CHAPTER 6*

ML ESTIMATION IN GEOMETRIC COMPETING RISK FAILURE MODEL WITH CHANGING PARAMETERS FROM TYPE-I AND PROGRESSIVELY TWO-STAGE TYPE-I CENSORED AND GROUP CENSORED SAMPLES

6.1 INTRODUCTION

In life test experiment, it is common practice to terminate the experiment when either a certain number of failures occur or certain stipulated time is elapsed. Cohen and Whitten (1988) investigated the problems of maximum likelihood estimation in case of exponential, Weibull and normal distributions when samples are progressively censored. In such tests after the first stage of experiment is over, the times of failure are observed and some surviving items are withdrawn from the test for other related experimentation and test is continued with the remaining items. This process is repeated at each stage, giving rise to progressive censoring. However, in services, certain stores and equipments are subjected to regular checkup even though they are functioning normally. When such items are placed on life tests, at each stage of censoring items that have not failed are checked up and overhauled repairing minor defects at each stage of censoring.

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This would lead to changes in parameters of failure time distribution of the item under test. Srivastava (1967), Gajjar and Khatri (1969), and Patel and Gajjar (1979, 1982, 1995) have extensively studied the problem of estimation based on exponential, lognormal, logistic and Makeham distributions with changing parameters at each stage of censoring. The references cited above and in the earlier chapters 3, 4 and 5 deals with estimation of parameters for failure time models when only one type or cause of failure is assumed to operate. Anello (1968) specifically, in the areas of life testing and reliability analysis has explored more like the mixed distribution model, the competing risk model allows flexibility in explaining failure data when there are multiple modes of failure.

Suppose that a device exhibits $K$ modes (risks) of failures $m_1, m_2, ..., m_K$ and that a random life time on this item occurs as follows:

When the device begins operation, each failure mode simultaneously generated a random lifetime that is independent of the other modes. Thus, in effect, $K$ life times denoted by $X_1, X_2, ..., X_K$ simultaneously begin, life time $X_i$ corresponds to the $i^{th}$ mode of failure. Failure of the device occurs, as soon as any one of the life times, say $X_i$ is realized. In effect, if the life length of the device is denoted by a random variable $X$ then $X = \min(X_1, X_2, ..., X_K) \equiv X_i$. If $F_{X_i}(x)$ is the cumulative distribution function of $X_i$, the c.d.f. of $X$ $F_X(x)$ is given by

$$F_X(x) = 1 - \prod_{i=1}^{K} [1 - F_{X_i}(x)]$$

The above derivation of the competing risk failure model not only is dependent of the functional form of the $F_{X_i}(x)$, but also allows for the $F_{X_i}(x)$ to be all different. This means that each failure mode can have any failure distribution and that not all the failure distributions need be alike. However, derivation does require that all the $K$ modes operate independently of each other. In competing risk model all the $K$ modes begin to generate, random life times simultaneously whereas in the mixed population model only one of the $K$ possible modes generate a random lifetime that causes part failure. Different identifiable causes of failure may be attributed to electrical, thermal, climatic and mechanical stresses applied to an item. Peck (1966) has considered transistor failure studies relating to two types of failure namely electrical degradation of certain parts (cause-1) and faulty bonding of the leads (cause-2) leading to failures of transistors. Boardman and
Kendell (1970) have considered the problem of estimation with respect to Type-I censoring when an item is subjected to only one of the two causes of failure assuming that time to failure of an item is due to \(i^{th}\) cause is distributed as exponential with failure rate \(\lambda_i\) (\(i = 1, 2\)). Some other continuous lifetime models on similar lines are proposed and motivated by Mendenhall and Hader (1958), Cox (1959), Boardman and Kendell (1970), David and Moeschberger (1978) and Patel and Gajjar (1992). Nelson, W. B. (1970) has considered a testing of electrical appliances. The life times of appliances are considered in terms of the number of completed cycles to failure. Moreover, appliance could fail due to different 18 causes of failure. This shows the use of discrete failure time distribution in competing risk failure model.

