CHAPTER 10

BEST LINEAR UNBIASED ESTIMATION BASED ON ORDER STATISTICS FROM GEOMETRIC DISTRIBUTION

10.1 INTRODUCTION

Reliability studies frequently involve testing of items that are designed to last for long period. In such studies, constraints in the form of truncation and/or censoring would be deemed essential as a means of obtaining information within reasonable time limitations. While there are several means of censorship (see: Gajjar and Khatri (1969)) of which two are of common usage. These are commonly referred to as Type-I and Type-II censorships. In life-testing experiments, an experimenter may often have to terminate the experiment after a certain number of units fail instead of waiting for all the units to fail. This is naturally both time and cost effective. Samples observed in this manner are called Type-II censored samples.

Under Type-II censored scheme when the failure times of some units were not observed due to mechanical or experimental difficulties or some units failed between two points of observation with exact times of failure of these units unobserved. Sample obtain under such situations is known as multiply Type-II censored sample.

Best linear unbiased estimation is one of the commonly used methods of estimation for the location and scale parameters of a population when the available sample is either complete or Type-II censored; see, for example, David(1981), Balakrishnan and Cohen(1991) and Arnold, Balakrishnan and Nagaraja(1992).

Balakrishnan and Rao (1997) have discussed a simple method of derivation of best linear unbiased estimators based on order statistics from
exponential distribution under Type-II and doubly Type-II censored samples. Here we have considered best linear unbiased estimators for a geometric distribution based on order statistics under censored samples following Balakrishnan and Rao (1997). An illustrative example is cited to exemplify the method in the section: 10.4.

10.2 ESTIMATION BASED ON TYPE-II CENSORED SAMPLES

Let \( X^{(1)} \leq X^{(2)} \leq \ldots \leq X^{(n-s)} \) be a Type-II right censored ordered sample for a test with sample size \( n \) from a geometric distribution given by

\[
P(X = x) = \left( \frac{1}{1 + \theta} \right) \left( \frac{\theta}{1 + \theta} \right)^{x-1}, x = 1, 2, \ldots; \theta > 0. \tag{10.2.1}
\]

It is well known that

\[
Z_i = (n - i + 1)(X^{(i)} - X^{(i-1)}), \quad i = 1, 2, \ldots, n \tag{10.2.2}
\]

are all independent and identically distributed random variables having geometric distribution with probability distribution,

\[
P(Z = z) = \left( \frac{1}{1 + \theta} \right) \left( \frac{\theta}{1 + \theta} \right)^z, \quad z = 0, 1, 2, \ldots; \theta > 0. \tag{10.2.3}
\]

In order to derive the BLUE of \( \theta \), let us write

\[
\theta^* = \sum_{i=1}^{n-s} a_i X^{(i)} \tag{10.2.4}
\]
or equivalently

\[ \theta^* = \sum_{i=1}^{n-s} a_i' z_{(i)} \]  

(10.2.5)

where

\[ a_i = a_i'(n - i + 1) - a_{i+1}'(n - i); i = 1, 2, \ldots, n - s - 1 \text{ and } a_{n-s} = a_{n-s}'(s + 1). \]

It is clear that

\[ E(\theta^*) = \theta \sum_{i=1}^{n-s} a_i' \quad \text{and} \quad V(\theta^*) = \theta(1 + \theta) \sum_{i=1}^{n-s} a_i'^2. \]  

(10.2.6)

Since \( \theta^* \) needs to be unbiased for \( \theta \), subject to minimum \( V(\theta^*) \), we need to do is

Minimize \( \sum_{i=1}^{n-s} a_i'^2 \) subject to the condition \( \sum_{i=1}^{n-s} a_i' = 1. \)

Now consider a function \( L \) as follow:

\[ L = \sum_{i=1}^{n-s} a_i'^2 + \lambda \left(1 - \sum_{i=1}^{n-s} a_i'\right) \]

where \( \lambda \) is Langrange's multiplier.

