CHAPTER 7

ESTIMATION OF TWO PARAMETER GEOMETRIC DISTRIBUTION UNDER TYPE-II AND MULTIPLY TYPE-II CENSORING

7.1 INTRODUCTION

The one parameter geometric distribution has an important position in discrete lifetime models. Statistical methods were extensively developed for the simple lifetime model. Many authors have contributed for the methodology and estimation of the parameters of this distribution as discussed in earlier chapters. Here we assume the survival times of certain items have a two parameter geometric distribution with probability mass function (pmf)

\[ f(x; \theta, N) = P(X = x) = \left( \frac{1}{1+\theta} \right)^{\frac{\theta}{1+\theta}}^{x-N-1}, \quad x = N+1, N+2, ..., \theta > 0 \]

and \( N \in I = \{1,2,3,...\} \cup \{0\} \) (7.1.1)

Here \( N \) and \( \theta \) are unknown parameters.

This model is employed in situations where it is believed that death or failure cannot occur before certain cycles \( N \); where \( N \geq 0 \) is the warranty time or threshold parameter and \( \theta + N \) is the expected lifetime of an item. When \( N \) is known, it can be analyzed as for the one-parameter geometric distribution, because \( X - N \) has a one parameter geometric distribution. One should be cautious when the model includes a threshold parameter \( N \), since the data usually does not provide enough information about \( N \) and including \( N \) could cause rather special statistical problems. Fortunately, the problem has been satisfactorily solved for the geometric distribution. In addition, we have discussed ML estimation of the
parameters based on multiply Type-II censored samples from the lifetime model given in (7.1.1). ML estimators of expected life and reliability of an item with their asymptotic standard errors are also derived. Hence, we deduced asymptotic variance of an estimator of reliability of an item at time t. We also present three examples involving failure times to first break down of 20 machines considered by Dunsmore (1974) in order to illustrate the methods of estimation.

7.2 ML ESTIMATION OF THE PARAMETERS UNDER TYPE-II CENSORING

Consider an experiment in which n components are put up to test simultaneously and the failure times (in terms of completed cycles or trials) of these components are recorded. Suppose the experimenter terminates the experiment as soon as \( r \) th component has failed and let ordered failure times of \( r \) components are

\[
x_{(1)}, x_{(2)}, \ldots, x_{(r)}
\]

which constitute a Type-II censored sample.

Then using the pmf (7.1.1) the likelihood function under Type-II censoring is given by

\[
L = \text{const} \prod_{i=1}^{r} f_X(x_{(i)}) \left[ 1 - F_X(x_{(r)}) \right]^{n-r}
\]

where

\[
F_X(x_{(r)}) = P(X \leq x_{(r)}) = 1 - \left( \frac{\theta}{1+\theta} \right)^{x_{(r)}-N}.
\]

Hence the log likelihood function will be

\[
\log L = \text{const} + r \log \left( \frac{1}{1+\theta} \right) + \left\{ \sum_{i=1}^{r} (x_{(i)} - N - 1) + (x_{(r)} - N)(n-r) \right\} \log \left( \frac{\theta}{1+\theta} \right).
\]

(7.2.1)
Then it is a simple exercise to derive the MLE of $N$ and $\theta$ as

$$\hat{N} = x_{(0)} - 1$$

(7.2.2)

and

$$\hat{\theta} = \frac{\sum_{i=1}^{r}(x_{(i)} - x_{(0)}) + (x_{(r)} - x_{(0)} + 1)(n - r)}{r}.$$ 

(7.2.3)

**Theorem:**

Let $X$ be a random variable having two-parameter geometric distribution with parameter, $N$ and $\theta$ as given in (7.1.1), then

$$Z_{(i)} = (n - i + 1)(X_{(i)} - X_{(i-1)}) - 1, \quad i = 1, 2, \ldots, n; \quad X_{(0)} = N$$

has also geometric distribution with parameter $\theta$ where $X_{(i)}$ is the $i^{th}$ order statistic of failure observations.