In this chapter, we have considered the problem of maximum likelihood estimation with respect to Type-I censoring and group censoring without replacement, when an item is subjected to one and only one of the two independent causes of failure assuming geometric competing risk failure model. In addition, the problem is extended to two stage Type-I progressive censoring scheme. The asymptotic standard errors of the estimates are obtained for both the types of censoring as well as two stage Type-I progressive censoring. Some examples are included to show possible application of estimation from these types of competing risk failure models.

### 6.2 COMPETING RISK FAILURE MODEL FROM TYPE-I CENSORED SAMPLES

Every item on test will be subject to each of causes of failure; however, failure of an item will result from only one cause. The unit will be withdrawn from the test to determine the exact type of failure. It is further assumed that the various types of causes of failure act independently on each of them. To illustrate this problem consider the example given by Nelson (1970). Life test is carried out during the development of small electrical appliances. Appliances were essentially operated repeatedly until failure. There are 18 unrelated possible causes of failure for code numbers 1 to 18 denoted the appliances. Other example is given by Peck (1966)
related to epitomical transistors used in the telephone industry. Two apparently unrelated types of failures are

(i) electrical degradation of certain parts
(ii) faulty bonding of certain parts of the leads.

Let an item be placed on a life test without replacement. Suppose that each item can fail due to one of the two identifiable causes, which operate independently of each other. Suppose that the life of an item is measured on a discrete scale (e.g. in completed number of cycles) and failure time distribution of an item due to \( i \)th cause is geometric with failure rate \( \frac{1-q_i}{q_i} \), \( 0 < q_i < 1 \), \( i = 1, 2 \).

If \( X_i \) is the time of failure of an item due to cause \( i \) then the probability mass function (pmf) of \( X_i \) is

\[
f_i(x) = \left(1-q_i\right)q_i^{x-1}, \quad 0 < q_i < 1, x = 1, 2, 3, \ldots; \quad i = 1, 2.
\]

Moreover, corresponding distribution function is

\[
F_i(x) = 1-q_i^x; \quad i = 1, 2
\]

As \( X_1 \) and \( X_2 \) denote times of failure due to causes 1 and 2 with pmf \( f_1(x) \) and \( f_2(x) \) given in (6.2.1) then the time of failure of an item regardless of cause is given by

\[T = \min(X_1, X_2)\]

and its pmf is given by

\[
h_T(t) = (1-q_1q_2)(q_1q_2)^{t-1}; \quad t = 1, 2, 3, \ldots
\]

with the distribution function

\[
H_T(t) = 1-(q_1q_2)^t; \quad t = 1, 2, 3, \ldots
\]

Let \( p_i(t) \) be the probability that an item failed by cause \( i \) at time \( t \) and it must not failed by the cause \( i' \neq i \) up to time \( t \) with two independent causes only. Then \( p_i(t) \) is given by
\begin{equation}
\pi_i(t) = f_i(t)[1 - F_i(t)]; \quad i' \neq i = 1, 2.
\end{equation}

Hence, the probability of failure of an item is given by

\[ p(t) = p_1(t) + p_2(t). \]

Using (6.2.1) and (6.2.2) in (6.2.5) we get

\[ p(t) = q_2(1 - q_1)(q_1q_2)^{i-1} + q_1(1 - q_2)(q_1q_2)^{i-1}. \]

Hence, the pmf of \( t \) regardless of cause of failure defined in (6.2.3) can be obtained from (6.2.6) after adjusting the normalizing factor \( C \) as

\begin{equation}
h_r(t) = Cp(t)
\end{equation}

such that \( \sum_{i=1}^{\infty} h_r(t) = \sum_{i=1}^{\infty} Cp(t) = 1 \)

which gives

\[ C = \frac{1 - q_1q_2}{q_2(1 - q_1) + q_1(1 - q_2)}. \]

Substituting \( C \) in (6.2.7), we have

\[ h_r(t) = \frac{q_2(1 - q_1)}{q_2(1 - q_1) + q_1(1 - q_2)}(1 - q_1q_2)(q_1q_2)^{i-1} + \frac{q_1(1 - q_2)}{q_2(1 - q_1) + q_1(1 - q_2)}(1 - q_1q_2)(q_1q_2)^{i-1} \]

\[ = g_1(t) + g_2(t) \]

which would be same as obtained in (6.2.3) with distribution function given in (6.2.4). The failure model given by (6.2.8) with distribution function given in (6.2.4) is called the competing risk failure model in which time to failure of an
item is due to two independent causes of failure and represents the pmf of time to failure which occurs first irrespective of the cause. Probability models on similar lines were proposed by Boardman and Kendell (1970) and Patel and Gajjar (1992).