Upon differentiating \( L \) with respect to \( a_i' \) and \( \lambda \) separately and comparing them with zero, we get \( a_i' = \frac{1}{n-s}, i = 1, 2, \ldots, n-s \) and hence

\[ a_i = \frac{1}{n-s} \text{ for } i = 1, 2, \ldots, n - s - 1 \text{ and } a_{n-s} = \frac{(s+1)}{(n-s)}. \]

Then from (10.2.4) BLUE of \( \theta \) can be obtained as

\[ \theta^* = \frac{1}{n-s} \left[ \sum_{i=1}^{n-s-1} X_{(i)} + (s+1)X_{(n-s)} \right] \]

\[ = \frac{1}{n-s} \left[ \sum_{i=1}^{n-s} X_{(i)} + sX_{(n-s)} \right] \]  

(10.2.7)
with
\[ V(\theta^*) = \frac{\theta(1+\theta)}{n-s} \]  
(10.2.8)

Note that here BLUE $\theta^*$ of $\theta$ is identical to the maximum likelihood estimator.

Next, let us consider $X_{(r+1)} \leq X_{(r+2)} \leq \cdots \leq X_{(n-s)}$ be a Type-II doubly censored ordered sample for a test with sample size $n$ from geometric distribution in (10.2.1). Then in order to derive the BLUE of $\theta$ of the form

\[ \theta^* = \sum_{i=r+1}^{n-s} a_i X_{(i)} \]  
(10.2.9)

which we write as

\[ \theta^* = a_{r+1} X_{(r+1)} + \sum_{i=r+2}^{n-s} a_i Z_i \]  
(10.2.10)

Since $X_{(r+1)}$ and $Z_i (i = r+2, \ldots, n-s)$ are statistically independent and as

\[ X_{(r+1)} = \sum_{i=1}^{r+1} \left( \frac{Z_i}{n-i+1} \right) \]

we have

\[ E(\theta^*) = \theta \left[ a_{r+1} \alpha_{r+1} + \sum_{i=r+2}^{n-s} a_i \right] \]  
(10.2.11)

and

\[ V(\theta^*) = \theta(1+\theta) \left[ \sum_{r+1} a^2_{r+1} + \sum_{i=r+2}^{n-s} a_i^2 \right] \]  
(10.2.12)

where

\[ \alpha_j = \sum_{i=1}^{j} \left( \frac{1}{n-i+1} \right) \text{ and } \sum_{j,j} = \sum_{j,k} = \sum_{i=1}^{j} \left( \frac{1}{(n-i+1)^2} \right) \text{ for } k > j. \]  
(10.2.13)
Now minimizing the variance in (10.2.12) subject to the condition that $E(\theta^*)$ in (10.2.12) equals $\theta$, we immediately obtain

$$a_{r+1} = \frac{\alpha_{r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2},$$

$$a_i = \frac{\sum_{r+1,r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2}; \quad i = r+2, \ldots, n-s-1.$$

Hence, we have

$$a_{r+1} = \frac{\alpha_{r+1} - (n-r-1)\sum_{r+1,r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2},$$

$$a_i = \frac{\sum_{r+1,r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2}; \quad i = r+2, \ldots, n-s-1.$$

and

$$a_{n-s} = \frac{(s+1)\sum_{r+1,r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2}.$$

Then the BLUE of $\theta$ from (10.2.9) is given by

$$\theta^* = \frac{[\alpha_{r+1} - (n-r-1)\sum_{r+1,r+1}]X_{(r+1)} + \sum_{j=r+2}^{n-s-1} X_{(j)} \sum_{r+1,r+1} + (s+1)\sum_{r+1,r+1} X_{(n-s)}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2}$$

and

$$V(\theta^*) = \frac{\theta(\theta+1)\sum_{r+1,r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2}$$

and

$$V(\theta^*) = \frac{\theta(\theta+1)\sum_{r+1,r+1}}{(n-s-r-1)\sum_{r+1,r+1} + \alpha_{r+1}^2}$$

(10.2.14)

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10.3 ESTIMATION BASED ON MULTIPLY TYPE-II CENSORED SAMPLE

In this section we consider a general multiply Type-II censored data from the geometric distribution in (10.2.1). Out of n components put to test, suppose the experimenter fails to observe the first r, last s, and the middle t observations. Now, let

\[ Y_{(r+1)} \leq \ldots \leq Y_{(r+k)} \leq Y_{(r+k+t+1)} \leq \ldots \leq Y_{(n-s)} \]

be the available multiply Type-II censored sample from a geometric distribution in (10.2.1) (Balakrishnan, 1990).

