CHAPTER 1

INTRODUCTION

1.1 PRELIMINARY

Reliability theory developed apart from the mainstream of probability and statistics. It was used primarily as a tool to help nineteenth century maritime and life insurance companies compute profitable rates to charge their customers. In today's technological world nearly everyone depends upon the continued functioning of a wide array of complex machinery and equipment for their everyday health, safety, mobility and economic welfare. We expect our cars, computers, electrical appliances, lights, televisions etc. to function whenever we need them day after day, year after year. When they fail the results can be catastrophic: Injury, loss of life and/or costly lawsuits can occur. More often, repeated failure leads to annoyance, inconvenience and a lasting customer dissatisfaction that can play havoc with the responsible company's marketplace position. It takes a longtime for a company to build up a reputation for reliability and only a short time to be branded as "unreliable" after shipping a flawed product. Continual assessment of new product reliability and ongoing control of the reliability of everything shipped are critical necessities in today's competitive business arena.

The everyday usage term "quality of a product" is loosely taken to mean its inherent degree of excellence. In industry, this is made more precise by defining quality to be "conformance to requirements at the start of use". Assuming the product specifications adequately capture customer requirements the quality level can now be precisely measured by the fraction of units shipped that meet specifications. But, how many of these units still meet specifications after a week of operation? Or after a month or at the end of a one-year warranty period? That is where "reliability" comes in. Quality is a snapshot at the start of life and reliability is a motion picture of the day-by-day operation. The quality level might be described by a single fraction defective. To describe reliability fallout a probability model that describes the fraction fallout over time is needed. This is
known as life distribution model. A life distribution does find its frequent application in the engineering and biomedical sciences.

The times to the occurrences of events, which are of interest for some population of individuals, are termed as “life times”. Some times the events of interest are deaths of individual in the real sense and “life time” is the actual length of life of an individual or perhaps a survival time, measured from some particular starting point. In other instances “life time” is used in a figurative sense. Mathematically, one can think of “life time” as merely meaning “non-negative valued variable”. For e.g. manufactured items such as mechanical or electronic components are often subjected to life tests in order to obtain information on their endurance. This involves putting items in operation, often in a laboratory setting and observing them until they fail. It is common here to refer to life times as “failure times”, since when an item ceases operating satisfactorily, it is said to have “failed”.

The theoretical population models used to describe unit life times are known as lifetime distribution models. The population is generally considered to be all of the possible unit life times that could be manufactured based on a particular design and choice of materials and manufacturing process. A random sample of size n from this population is the collection of failure times observed for a randomly selected group of n units. A lifetime distribution model can be any probability density function $f(t)$ defined over the range of time from $t = 0$ to $t = \infty$. The corresponding cumulative distribution function $F(t)$ is a very useful function as it gives the probability that a randomly selected unit will fail by time $t$. The data, to which statistical methods are applied in order that parameters of interest can be estimated in this reliability context, usually result from life tests. A typical life test is one in which prototypes of the item or organism of interest is subjected to stresses and environmental conditions that typify the intended operating conditions. During the test successive times to failures are noted. Since the failures occur in order, the theory of order statistics plays an important role in the analysis of the life test data.

Literature related to statistical methods used in the analysis of life test data lies scattered in a number of professional journals and books. Reliability studies frequently involve testing of items (say n in number) that are designed to last for long periods of time. In such studies, constraints in the form of truncation
and/or censoring would be deemed essential as a means of obtaining information within reasonable time limitations, while there are several means of censorship (see Gajjar and Khatri (1969)) two types of common usage. These are commonly referred to as Type-I and Type-II censorships. Type-I censorship or censoring occurs when the researcher sets a time limit on terminating the life test even though some of the test items remain operational. Type-II censoring occurs when the life test is terminated at the particular (the \( r^{th} \), say \( r < n \)) failure. In Type-I censoring the number of failures and all the failure times are random variables, whereas in Type-II censoring the failure times are random variables, the number of failures being considered fixed. Type-II censoring has the advantage of providing more or less uniform amount of information in repeated sampling with the disadvantage that the length of testing time varies from test to test. Type-I censoring provides a constant length of testing time in repeated sampling with amount of information varying from test to test. One advantage of Type-I censoring is that it simplifies the problem of test scheduling in a production process where information from periodic production of lots has to be obtained at regular intervals.