### 6.3 LIKELIHOOD ESTIMATION FOR TYPE-I CENSORING

Let $n$ items be placed on a life test without replacement. Let $r_i$ be the number of failures and let $t_{i1}, t_{i2}, ..., t_{ir_i}$ be the failure times due to $i^{th}$ cause, $i = 1, 2$.

Let $N$ be the fixed point of censoring, i.e. the test is terminated after $N$ completed cycles, then the likelihood function under competing risk failure model with Type-I censoring without replacement is given by

$$L \propto \prod_{i=1}^{2} \prod_{j=1}^{r_i} \left[ g_i(t_{ij})^{r_i} - H(N) \right]^{n-r}$$  \hspace{1cm} (6.3.1)

where $r_i =$ number of failures due to $i^{th}$ cause, $i = 1, 2$.

$r = r_1 + r_2 =$ total number of failures for the entire duration of the test.

When the life of an item obeys the probability law given by (6.2.8) then using (6.3.1) we get

$$LogL = \text{const} + r_1 \log(1 - q_1) + r_1 \log q_2 + r \log(1 - q_1 q_2) + r_1 (\bar{l}_1 - 1) \log(q_1 q_2) -$$
$$- r \log(p_1 q_2 + p_2 q_1) + r_2 \log(1 - q_2) + r_2 \log(q_1) +$$
$$r_2 (\bar{l}_2 - 1) \log(q_1 q_2) + N(n-r) \log(q_1 q_2)$$  \hspace{1cm} (6.3.2)
where

\[ \tilde{t}_1 = \frac{\sum_{j=1}^{r_1} t_{1j}}{r_1} \quad \text{and} \quad \tilde{t}_2 = \frac{\sum_{j=1}^{r_2} t_{2j}}{r_2}; \quad r = r_1 + r_2. \]  

(6.3.3)

Thus

\[ \frac{\partial \log L}{\partial q_1} = -\frac{r_1}{1 - q_1} - \frac{rq_2}{1 - q_1q_2} + \frac{A - r_1}{q_1} - \frac{r(1 - 2q_2)}{q_1 + q_2 - 2q_1q_2} \]  

and

\[ \frac{\partial \log L}{\partial q_2} = -\frac{r_2}{1 - q_2} - \frac{rq_1}{1 - q_1q_2} + \frac{A - r_2}{q_2} - \frac{r(1 - 2q_1)}{q_1 + q_2 - 2q_1q_2} \]  

(6.3.4)

(6.3.5)

where

\[ A = r_1 \tilde{t}_1 + r_2 \tilde{t}_2 + N(n - r). \]  

(6.3.6)

On equating (6.3.4) and (6.3.5) with zero and solving them according to the method described in Appendix we get the ML estimates \( \hat{q}_1 \) and \( \hat{q}_2 \) of \( q_1 \) and \( q_2 \) respectively.

6.4 EXPECTATIONS AND STANDARD ERRORS OF ESTIMATES

In this section we shall obtain the expected values of \( r_1 \) and \( r_2 \) and standard errors of the estimates \( \hat{q}_1 \) and \( \hat{q}_2 \).

Since the distribution of \( t \) is regardless of causes, we have

\[ E(\tilde{t}_1) = E(\tilde{t}_2) = E(\tilde{t}) = E(\tilde{t}|1 \leq t \leq N) \]

In addition, using (6.2.3) and (6.2.4) it is easy to verify that
\[ E(i) = E(i_1) = E(i_2) = \frac{(1-Q)(N+1)Q^N}{H(N) \left[ \frac{1-Q^{N+1}}{(1-Q)^2} - \frac{(N+1)Q^N}{(1-Q)} \right]} \]  

(6.4.1)

Where \( Q = q_1q_2 \).