Then in order to derive the BLUE of \( \theta \) of the form

\[
\theta^* = \sum_{i=r+1}^{r+k} a_i X_{(i)} + \sum_{i=r+k+t+2}^{n-s} a_i X_{(i)}
\]

(10.3.1)

let us write \( \theta^* \) as

\[
\theta^* = a_{r+1} X_{(r+1)} + \sum_{i=r+2}^{r+k} a_i Z_i + a_{r+k+t+1} X_{(r+k+t+1)} + \sum_{i=r+k+t+2}^{n-s} a_i Z_i.
\]

Because \( X_{(r+1)} \) and \( Z_i \) ( \( i = r + 2, \ldots, r+k \)) and \( X_{(r+k+t+1)} \) and \( Z_i \),

( \( i = r + k + t + 2, \ldots, n-s \)) are statistically independent we readily have

\[
E(\theta^*) = \theta \left[ a_{r+1} a_{r+1} + \sum_{i=r+2}^{r+k} a_i a_{r+k+t+1} \alpha_{r+k+t+1} + \sum_{i=r+k+t+2}^{n-s} a_i \right]
\]

(10.3.2)

and

\[
V(\theta^*) = \theta(1+\theta) \left[ \Sigma_{r+1, r+1} a_{r+1}^2 + \sum_{i=r+2}^{r+k} a_i^2 + a_{r+k+t+1}^2 \Sigma_{r+k+t+1, r+k+t+1} + \sum_{i=r+k+t+2}^{n-s} a_i^2 \right]
\]

(10.3.3)
where
\[ a_{r+1} = a_{r+1} - (n-r-1)a_{r+2}, \]
\[ a_i = a_i(n-i+1) - a_{i+1} (n-i); i = r+2, ..., r+k+1, r+k+t+2, ..., n-s-1, \]
\[ a_{r+k} = a_{r+k} (n-r-k+1), \]
\[ a_{r+k+t+1} = a_{r+k+t+1} - (n-r-k-t-1)a_{r+k+t+2}, \]
and
\[ a_{n-s} = a_{n-s} (s+1). \] (10.3.4)

Because \( \theta^* \) needs to be unbiased for \( \theta \), we must have
\[ a_{r+1}a_{r+1} + \sum_{i=r+2}^{r+k} a_i + a_{r+k+t+1}a_{r+k+t+1} + \sum_{i=r+k+t+2}^{n-s} a_i = 1 \]
Let us assume that
\[ a_{r+1}a_{r+1} + \sum_{i=r+2}^{r+k} a_i = \phi \]
so that
\[ a_{r+k+t+1}a_{r+k+t+1} + \sum_{i=r+k+t+2}^{n-s} a_i = 1 - \phi, \text{ where } \alpha < \phi < 1. \] (10.3.5)

Now minimizing variance in (10.3.3) subject to the condition that \( E(\theta^*) \) in (10.3.2) equals \( \theta \) and the assumptions made in (10.3.5), we immediately obtain
\[ a_{r+1} = \frac{\phi a_{r+1}}{A}, \]
\[ a_i = \frac{\phi \sum_{r+1}^{r+1} a_{r+1}}{A}; i = r+2, ..., r+k, \]
\[ a_{r+k+t+1} = \frac{(1-\phi)a_{r+k+t+1}}{B}, \]
and
\[ a_j = \frac{(1-\phi) \sum_{r+k+t+1}^{r+k+t+1}}{B}; \quad j = r+k+t+1, \ldots, n-s, \]

where

\[ A = \alpha_{r+1}^2 + (k-1) \sum_{r+1}^{r+1} \quad \text{and} \quad B = \alpha_{r+k+t+1}^2 + (n-s-r-k-t-1) \sum_{r+k+t+1}^{r+k+t+1} \]

