolated vertices, $\gamma(G) \leq dilw(G)$. Threshold graphs were introduced by Chvátal.V and Hammer.P.L in [16], where different characterizations for such graphs are given. Block graphs, geodetic graphs and weakly geodetic graphs are studied in detail in [44], [48]. Stemple et.al [74] showed that a graph is geodetic if and only if each of its block is geodetic. It is known that block graphs $\subseteq$ geodetic graphs $\subseteq$ weakly geodetic graphs [13].

In
[19], eight characterizations of cographs which include the re­
cursive characterization and the forbidden subgraph characteri­
zation are given. The median and the anti-median of cographs 
are discussed in [69]. The recent Ph.D thesis by Ms. Aparna 
Lakshmanan [3] contains results regarding cographs and other 
graph classes such as the Gallai and the anti- Gallai graphs, the 
clique irreducible graphs, the clique vertex irreducible graphs 
and the weakly clique irreducible graphs.

The concept of wide diameter has been discussed and used 
in distributed and parallel computer networks [45]. In [43], Hou 
and Wang defined generalized wide diameter and calculated it 
for any $k$- regular $k$-connected graph. A generalized $p$-cycle is a 
digraph whose set of vertices is partitioned into $p$ parts that can 
be ordered in such a way that a vertex is adjacent only to the 
vertices in the next part. The bounds for the wide diameter of 
the generalized $p$-cycle is obtained in [28]. The wide diameter 
of butterfly networks is studied in [53]. Bolian Liu and Xiankun 
Zhang studied some problems on the relations between $D_w(G)$ 
and $diam(G)$ in [54]. In this paper they characterized the graphs 
$G$ for which $D_w(G) = diam(G), w > 1$. 
The diameter of a graph is an important factor for communication as it determines the maximum communication delay between any pair of processors in a network. The diameter of a graph may be affected by the addition or deletion of edges. In [33], Graham and Harary studied this aspect in hypercubes and proved that $D^{-1}(Q_n) = 2$, $D^{+1}(Q_n) = n - 1$ and $D^{+0}(Q_n) \geq (n-3)2^{n-1} + 2$. Bouabdallah et al. [12] improved the lower bound of $D^{+0}(Q_n)$ and furthermore gave an upper bound, $(n-2)2^{n-1} - nC_{[n/2]} + 2 \leq D^{+0}(Q_n) \leq (n-2)2^{n-1} - [2^{n-1}/2n - 1] + 1$.

The diameter variability arising from the addition or deletion of edges of a graph $G$ is defined in [77] and in this paper, Wang et al. proved that $D^{-1}(C_m) \geq 2$, $D^{-1}(T_{m,n}) \geq 2$, $D^{-2}(T_{m,n}) = 2$ for $m \geq 14$ and $m \neq 15$. Also they obtained the exact value of $D^{+1}(T_{m,n})$. 
1.5 Summary of the thesis

This thesis entitled 'Studies on some graph operators and related topics' is divided into five chapters including an introductory chapter. We shall now give a summary of each chapter.

The first chapter is an introduction and contains literature on graph operators. It also includes some basic definitions and terminology used in this thesis.

In the second chapter, the $P_3$ intersection graph of a graph $G$ which is the intersection graph of all induced 3-paths in $G$ is studied in detail. The following are some of the results proved:

- For a connected graph $G$, $P_3(G)$ is bipartite if and only if $G$ is $P_3$, $P_4$, $K_4 - \{e\}$ or a paw.
- $K_{1,4}$ is a forbidden subgraph for a graph to be the $P_3$ intersection graph.
- There exist only a finite family of forbidden subgraphs for the $P_3$ intersection graphs to be $H$-free for any finite graph.
H.

- For a connected graph $G$, $\chi(P_3(G)) \geq \chi(G) - 1$. The equality holds if and only if $G$ is either $K_n - \{e\}$ or a complete graph with a pendant vertex attached to it.

- The relationship between the chromatic number, clique number, connectivity, independence number, domination number, the radius and the diameter of a graph and its $P_3$ intersection graph.