**Proof:**

The joint distribution of order statistic $X_{(i)}, i = 1, 2, \ldots, n$ is

$$f(x_{(1)}, \ldots, x_{(n)}) = \frac{n!}{(1 + \theta)^n} \left( \frac{\theta}{1 + \theta} \right)^n \frac{\sum_{i=1}^{n}(x_{(i)} - N - 1)}{n^r}.$$ 

(7.2.4)

We may write

$$\sum_{i=1}^{n}(x_{(i)} - N - 1) = \sum_{i=1}^{n}(n - i + 1)(x_{(i)} - x_{(i-1)}) - n \quad \text{with} \quad x_{(0)} = N.$$

Thus, equation (7.2.4) can be written as

$$f(x_{(1)}, \ldots, x_{(n)}) = \frac{n!}{(1 + \theta)^n} \left( \frac{\theta}{1 + \theta} \right)^n \frac{\sum_{i=1}^{n}(n - i + 1)(x_{(i)} - x_{(i-1)}) - n}{n^r}.$$
Making the transformation

\[ Z_i = (n-i+1)(X_{(i)} - X_{(i-1)}) - 1, \ i = 1, 2, \ldots n \]

Then

\[
f(z_1, z_2, \ldots, z_n) = n! \prod_{i=1}^{n} \left\{ \left( \frac{1}{1+\theta} \right) \left( \frac{\theta}{1+\theta} \right)^{z_i} \right\}
\]

\[
= n! \prod_{i=1}^{n} f(z_i, \theta)
\]

and noting that possible values of \( Z_i \) are 0, 1, \ldots, we conclude that \( Z_i \)'s are statistically independent and each has geometric distribution with p.m.f.

\[
f(Z = z) = \left( \frac{1}{1+\theta} \right) \left( \frac{\theta}{1+\theta} \right)^{z}, \ z = 0, 1, 2, \ldots; \ \theta > 0.
\]

Hence, the proof.

It is an easy task to estimate \( \theta \) in terms of order statistic.

As,

\[
\sum_{i=1}^{r} Z_i = \sum_{i=1}^{r} (x_{(i)} - x_{(1)}) + (n-r)(x_{(r)} - x_{(1)} + 1) = y \text{(say)}
\]

(7.2.5)

The ML estimator of \( \theta \) given in (7.2.3) becomes

\[
\hat{\theta} = \frac{y}{r}
\]

(7.2.6)

Note that the distribution of \( Y = \sum_{i=1}^{r} Z_i \) will be negative binomial with the pmf

\[
P(Y = y) = \binom{r+y-1}{y} \left( \frac{1}{1+\theta} \right) \left( \frac{\theta}{1+\theta} \right)^{y}, \ y = 0, 1, 2, \ldots; \ \theta > 0
\]

with \( E(Y) = r\theta \) and \( V(Y) = r\theta(1+\theta) \).

Thus

\[
E(\hat{\theta}) = \theta
\]
and

\[ V(\hat{\theta}) = \frac{\theta(1+\theta)}{r}. \quad (7.2.7) \]

As \( X_{(1)} \) is the first order statistic from the distribution given in (7.1.1), the pmf of \( X_{(1)} \) will be

\[ f_{X_{(1)}}(x) = PQ^{x-N-1}, x = N+1, N+2, \ldots; \quad Q = \left( \frac{\theta}{1+\theta} \right)^n, \quad P = 1 - Q. \]

Hence

\[ E(\hat{N}) = \frac{Q}{P} + N \quad \text{and} \quad V(\hat{N}) = \frac{Q}{P^2}. \quad (7.2.8) \]

As \( \hat{N} \) and \( \hat{\theta} \) are independent, \( \text{cov}(\hat{N}, \hat{\theta}) = 0 \).