Now \( E\left( \frac{\partial \log L}{\partial q_1} \right) = 0 \) and \( E\left( \frac{\partial \log L}{\partial q_2} \right) = 0 \) give

\[
\frac{-E(r_1)}{1-q_1} + E(r) \left\{ \frac{(E(i) - N)(1-Q) - Q}{1-Q} + \frac{q_1(2q_2 - 1)}{q_1 + q_2 - 2q_1q_2} \right\} + N_t = 0
\]

(6.4.2)

and

\[
\frac{-E(r_2)}{1-q_2} + E(r) \left\{ \frac{(E(i) - N)(1-Q) - Q}{1-Q} + \frac{q_2(2q_1 - 1)}{q_1 + q_2 - 2q_1q_2} \right\} + N_t = 0
\]

(6.4.3)

Solving (6.4.2) and (6.4.3) we get \( E(r_1) \) and \( E(r_2) \).

Also

\[
E\left( \frac{\partial^2 \log L}{\partial q_1^2} \right) = \frac{-E(r_1)}{(1-q_1)^2} + \frac{q_1^2 E(r)}{(1-q_1q_2)^2} - \frac{E(A-r_1)}{q_1^2} + \frac{(1-2q_2)^2 E(r)}{(q_1 + q_2 - 2q_1q_2)^2}
\]

(6.4.4)

\[
E\left( \frac{\partial^2 \log L}{\partial q_2^2} \right) = \frac{-E(r_2)}{(1-q_2)^2} + \frac{q_2^2 E(r)}{(1-q_1q_2)^2} - \frac{E(A-r_2)}{q_2^2} + \frac{(1-2q_1)^2 E(r)}{(q_1 + q_2 - 2q_1q_2)^2}
\]

(6.4.5)
and

\[
E\left( \frac{\partial^2 \log L}{\partial q_1 q_2} \right) = E(r) \left[ \frac{1}{(q_1 + q_2 - 2q_1 q_2)^2} - \frac{1}{(1-q_1 q_2)^2} \right].
\]  
(6.4.6)

Hence asymptotic variance- co variance matrix of the estimates is given by

\[
\Sigma = \begin{pmatrix} V(\hat{q}_1) & \text{cov}(\hat{q}_1, \hat{q}_2) \\ \text{cov}(\hat{q}_1, \hat{q}_2) & V(\hat{q}_2) \end{pmatrix}^{-1} \begin{pmatrix} E\left( \frac{\partial^2 \log L}{\partial q_1^2} \right) & -E\left( \frac{\partial^2 \log L}{\partial q_1 q_2} \right) \\ -E\left( \frac{\partial^2 \log L}{\partial q_1 q_2} \right) & E\left( \frac{\partial^2 \log L}{\partial q_2^2} \right) \end{pmatrix}.
\]  
(6.4.7)

\section*{6.5 ESTIMATION OF PARAMETERS OF MODEL (6.2.8) FROM GROUP- CENSORED SAMPLES}

In this section, we shall consider the maximum likelihood estimation of parameters of model (6.2.8) using group- censoring. Life tests are sometimes performed in group censoring so that only the total number of failures rather than times of failures are observed during the time interval \((0, N]\), where \(N\) is fixed censoring time.
The likelihood function for the Type-1 group-censored sample of the model (6.2.8) is

\[ L \propto \prod_{i=1}^{2} \left( \sum_{i=1}^{N} g_i(t) \right)^{\eta_i} \left( 1 - H(N) \right)^{n-r}. \]

Hence

\[ L = \text{const} \left\{ \frac{q_2(1-q_1)}{q_2(1-q_1) + q_1(1-q_2)} \right\}^{\eta_1} \left\{ \frac{q_2(1-q_1)}{q_2(1-q_1) + q_1(1-q_2)} \right\}^{\eta_2} \left( q_2q_1 \right)^{N(n-r)}. \]  

(6.5.1)

Differentiating with respect to \( q_1 \) and \( q_2 \) we get

\[
\frac{\partial \log L}{\partial q_1} = \frac{-r_1}{1-q_1} - \frac{rNq_1^{N-1}q_2^N}{1-(q_1q_2)^N} + \frac{N(n-r)+r_2}{q_1} - \frac{r(1-2q_2)}{q_1+q_2-2q_1q_2}.
\]  

(6.5.2)

and

\[
\frac{\partial \log L}{\partial q_2} = \frac{-r_2}{1-q_2} - \frac{rNq_1^Nq_2^{N-1}}{1-(q_1q_2)^N} + \frac{N(n-r)+r_1}{q_2} - \frac{r(1-2q_1)}{q_1+q_2-2q_1q_2}.
\]  

(6.5.3)

Maximum likelihood estimates \( \hat{q}_1 \) and \( \hat{q}_2 \) of \( q_1 \) and \( q_2 \) respectively can be obtained by the method describe in section 6.3.