The third chapter is the study of another graph operator - the edge $C_4$ graph of a graph. If $G$ does not contain $C_4$ as a subgraph, then the edge $C_4$ graph of a graph coincides with its line graph. So if $G$ is an Eulerian graph which does not contain $C_4$ as a subgraph, then $E_4(G)$ is Eulerian. Following are some of the results obtained:

- There exist infinitely many pairs of non isomorphic graphs whose edge $C_4$ graphs are isomorphic.

- Characterizations for $E_4(G)$ being connected, complete, bipartite etc.
• There is no forbidden subgraph characterization for $E_4(G)$.

• The relationships between the diameter, radius, center, domination number of $G$ and those of $E_4(G)$.

• Relationships between different types of dominations of $G$ and that of $E_4(G)$.

• For any connected graph $G$, $diam(G) - 2 \leq \gamma_c(E_4(G)) \leq 2\beta(G) - 1$.

• A bound for the domination number of $E_4(G)$ in terms of the order of $G$. Further for a graph $G$, which is a tree or a unicyclic graph, characterization is obtained for the strict bound of the domination number of $E_4(G)$.

• Conditions for the $E_4(G)$ being a clique dominated graph, threshold graph, cograph, geodetic graph, weakly geodetic graph and block graph.

The dynamics such as convergence, divergence, periodicity, fixedness etc of the $P_3$ intersection graph and the edge $C_4$ graph are included in chapter four. The following are some of the results proved:
1.5. Summary of the thesis

- There are no $P_3$-periodic graphs.

- If a graph $G$ is $P_3$-convergent, then it is $P^n_3(G)$-complete for some $n \geq 1$ and hence all the $P_3$-convergent graphs are $P_3$-mortal graphs.

- The relationship between the touching number of $P_3(G)$ and the vertex touching number of $G$.

- Characterization of the $E_1$-convergent graphs.

- The relationship between the touching number of $G$ and that of $E_1(G)$.

In chapter five of this thesis, the diameter variability and the $w$-wide diameter of the three graph operators - the $P_3$ intersection graph, the edge $C_4$ graph and the line graph and some graph operations such as join and corona are studied. Some of the results are listed below:

- Corresponding to a $w$-container in $G$, there exists $w$-containers in $P_3(G)$ and $E_4(G)$.

- Strict bound for the $w$-wide diameter of $P_3(G)$, $L(G)$ and $E_4(G)$. 
• Strict bounds for $D^{+i}$ of $P_3(G)$, $E_4(G)$ and $L(G)$.

• The diameter variability of join and corona of two graphs.

All the graphs considered in this thesis are finite, undirected and simple. Some results of this thesis are included in [55] - [60]. We conclude the thesis with some suggestions for further study and a bibliography.

1.6 List of publications

Papers presented


(2) The edge $C_4$ graph of a graph, International Conference on Discrete Mathematics, December 15 – 18, 2006, IISc, Bangalore, India.
(3) The Dynamics of the $P_3$ intersection graph, National Seminar on Algebra and Discrete Mathematics, November 14–16, 2007, University of Kerala, Trivandrum, India.

(4) The edge $C_4$ graph of some graph classes, International Conference on Discrete Mathematics, June 6–10, 2008, University of Mysore, Mysore, India.

(5) Some Domination parameters in $E_4(G)$, National Seminar on Discrete Mathematics and its Applications, August 7–9, 2008, St. Pauls College, Kalamassery, India.


**Papers published / communicated**


(2) Manju K. Menon, A. Vijayakumar, The Edge $C_4$ graph of a graph, Proceedings of the International Conference
(3) Manju K. Menon, A. Vijayakumar, Dynamics of the $P_3$ intersection graph (communicated).

(4) Manju K. Menon, A. Vijayakumar, The Edge $C_4$ graph of some graph classes (communicated).

(5) Manju K. Menon, A. Vijayakumar, Some domination parameters in $E_4(G)$ (communicated).