Now Reliability of an item up to trial \( t \) is,

\[ R_x(t) = \left( \frac{\theta}{1+\theta} \right)^{-N} \]

whose MLE is given by

\[ \hat{R} = \hat{R}_x(t) = \left( \frac{\hat{\theta}}{1+\hat{\theta}} \right)^{-\hat{N}}. \quad (7.2.9) \]

Here,

\[ \left( \frac{\partial \hat{R}}{\partial \theta} \right)_{(\hat{\theta}, \hat{N})=\left(\theta, N\right)} = (t-N) \left( \frac{\theta}{1+\theta} \right)^{-N-1} \left( \frac{1}{(1+\theta)^2} \right) \]

\[ \left( \frac{\partial \hat{R}}{\partial \hat{N}} \right)_{(\hat{\theta}, \hat{N})=\left(\theta, N\right)} = - \left( \frac{\theta}{1+\theta} \right)^{-N} \ln \left( \frac{\theta}{1+\theta} \right) \]

\[ (7.2.10) \]
Using (7.2.7) to (7.2.10) \( V(\hat{R}(t)) \) can be obtained as (see: Cassela and Berger (2001))

\[
V(\hat{R}_x(t)) = \left[ \left( \frac{\partial \hat{R}}{\partial \hat{\theta}} \right)^2 V(\hat{\theta}) + \left( \frac{\partial \hat{R}}{\partial \hat{N}} \right)^2 V(\hat{N}) \right] \\
\theta, N = (\theta, N)
\]

(7.2.11)

\[
= \left( \frac{\theta}{1 + \theta} \right)^{2(t-N)} \left[ \frac{(t-N)^2}{\theta(1+\theta)^r} + \left\{ \ln \left( \frac{\theta}{1+\theta} \right) \right\}^2 \frac{1}{\left( \frac{1+\theta}{\theta} \right)^n - 1} \right] .
\]

(7.2.12)

### 7.3 ML ESTIMATION UNDER MULTIPLY TYPE-II CENSORING

Suppose \( n \) items are placed on a life testing experiment and the experimenter fails to observe some of the observations.

Let the \( r_1, r_2, ..., r_K \) failure times (in completed cycles) are only made available.

That is,

\[
X_{r_1}, X_{r_2}, ..., X_{r_K}
\]

(7.3.1)

is the multiply Type-II censored sample from a sample of size \( n \) available from (7.1.1), where \( 1 \leq r_1 < ..., < r_K \leq n \).

In this section, we derive the ML estimators of the parameters \( N \) and \( \theta \) with their asymptotic variance covariance matrix.

The likelihood function based on the multiply Type-II censored sample in (7.3.1) is given by

\[
L = \text{const.}(F_X(x_{r_1}))^{n-1} \prod_{i=1}^{K} f_X(x_{r_i}) \prod_{i=1}^{K-1} \left[ F(X_{r_{i+1}}) - F(X_{r_i}) \right]^{r_{i+1} - r_i - 1} \left[ 1 - F_X(x_{r_K}) \right]^{n-r_K}
\]
\[ L = \text{const} \left\{ 1 - \left( \frac{\theta}{1 + \theta} \right)^{x_N - N} \right\} \prod_{i=1}^{k} \left\{ \left( \frac{1}{1 + \theta} \right)^{x_i - N - 1} \right\} \times \]

\[ \prod_{i=1}^{k-1} \left\{ \left( \frac{\theta}{1 + \theta} \right)^{x_i - N} - \left( \frac{\theta}{1 + \theta} \right)^{x_{i+1} - N} \right\}^{S_i} \left( \frac{\theta}{1 + \theta} \right)^{x_N N(x_{i+1} - r_k)} \]

(7.3.2)

where \( s_i = r_{i+1} - r_i - 1 \).