Again using \( E\left( \frac{\partial \log L}{\partial q_1} \right) = 0 \) and \( E\left( \frac{\partial \log L}{\partial q_2} \right) = 0 \), \( E(r_1) \) and \( E(r_2) \) can be obtained by solving the equations

\[
E(r_1) = \frac{1}{q_1} + \frac{Nq_1^{N-1}q_2^N}{1-(q_1q_2)^N} + \frac{N}{q_1} + \frac{(1-2q_2)}{q_1+q_2-2q_1q_2}.
\]  

(6.5.4)
\[
E(r_1) \left\{ \frac{Nq_1^N q_2^{N-1}}{1-(q_1 q_2)^N} + \frac{N}{q_2} + \frac{(1-2q_1)}{q_1 + q_2 - 2q_1 q_2} - \frac{1}{q_2} \right\} + \\
E(r_2) \left\{ \frac{1}{1-q_2} + \frac{Nq_1^N q_2^{N-1}}{1-(q_1 q_2)^N} + \frac{(1-2q_1)}{q_1 + q_2 - 2q_1 q_2} + \frac{N}{q_2} \right\} - \frac{nN}{q_2} = 0
\]

(6.5.5)

Also,

\[
E\left( \frac{\partial^2 \log L}{\partial q_1^2} \right) = -\frac{E(r_1)}{(1-q_1)^2} \frac{E(r)N(q_1 q_2)^N}{q_1^2} \left\{ \frac{N - (1-(q_1 q_2)^N)}{1-(q_1 q_2)^N} \right\} - \\
\frac{E(r_2) + N(n-E(r))}{q_1^2} + \frac{E(r)(1-2q_2)^2}{(q_1 + q_2 - 2q_1 q_2)^2}
\]

(6.5.6)

\[
E\left( \frac{\partial^2 \log L}{\partial q_2^2} \right) = -\frac{E(r_2)}{(1-q_2)^2} \frac{E(r)N(q_1 q_2)^N}{q_2^2} \left\{ \frac{N - (1-(q_1 q_2)^N)}{1-(q_1 q_2)^N} \right\} - \\
\frac{E(r_1) + N(n-E(r))}{q_2^2} + \frac{E(r)(1-2q_1)^2}{(q_1 + q_2 - 2q_1 q_2)^2}
\]

(6.5.7)

and

\[
E\left( \frac{\partial^2 \log L}{\partial q_1 \partial q_2} \right) = -\frac{E(r)N^2(q_1 q_2)^{N-1}}{(1-(q_1 q_2)^N)^2} + \frac{E(r)}{(q_1 + q_2 - 2q_1 q_2)^2}.
\]

(6.5.8)

Again, variance co-variance matrix of the estimates can be obtained using (6.4.7).
6.6 COMPETING RISKS FAILURE MODEL 
FROM TYPE-I TWO-STAGE 
PROGRESSIVELY CENSORED SAMPLES

In this section, we expand the scheme used in the earlier section for two-stage 
progressive censoring. Let n items are placed on a life test without replacement. 
Suppose that each item can fail due to one of the two identifiable causes, which 
operate independently of each other. Suppose that failure time distribution of an 
item due to \( i \)th cause is geometric with probability mass function (pmf) 
\[
f_i(x) = (1 - q_i)q_i^{x-1} ; x = 1, 2, \ldots ; 0 < q_i < 1 ; i = 1, 2.
\]

Let \( X_i \) is the time (in completed cycles) of failure of an item due to cause \( i \) then 
composite probability mass function of \( X_i \) for two stage Type-I progressive 
censoring with changing parameters can be obtain in similar manner discussed in 
section:3.2 of chapter:3 as follow:

\[
f_i(x) = \begin{cases} 
  f_i^{(0)}(x) = (1 - q_i^{(0)})(q_i^{(0)})^{x-1}, & x = 1, 2, \ldots, N_i \\
  f_i^{(2)} = \left[ \frac{q_i^{(1)}}{q_i^{(2)}} \right]^{N_i} (1 - q_i^{(2)}) (q_i^{(2)})^{x-1}; & x = N_i + 1, N_i + 2, \ldots, \infty,
\end{cases}
\]

for \( i = 1, 2 \).