1.2 PRE- WORK

There is an extensive body of literature concerning properties of several estimators that are proposed for estimating parameters of probability models commonly used in reliability studies under Type-II censoring. Though some work in the area of reliability and life testing has been done under Type-I censoring but it is not as extensive as that under Type-II censoring. The early work concerning estimation of parameters from continuous life time distributions such as normal, exponential, weibull, extreme value distributions and discrete life time distribution particularly geometric distribution based on single stage Type-I and Type-II censoring was initiated by Gupta (1952), Epstein and Sobel (1953, 1954), Lieblein and Zelen (1956), Bartholomew (1957, 1963), Cohen (1965), Tiku (1967) and others. Recently, rather extensively the work has been studied by, Yaqub and Khan (1981), Patel and Gajjar (1990), Cohen (1991), Balakrishnan and Cohen (1991), Bain and Engelhardt (1991) and Hater and Balakrishnan (1996). These
authors have all considered lifetime studies in industrial as well as actuarial (human life time) contexts, in parametric and non-parametric cases.

In several situations, the initial censoring results only in withdrawal of a portion of the surviving items. Those which remain on test continue under further observation until an ultimate failure or until a subsequent stage of censoring is performed. For sufficiently large samples censoring is done through several stages. This leads to progressive censoring of Type I or Type II. Progressive censoring can be adopted for several reasons. Progressively censored sample arise, for instance, when certain items must be withdrawn from a life test prior to failure for use as test objects in related experimentation. They may also result from a compromise between the need for more rapid testing and the desire to include at least some extreme life spans in the sample data. When the test facilities are limited and when prolonged life tests are cost-prohibitive, the early censoring of a substantial number of items from the test frees facilities for other tests while items which are allowed to continue on test until subsequent failures provide information on extreme sample values.

Cohen (1963) considered type I progressively censored samples in case of normal and exponential distributions and obtained maximum likelihood estimates of the parameters of these distributions with the assumption that the parameters remain the same at each stage of censoring. But there are situations where it might be reasonable to assume that the parameters of a distribution under consideration might change at each stage of censoring. The justification of this reasoning lies in the fact that the surviving items entering the subsequent stage are checked and overhauled eliminating or repairing minor defects wherever possible. It may be noted that due to different parameters at different stages of censoring it leads to estimating parameters from truncated censored distributions. Srivastava (1967), Gajjar and Khatri (1969), Patel and Gajjar (1979) and Patel (1991) have considered type I progressively censored and group-censored samples from exponential, Weibull, inverse Gaussian, log-normal, power series and logistic distributions with different parameters at different stages of censoring and obtained maximum likelihood estimates of the parameters. The maximum likelihood estimates or estimating equations obtained by Gupta (1952) and Cohen (1963) can be deduced as special cases from these results.
1.3 PRESENT WORK

Sometimes certain items are installed at different times and each item has its own specific censoring time $L_i$. Stated more precisely, suppose $n$ items are subjected to limited period of observations $L_1, L_2, ..., L_n$ so that an individual's life time $T_i$ is observed only if $T_i \leq L_i$. The data from such a set up can be conveniently represented by the $n$ pairs of random variables $(t_i, \delta_i)$, where

$$t_i = \text{Min} \left( T_i, L_i \right)$$

and

$$\delta_i = \begin{cases} 1 & \text{if } T_i \leq L_i \\ 0 & \text{if } T_i > L_i \end{cases}$$

That is $\delta_i$ indicates whether the lifetime $T_i$ is censored or not, and $t_i$ is equal to $T_i$ if it is observed, and to $L_i$ if it is not. In this case the likelihood function is

$$L = \prod_{i=1}^{n} f(t_i)^{\delta_i} S(L_i)^{1-\delta_i}$$

It is to be noted that Bartholomew (1957) has discussed the estimation of mean life of an item using such life test experiment with the assumption that the lifetime distributions of an item remains the same through out the experiment. Srivastava (1967) has considered a problem of maximum likelihood estimation of the parameters with exponential distribution having failure rate $\lambda_1$ when lifetime is less than or equal to known time $T_0$ otherwise exponential with failure rate $\lambda_2$. This type schemes frequently arises in medical research where, for example, a decision is made to terminate a study at a date on which not all the patients/individuals' lifetimes will be known.