Hence, from (10.3.4) we have

\[ a_{r+1} = \frac{\phi (\alpha_{r+1} - (n-r+1) \sum_{r+1}^{r+1})}{A}, \]

\[ a_i = \frac{\phi \sum_{r+1}^{r+1}}{A}, \quad i = r+2, \ldots, r+k-1, \]

\[ a_{r+k} = \frac{\phi \sum_{r+1}^{r+1} (n-r-k+1)}{A}, \]

\[ a_{r+k+t+1} = \frac{(1-\phi) (\alpha_{r+k+t+1} - (n-r-k-t-1) \sum_{r+k+t+1}^{r+k+t+1})}{B}, \]

\[ a_j = \frac{(1-\phi) \sum_{r+k+t+1}^{r+k+t+1}}{B}, \quad j = r+k+t+1, \ldots, n-s, \]

\[ a_{n-s} = \frac{(1-\phi) (s+1) \sum_{r+k+t+1}^{r+k+t+1}}{B}. \]

Thus, we readily obtain the BLUE of \( \theta \) in this case as

\[ \theta^* = \frac{\phi (\alpha_{r+1} - (n-r+1) \sum_{r+1}^{r+1}) X_{(r+1)}}{A} + \frac{\phi \sum_{r+1}^{r+1} \sum_{i=r+1}^{r+k-1} X(i)}{A} + \frac{\phi (n-r-k+1) \sum_{r+1}^{r+1} X_{(r+k)}}{A} + \frac{\phi (1-\phi) (\alpha_{r+k+t+1} - (n-r-k-t-1) \sum_{r+k+t+1}^{r+k+t+1}) X_{(r+k+t+1)}}{B} + \frac{(1-\phi) \sum_{r+k+t+1}^{r+k+t+1} \sum_{i=r+k+t+2}^{n-s} X(i)}{B} + \frac{(1-\phi) (s+1) \sum_{r+k+t+1}^{r+k+t+1} X_{(n-s)}}{B} \]

and

\[ V(\theta^*) = \theta (1+\theta) \left[ \frac{\phi^2 \sum_{r+1}^{r+1}}{A} + \frac{(1-\phi)^2 \sum_{r+k+t+1}^{r+k+t+1}}{B} \right] \]

(10.3.6)
Furthermore it is obvious that $V(\theta^*)$ would be minimum for

$$\phi = \frac{A\sum_{r+k+t+1}^{r+k+t+1}}{B\sum_{r+1}^{r+1} + A\sum_{r+k+t+1}^{r+k+t+1}} \quad (10.3.7)$$

Upon substituting $\phi$ in (10.3.7) in $\theta^*$ and $V(\theta^*)$ given in (10.3.6), we then obtain the BLUE of $\theta$ and $V(\theta^*)$.

10.4 ILLUSTRATIVE EXAMPLE

**Example:**

The following data represent the number of cycles to failure for 100-centimeter specimens of yarn, tested at a particular strain level (Lawless: 1982, page-266):

(a) 15, 20, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, - - .

Here the last two observations are censored because the life testing experiment was terminated as soon as 23rd failure occurred.

Then the BLUE of $\theta^*$ and $SE(\theta^*)$ from (10.2.7) and (10.2.8) are

$$\theta^* = 160.47826 \quad \text{and} \quad SE(\theta^*) = 33.566125$$

(b) Now, we consider Type-II doubly censored sample as follows:

- - - 42, 61, 76, 86, 98, 121, 146, 149, 157, 175, 176, 180, 180, 198, 220, 224, 251, 264, 282, 321, - - .

Here the first three and last two observations are censored.

Then the BLUE of $\theta^*$ and $SE(\theta^*)$ from (10.2.14) and (10.2.15) can be obtain as
\[ \theta^* = 178.66071 \quad \text{and} \quad \text{SE} (\theta^*) = 37.365023. \]

(c) Now, we consider the above data as multiply Type-II censored in the sample following way:

\[-, -, -, 38, 42, 61, 76, 86, 98, 121, 146, 149, 157, -, -, 180, 180, 198, 220, 224, 251, 264, 282, 321, -, -\]

Where the first three, middle two and last two observations are censored or not available.

Where the symbol '-' indicates the censored or not available observations.

Then the BLUE of \( \theta^* \) and \( \text{SE} (\theta^*) \) from (10.3.6) and (10.3.7) can be obtain as

\[ \theta^* = 168.85934 \quad \text{and} \quad \text{SE}(\theta^*) = 29.02301. \]