(6.6.2)

If \( X_1 \) and \( X_2 \) denote times to failure due to causes 1 and 2 with pmf \( f_1(x) \) and 
\( f_2(x) \) given by (6.6.1) then the pmf for time to failure \( T = \min\{ X_1, X_2 \} \) of an 
item regardless of cause is given by 
\[
h_T(t) = (1 - q_1q_2)(q_1q_2)^{t-1} ; t = 1, 2, 3, \ldots ; 0 < q_1, q_2 < 1.
\]

(6.6.3)
Hence, the composite pmf of time $T$ regardless of cause for two-stage Type-I censoring is given

$$h_T(t) = \begin{cases} 
(1-q_1^{(1)}q_2^{(1)})(q_1^{(1)}q_2^{(1)})^{t-1} & ; t = 1,2,\ldots,N_1 ; 0 < q_1, q_2 < 1. \\
(1-q_1^{(2)}q_2^{(2)}) \left( \frac{q_1^{(1)}q_2^{(1)}}{q_1^{(2)}q_2^{(2)}} \right)^{N_1} (q_1^{(2)}q_2^{(2)})^{t-1} & ; t = N_1 + 1, N_1 + 2,\ldots,\infty.
\end{cases}$$

(6.6.4)

Now let

$$p_T(t) = \text{Probability of failure of an item at time } t \text{ regardless of cause of failure}$$

$$= P(\text{item is failed due to cause I at time } t) \times P(\text{item is not failed due to cause II up to cause time } t) + P(\text{item is failed due to cause II at time } t) \times P(\text{item is not failed due to cause I up to time } t)$$

$$= f_1(t)[1-F_2(t)] + f_2(t)[1-F_1(t)].$$

(6.6.5)

Hence, by adjusting normalizing factor $C^{(1)}$, the pmf of $T$ regardless of cause can be obtained from (6.6.5) as follow:

$$h_T(t) = C^{(1)}p_T(t) ; t = 1,2,\ldots.$$ 

(6.6.6)

such that

$$\sum_{t=1}^{\infty} h_T(t) = 1$$

which gives

$$C^{(1)} = \frac{1-q_1q_2}{q_1 + q_2 - 2q_1q_2}.$$  

Upon substituting $C^{(1)}$ in (6.6.6) the pmf of $T$ is given by
\[ h_r(t) = \frac{(1-q_1)q_2}{q_1 + q_2 - 2q_1q_2} (1-q_1q_2)(q_1q_2)^{-1} + \frac{(1-q_2)q_1}{q_1 + q_2 - 2q_1q_2} (1-q_1q_2)(q_1q_2)^{-1} \]
\[ t = 1, 2, \ldots \]

(6.6.7)

which is same as given in (6.6.3).

In similar way the composite pmf of time \( T \) for two stage Type-I censoring can be obtained as

\[
\begin{align*}
\{ & \frac{(1-q_1^{(0)})q_2^{(0)}}{q_1^{(0)} + q_2^{(0)} - 2q_1^{(0)}q_2^{(0)}} (1-q_1^{(0)}q_2^{(0)})(q_1^{(0)}q_2^{(0)})^{-1} + \\
& \frac{(1-q_2^{(0)})q_1^{(0)}}{q_1^{(0)} + q_2^{(0)} - 2q_1^{(0)}q_2^{(0)}} (1-q_1^{(0)}q_2^{(0)})(q_1^{(0)}q_2^{(0)})^{-1} ; t = 1, 2, \ldots, N_1 \\
& C^{(2)} p_r^{(2)}(t) ; t = N_1 + 1, N_1 + 2, \ldots, \infty
\}
\end{align*}
\]

(6.6.8)

where

\[
p_r^{(2)}(t) = f_1^{(2)}(t)[1-F_2^{(2)}(t)] + f_2^{(2)}(t)[1-F_1^{(2)}(t)].
\]