Suppose an item with failure rate $X$ follows the distribution $F_X(\theta)$ with density function $f_X(\theta)$ for $\theta$ is a vector valued parameter in a real
parameter space $\Omega$. Suppose $X$ has the distribution function $F(X_{i_0}; \theta)$ in the time interval $(N_{i-1}, N_i]$ for $i = 1, 2, \ldots, K (K > 1)$ with $N_0 = 0$ and $N_K = \infty$.

Let $n$ items are placed on a life test without replacement and let $n_i$ items be fail at the $i$th stage. Let $r_i$ be the number of items that withdrawn from the test immediately after the censoring time $N_{i-1}, i = 2, 3, \ldots, K$ so that $n(K) = n(K) - n_K$; where $n(K)$ denotes the number of item entering the $K$th stage of an experiment. Also, let $X_{1(i)} \leq X_{2(i)} \leq \ldots \leq X_{n_i(i)}$ be the times of failure for $i = 1, 2, \ldots, K (K > 1)$ then the likelihood function for $K$-stage Type-I progressive censoring without replacement is given by

$$L \propto \prod_{i=1}^{K} \left( \prod_{j=1}^{n_i} f_i(x_j^{(i)}) \right) \prod_{i=1}^{K} \left[ 1 - F_i(N_i) \right].$$

Taking $X$ as non-negative integer valued random variable and $N_i$’s can be chosen to be non-negative integers, a problem of estimating parameters at different stages of censoring can be considered. The method of maximum likelihood can be employed to estimate the properties of different types of estimators like MLE, shrinkage estimator, minimum mean square error estimator, and almost unbiased estimator can be investigated.

A generalization of Type-II censoring is progressive Type-II censoring. According to Balakrishnan and Aggarwala (2000) under progressive Type-II censoring scheme a total of $n$ units placed on a life test, only $m$ are completely observed until failure. At the time of first failure, $R_1$ of the $n - 1$ surviving units are randomly withdrawn from the test. At the time of next failure $R_2$ of the $n - 2 - R_1$ surviving units are censored, and so on. Finally, at the time of $m^{th}$ failure all the remaining $R_m = n - m - \sum_{i=1}^{m-1} R_i$ surviving units are censored.

Again a more generalization of such progressive Type-II censoring scheme is discussed by Lawless (1982). In this scheme, the first $n_1$ failures in a sample of $n$ items are observed. Then $r_1$ of the remaining $n - n_1$ not failed items are withdrawn from the experiment, leaving $n - n_1 - r_1$ on the test. When further $n_2$ items have failed $r_2$ of the still not failed items are withdrawn and so on. Finally, the experiment is terminated at the end of $n_K^{th}$ failure.
Let \((X^{(0)}_1, X^{(0)}_2, \ldots, X^{(0)}_r)\) are the failure times during the \(i^{th}\) stage of censoring, \(i = 1, 2, \ldots, K\) and \(X^{(1)}_r, X^{(2)}_r, \ldots, X^{(K)}_r\) are the censoring times for \(K\)-stages respectively. Then the likelihood function for \(K\)-stage Type-II progressive censoring without replacement is given by

\[
L = \prod_{i=1}^{K} \frac{n^{(i)}!}{(n^{(i)} - r)!} \left\{ \prod_{j=1}^{n} f_i(x_j^{(i)}) \right\} \prod_{i=1}^{K} \left[ 1 - F_i(x_i^{(r)}) \right]^{r}.
\]

Where \(f(\cdot)\) and \(F(\cdot)\) are probability density function and cumulative distribution function of life time random variable respectively. Using the method of maximum likelihood estimation of the parameters, expected waiting time of the test, expected total time of the test, sample size to minimize the total cost of the test can be considered for discrete or continuous lifetime models.

This thesis is concerned with the characterization of the geometric distribution, hypogeometric distribution - a discrete lifetime model and the problem of estimation under various types of censoring schemes. The problem of estimation contains the following aspects:

1. Estimation of the parameters under Type-I and Type-II progressive censoring scheme when samples are drawn from
   
   (a) Geometric and two parameter geometric
   
   (b) Compound geometric or competing risk geometric
   
   (c) Mixture of geometric distributions

2. Best linear unbiased estimation based on order statistics from geometric distribution.

Chapter-2 deals with the study of some basic results and characterizations of geometric lifetime model. Recently, a lot of interest is generated in the area of discrete lifetime models. When data in lifetimes are in grouped form, discrete models become very much handy in their analysis. Generally, it is inconvenient to measure the life length of a device continuously until its failure. In such circumstances, one measures the life of a device on a
discrete scale and considers the number of successful cycles or operations before failure. Therefore, the number of successful cycles before failure is more pertinent rather than the time of continuous failure.