It is very clear that \( L \) given in (7.3.2) is a monotonic increasing function in \( N \) and \( N + 1 \leq x_{r_1} \leq x_{r_2} \leq \ldots \leq x_{r_K} \). Thus, maximum value of \( N \) is \( x_{r_1} - 1 \) so \( L \) would be maximum at \( N = x_{r_1} - 1 \) which shows mle of \( N \) as

\[ \hat{N} = x_{r_1} - 1 \]

(7.3.3)

Now using the results of Margolin and Winokur (1967) we find that

\[ E(\hat{N}) = n \left( \frac{r_1 - 1}{r_1 - 1} \right) \sum_{j=0}^{n-1} \left( \frac{(-1)^j \binom{r_1 - 1}{j}}{(n - r_1 + j + 1)} \frac{1}{1 - \left( \frac{\theta}{1 + \theta} \right)^{n-r_1+j+1}} \right) + N - 1 \]

(7.3.4)

\[ V(\hat{N}) = n \left( \frac{r_1 - 1}{r_1 - 1} \right) \sum_{j=0}^{n-1} \left( \frac{(-1)^j \binom{r_1 - 1}{j}}{n - r_1 + j + 1} \left( \frac{1 + \left( \frac{\theta}{1 + \theta} \right)^{n-r_1+j+1}}{1 - \left( \frac{\theta}{1 + \theta} \right)^{n-r_1+j+1}} \right)^2 \right) \left[ E(\hat{N}) - N + 1 \right]^2. \]

(7.3.5)

From (7.3.2) the likelihood equation for \( \theta \) is obtained as
\[
\frac{\partial \log L}{\partial \theta} = \frac{1}{\theta(1+\theta)} \left[ -\frac{(\tau_1 - 1)(x_\eta - N)}{1 - \left(\frac{\theta}{1+\theta}\right)^{x_\eta - N}} - K\theta - K + nxr_K - Nn \right] 
\]

\[
- \frac{1}{\theta(1+\theta)} \left[ \sum_{i=1}^{K-1} (x_{\eta+1} - x_\eta) \tau_i + \sum_{i=1}^{K-1} \frac{s_i(x_{\eta+1} - x_\eta)}{1 - \left(\frac{\theta}{1+\theta}\right)^{x_{\eta+1} - x_\eta}} \right] = 0
\]

(7.3.6)

The LHS of equation (7.3.6) is monotone function of \(\theta\), which can be observed by numerical calculations. Hence it provides a unique maximum likelihood estimate, which although not expressible in an explicit form, may be determined by numerical methods. Solving (7.3.6) one can find easily ML estimator of \(\hat{\theta}\).

Now,

\[
\frac{\partial^2 \log L}{\partial \theta^2} = \frac{1}{\theta^2(1+\theta)^2} \left[ -\frac{(\tau_1 - 1)(x_\eta - N)^2}{1 - \left(\frac{\theta}{1+\theta}\right)^{x_\eta - N}} \left\{ 1 - \left(\frac{\theta}{1+\theta}\right)^{x_\eta - N} \right\}^2 - K\theta(1+\theta) - \sum_{i=1}^{K-1} \frac{s_i(x_{\eta+1} - x_\eta)^2}{1 - \left(\frac{\theta}{1+\theta}\right)^{x_{\eta+1} - x_\eta}} \left\{ 1 - \left(\frac{\theta}{1+\theta}\right)^{x_{\eta+1} - x_\eta} \right\}^2 \right]
\]

\[
+ (1+2\theta) \left[ -\frac{(\tau_1 - 1)(x_\eta - N)}{1 - \left(\frac{\theta}{1+\theta}\right)^{x_\eta - N}} - K\theta - K + nxr_K - Nn - \sum_{i=1}^{K-1} (x_{\eta+1} - x_\eta) \tau_i - \sum_{i=1}^{K-1} \frac{s_i(x_{\eta+1} - x_\eta)}{1 - \left(\frac{\theta}{1+\theta}\right)^{x_{\eta+1} - x_\eta}} \right] \right]
\]

(7.3.7)

Due to Cohen (1963) the asymptotic variance of ML estimator of \(\theta\) is given by

\[
V(\hat{\theta}) = -\frac{1}{E\left(\frac{\partial^2 \log L}{\partial \theta^2}\right) \left(\frac{\partial^2 \log L}{\partial \theta^2}\right)} \quad (7.3.8)
\]

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Now using the results (7.2.10), (7.3.5) and (7.3.8) in (7.2.11) variance of

\[ \hat{R}_X(t) = \left( \frac{\hat{\theta}}{1+\hat{\theta}} \right)^{N-\hat{N}} \]

can be easily obtained.