Using (6.6.2), we get

\[
p_r^{(2)}(t) = (1-q_1^{(2)})q_2^{(2)} (q_1^{(1)}q_2^{(1)})^{N_1} (q_1^{(2)}q_2^{(2)})^{t-N_1-1} + (1-q_2^{(2)})q_1^{(2)} (q_1^{(2)}q_2^{(2)})^{N_1} (q_1^{(2)}q_2^{(2)})^{t-N_1-1}
\]

As \( h_r(t) \) given in (6.6.8) is pmf of \( T \),

\[
\sum_{t=1}^{\infty} h_r(t) = 1
\]

which gives

\[
C^{(2)} = \frac{1-q_1^{(2)}q_2^{(2)}}{q_1^{(2)} + q_2^{(2)} - 2q_1^{(2)}q_2^{(2)}}.
\]
Finally substituting \( C^{(2)} \) in (6.6.8) we get the composite pmf of time \( T \) for two stage Type-I progressive censoring as,

\[
    h_r(t) = \begin{cases} 
        g_1^{(1)}(t) + g_2^{(1)}(t) & ; t = 1, 2, ..., N_1 \\
        g_1^{(2)}(t) + g_2^{(2)}(t) & ; t = N_1 + 1, N_1 + 2, ..., \infty
    \end{cases}
\]

where

\[
    g_i^{(k)}(t) = \frac{(1 - q_i^{(k)})q_j^{(k)}}{q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)}q_j^{(k)}} \left( \frac{q_i^{(k-1)}q_j^{(k-1)}}{q_i^{(k)}q_j^{(k)}} \right)^{N_{k-1}} \left( 1 - q_i^{(k)}q_j^{(k)}q_i^{(k)}q_j^{(k)} \right)^{t-1}
\]

(6.6.9)

where \( j = 2(1) \) when \( i = 1(2); i = 1, 2; k = 1, 2 \) and \( N_0 = 0 \).

In addition, the corresponding cumulative distribution function is given by

\[
    H_r(t) = \begin{cases} 
        H^{(1)}(t) = 1 - (q_1^{(1)}q_2^{(1)})^t & ; t = 1, 2, ..., N_1 \\
        H^{(2)}(t) = 1 - \left( \frac{q_1^{(2)}q_2^{(2)}}{q_1^{(3)}q_2^{(3)}} \right)^{N_1} (q_1^{(2)}q_2^{(3)})^t & ; t = N_1 + 1, N_1 + 2, ..., \infty
    \end{cases}
\]

(6.6.10)

The failure model given by (6.6.9) or (6.6.10) are called discrete multiple risk failure model in which time to failure of an item is due to two causes of failure and represents the pmf of time to failure, which occurs first irrespective of cause. Probability models on similar lines were proposed and motivated by David and Moeschbrger (1978) and Patel and Gajjar (1992).
6.7 LIKELIHOOD ESTIMATION FOR TWO-STAGE TYPE-I PROGRESSIVE CENSORING

Let n items are placed on a life test without replacement. Each item can fail either by causes 1 or 2, and it is removed from the test with the time and cause of failure. Let \( N_2 \) be the final time up to which the test is continued. We introduce the following notations:

\[
\begin{align*}
    k &= \text{number of stages}, \ i = \text{number of causes}. \\
    r_i &= \text{number of failure due to cause } i = r_i^{(1)} + r_i^{(2)}; \ i = 1, 2. \\
    r^{(k)} &= \text{number of failure due to case } 1 \text{ and } 2 \text{ both at the } k^{\text{th}} \text{ stage of censoring} \\
        &= r_i^{(k)} + r_2^{(k)}; \ k = 1, 2. \\
    r_i^{(k)} &= \text{number of failures due to } i^{\text{th}} \text{ cause at } k^{\text{th}} \text{ stage of censoring}, \ i = 1, 2; \ k = 1, 2. \\
    r &= \text{total number of failures up to time } N_2 = r_1 + r_2. \\
    m^{(k)} &= \text{the fixed number of surviving items that are withdrawn from the experiment immediately after the } k^{\text{th}} \text{ stage}, \ k = 1, 2. \\
    m^{(2)} &= n - m^{(1)} - r_1 - r_2 = n - m^{(1)} - r^{(1)} - r^{(2)}. \\
    t_i^{(k)} &= \text{the failure times due to } i^{\text{th}} \text{ cause at the } k^{\text{th}} \text{ stage for } i = 1, 2; \ k = 1, 2.
\end{align*}
\]