When \( n \) items are under test without replacement and failure time distribution is geometric we have derived the distribution of between failure times and number of failures before trial \( t \).

Rajesh and Nair (1988) have considered a characterization of the geometric distribution using residual entropy function. Pathak and Sreehari (1981), Sreehari (1981) have considered some characterizations of the geometric and bivariate geometric distribution based on memory less property and other methods. Failure rate and mean residual life function are the important tools in reliability studies. We have considered some characterization results based on failure rate and mean residual life function of geometric distribution.

**Chapter- 3** is concerned with the estimation of parameters of the geometric distribution based on Type-I progressively censored samples with changing parameters at each stage of censoring. Also, we have studied relative bias, mean square error of the maximum likelihood estimates. Shrinkage estimator is considered and minimum mean square shrinkage estimator is derived. We have also obtained alternative and almost unbiased estimator of the parameters and made comparisons of these estimators. In establishing all these results, it is assumed that a number of items entering each stage of censoring is known.

Sometimes in life testing experiment the items are put on test at different times and at one particular instant, we know only the time of failure or the time since it is on test if it has not failed up to that instant can be observed. This is important from the point of view that the items are to be tested without interfering unduly with the normal running of a factory process. Such a situation frequently arises in medical research where a decision is made to terminate a study at the time at which not all individual’s life times will be known. For e.g. patients may enter the study in a more or less random fashion according to their time of diagnosis. If the study is terminated at some prearranged time; then
censoring times, i.e. the lengths of time from an individual's entry into study until the termination of the study are random. Bartholomew (1957) and Srivastava (1967) consider such problems.

The estimation of parameters of the geometric lifetime model under two stage Type-I progressive censoring with changing failure rate is considered under different installation time. The method of maximum likelihood is used and biases of the ML estimators are derived. The numerical example is cited to exemplify the estimation procedure. These results are presented in chapter -4.

A population is postulated to be composed of two subpopulations representing failure types mixed in proportion $p : (1-p)$ where $0 \leq p \leq 1$. Each unit in the population contains a tag, which signifies the subpopulation to which it belongs. The information on the tag (i.e. the cause of failure) is obtained only after a failure has occurred. Mendenhall and Hader (1958) and Lee and Sinha (1976) considered the mixture when the components have exponential and weibull distributions as life testing models and estimated the parameters by the method of maximum likelihood based on single stage Type-I censoring. Patel (1998) has considered extension of the work done by Mendenhall and Hader (1958) for two stage Type-I progressive censoring. When samples are progressively Type-I censored with different parameters of component geometric distribution and different mixing proportions, the maximum likelihood estimation of the parameters in case of two-stage progressive censoring and group progressive censoring is investigated in the chapter-5. Numerical examples are given.

Compound exponential model or competing risk exponential model like the mixture of exponential model allows flexibility in explaining failure data when there are multiple causes of failure which may be attributed to electrical, thermal, climatic and mechanical stresses operation upon the item or device. Boardman and Kendell (1970) have considered the problem of Type-I censoring when an item is subjected to only of two causes of failure assuming that time to failure of an item due to the $i^{th}$ failure is distributed as exponential with parameters $\lambda_i$, ($i = 1,2$). Patel and Gajjar (1991) have considered an extension of the compound exponential model with different parameters at each stage of censoring and an item is subjected to only one of the two causes of failure. Nelson (1970) has considered a testing of electrical appliances. The lifetimes of appliances are
considered in terms of the number of completed cycles to failure. Moreover, appliances could fail due to different 18 causes of failure. This shows the use of discrete failure time distribution in competing risk failure model.

Chapter-6 deals with the problem of maximum likelihood estimation with respect to Type-I censoring and group censoring without replacement, when an item is subjected to one and only one of the two causes of failure assuming geometric competing risk failure model. In addition, the problem is extended to two-stage Type-I progressive censoring scheme. The asymptotic standard errors of the estimates are obtained for both the types of censoring as well as two-stage Type-I progressive censoring. Some examples are included to show application of the estimation from these types of competing risk failure models.