### 7.4 Expected waiting time of the test

The costs associated with the life tests are directly related to the duration of an experiment. Therefore it is important for the experimenter to have some idea about the expected duration of an experiment. In multiply Type-II censoring scheme the test is terminated as soon as \( r_K \) failure is observed, i.e. \( X_{r_K} \). Using the result of Margolin and Winokur (1967) the expected value of \( X_{r_K} \), i.e. the expected time required to complete an experiment under multiply Type-II censoring is

\[
\hat{E}(X_{r_K}) = n \left( \frac{n-1}{r_K-1} \right) \sum_{j=0}^{r_K-2} \frac{(-1)^j \binom{r_K-1}{j}}{(n-r_K+j+1) \left( 1 - \left( \frac{\theta}{1+\theta} \right)^{n-r_K+j+1} \right)} + N .
\]  

(7.3.9)

### 7.5 Illustrative Examples

In this section we consider some examples to illustrate the estimation procedures. The ordered times to first break down of 20 machines were measured in an informative experiment, considered by Dunsmore (1974) are shown below.

**Example:**

4, 10, 13, 13, 18, 19, 19, 23, 24, 29, 46, 47, 57, 62, 74, 119, 188, 208, -.

Here last two observations are censored and we assume geometric life time model given in (7.1.1) for the data.

In case of Type-II censoring scheme, discussed in section-2, we get
\[ n = 20, \ r = 18, X(1) = 4, X(r) = 208 \]

\[ \hat{N} = 3, \ \hat{\theta} = 72.833, \ \hat{R}_X(t = 30) = 0.6919859 \]

\[ E(\hat{N}) = 6, V(\hat{N}) = 13.360851, V(\hat{\theta}) = 298.74882 \quad \text{and} \]

\[ V(\hat{R}_X(t)) = 4.4963186E-03 \]

**Example: 2**

Here we again consider the same example given by Dunsmore (1974), but here we assume that some of the observations were not recorded due to experimental difficulties and the experiment was stopped as soon as the eighteenth item failed. The data is as follows:

\[-, 10, 13, 18, 19, -, -, - , 29, 46, 47, 57, 62, 74, 119, 188, 208, -,-.\]

That is we consider the data under multiply Type-II censoring scheme.

Using the results of section -3 and 4 we find that

\[ n = 20, k = 14, r_1 = 2, r_2 = 3, r_3 = 4, r_4 = 5, r_5 = 6, r_6 = 10, r_7 = 11, r_8 = 12, \]

\[ r_9 = 13, r_{10} = 14, r_{11} = 15, r_{12} = 16, r_{13} = 17, r_{14} = 18. \]

\[ X_{r_1} = 10, X_{r_2} = 13, X_{r_3} = 13, X_{r_4} = 18, X_{r_5} = 19, X_{r_6} = 29, X_{r_7} = 46, X_{r_8} = 47, X_{r_9} = 57, \]

\[ X_{r_{10}} = 62, X_{r_{11}} = 74, X_{r_{12}} = 119, X_{r_{13}} = 188, X_{r_{14}} = 208. \]

\[ \hat{N} = 9, \ \hat{\theta} = 67, \ \hat{R}_X(t = 30) = 0.7326281, V(\hat{N}) = 24.093688, \]

\[ V(\hat{\theta}) = 229.7058341, \ E(X_{r_k}) = 147 \quad \text{and} \quad V(\hat{R}_X(t)) = 0.05457887. \]