One parameter geometric distribution has been cosidered by numerous authors like Yaqub and Khan (1981), Patel and Gajjar (1990) as a lifetime model. A two parameter geometric distribution with p.m.f.

\[ f(x; \theta, N) = P(X = x) = \left( \frac{1}{1 + \theta} \right)^{x-N} \left( \frac{\theta}{1 + \theta} \right) ; x = N+1, N+2, ..., \theta > 0 \text{ and } N \text{ is non-negative integer} \]

is employed in situations where it is believed that death or failure cannot occur before certain cycles \( N \). Here \( N \) and \( \theta \) are unknown parameters and \( N \) is known as warranty time or threshold parameter. In chapter-7, we have discussed maximum likelihood estimation of the parameters based on Type-II and multiply Type-II censored samples from a two parameter geometric distribution. Maximum likelihood estimators of expected life and reliability of an item are also derived with the asymptotic standard errors. To illustrate the methods of estimation three examples involving simulated failure time data are presented.

A generalization of Type-II censoring scheme is considered by Balakrishnan and Aggarwala (2000), which is known as progressive Type-II censoring scheme. Under this scheme of censoring from a total of \( n \) units placed on a life test, only \( m \) are completely observed until failure. At the time of first failure \( R_1 \) of the \( n-1 \) surviving units are randomly withdrawn from the test. At the time of next failure, \( R_2 \) of the \( n-2 - R_1 \) surviving units are censored, and so on. Finally, at the time of \( m \)th failure all the remaining \( R_m = n - m - \sum_{i=1}^{m-1} R_i \) surviving units are censored. The ordered failure times arising from such a progressively Type-II right censored samples are called progressively Type-II right censored...
order statistics. We have discussed the method of estimation of parameter(s) from various continuous lifetime models with the same parameter at each stage of progressive Type-II censoring.

Lawless (1982) has discussed a more generalization of such progressive Type-II censoring scheme, in which the first $n_1$ items are observed. Then $r_1$ of the remaining $n - n_1$ unfailed items are withdrawn from the experiment, leaving $n - n_1 - r_1$ on the test. When further $n_2$ items have failed $r_2$ of the still unfailed items are withdrawn and so on. Finally, the experiment is terminated at the end of $n_K$ failure. **Chapter-8** is concerned with the maximum likelihood estimation of the parameters of the geometric lifetime model under progressive Type-II censoring with changing parameters at each stage of censoring. In addition, MLE of reliability function at time $t$, expected total termination time, expected total test time and expression for minimum sample size required for the test with the given total cost of the experiment are considered. In addition, prediction of the $n_i$ failure for the $i$ stage of censoring is discussed on the basis of available $n_s$ failures at the $i$ stage of censoring. A test for homogeneity of a selected set of parameters out of $K$ parameters of the $K$ stage Type-II progressive censoring scheme is also included.

Numerous parametric models are used in the analysis of lifetime data and in the analysis of lifetime data and in the study of system reliability. Among univariate models, a few particular distributions occupy a central role because of their usefulness in a wide range of situations. Some of the important models in this category are the exponential, Weibull, gamma, log-normal. Shooman (1968) has examined aging and failure process of such lifetime models. Using probabilistic and/or statistical techniques analytical models of the reliability of a system can be derived. A system may be collections of equipments arrange to perform a function. The equipments arranged in the system may be in series, parallel, stand by or m out of n mode. As for e.g. CPU, missile, T.V. etc. The prediction of the reliability is based on the failure time distribution used. Trivedi (2002) has discussed hypoexponential distribution for a two component stand by system with statistically independent components having different exponential failure rate. Here it is assumed that switching equipment is perfect.

In **chapter-9**, we have considered a two-stage hypogeometric failure distribution as a discrete lifetime model. Here the failure trial $X$ of such a
lifetime model is given by some of the failure trials $X_1$ and $X_2$ of individual components, where individual component have geometric lifetime model with parameters $q_1$ and $q_2$ respectively. Statistical properties of the model along with estimation of the parameters as well as applications of the model are considered.

Chapter-10 deals with best linear unbiased estimation of the parameter of geometric distribution based on Type-II censored ordered statistics. Best linear unbiased estimation is one of the most commonly used methods of estimation for the location and scale parameter of a population when the available sample is either complete or Type-II censored. Balakrishnan and Rao (1997) have discussed a simple method of derivation of the best linear unbiased estimators based on order statistics from exponential distribution under Type-II and doubly Type-II censored samples. Following Balakrishnan and Rao (1997) we have derive the best linear unbiased estimators for a geometric distribution in case of Type-II and multiply Type-II censored samples. Illustrative examples are included for both the schemes.