For two stage Type-I progressive censoring the likelihood function is

\[
L \propto \prod_{k=1}^{2} \prod_{i=1}^{2} \prod_{s=1}^{2} \left[ \frac{g_i^{(k)}(t_i^{(k)})}{r_i^{(k)}} \right] \prod_{i=1}^{2} \left[ 1 - H_i^{(k)}(N_1) \right] m^{(k)}
\]

where \( N_1 \) and \( N_2 \) (\( N_1 < N_2 \)) are the fixed number of cycles of censoring.

When the life of an item obeys the probability mass function given by (6.6.2) and (6.6.10) then using (6.7.1) we get
Differentiating (6.7.2) with respect to $q_{ik}$ for $i, k = 1,2$ and equating to zero we get maximum likelihood equations as

\[
L \propto \prod_{k=1}^{2} \prod_{i=1}^{2} \prod_{j=1}^{2} \left( \frac{1 - q_{ik}}{q_{ik} + q_{jk} - 2q_{ik} q_{jk}} \right) \left( \frac{q_{ik}^{(k-1)} q_{jk}^{(k-1)}}{q_{ik}^{(k)} q_{jk}^{(k)}} \right) \left( 1 - q_{ik}^{(k)} q_{jk}^{(k)} \right) \left( q_{ik}^{(1)} q_{jk}^{(1)} \right)^{N_{ik}^{(1)}} \left( q_{ik}^{(2)} q_{jk}^{(2)} \right)^{N_{ik}^{(2)}}. 
\]

(6.7.2)

Differentiating (6.7.2) with respect to $q_{ik}$ for $i, k = 1,2$ and equating to zero we get maximum likelihood equations as

\[
\frac{\partial \log L}{\partial q_{ik}^{(k)}} = \frac{-r_{i}^{(k)}}{1 - q_{ik}^{(k)}} + \frac{A^{(k)} - r_{i}^{(k)}}{q_{ik}^{(k)}} - \frac{r_{j}^{(k)} q_{jk}^{(k)}}{1 - q_{ik}^{(k)} q_{jk}^{(k)}} - \frac{r_{i}^{(k)} (1 - 2q_{ik}^{(k)})}{q_{ik}^{(k)} + q_{jk}^{(k)} - 2q_{ik}^{(k)} q_{jk}^{(k)}} = 0
\]

for $i, k = 1,2; j = 2(1)$ when $i = 1(2)$

(6.7.3)

and

\[
A^{(k)} = r_{1}^{(k)} r_{1}^{(k)} + r_{2}^{(k)} r_{2}^{(k)} + (N_{k} - N_{k-1}) n^{(k)} - N_{k} r^{(k)} \text{ and } k = 1,2 \text{ with } N_{0} = 0, n^{(1)} = n.
\]

Using the method given in appendix one can find ML estimates of $q_{ik}^{(k)}$ for $i, k = 1,2$.

### 6.8 EXPECTATIONS AND STANDARD ERRORS OF ESTIMATES

In this section we shall obtain the expected values of $r_{i}^{(k)}$; $i, k = 1,2$ and standard errors of the estimates of $q_{ik}^{(k)}$ obtained in section: 6.7 Using (6.6.2) it is easy to verify that

\[
E(t_{i}^{(0)}) = E[t_{i}^{(0)} | t_{j}^{(0)} = 1,2,...,N_{1}] = \frac{1}{1 - \varphi^{(0)}} \frac{N_{i} [1 - H^{(0)}(N_{1})]}{H^{(0)}(N_{1})}
\]
and \( E(r_i^{(2)}) = \frac{1}{1-\varphi^{(2)}} + \frac{N_1[1-H^{(0)}(N_1)] - N_2[1-H^{(2)}(N_2)]}{H^{(2)}(N_2) - H^{(0)}(N_1)} \)

where \( \varphi^{(k)} = q_1^{(k)}q_2^{(k)} \); \( k = 1,2 \).

Finally solving the equations \( E\left( \frac{\partial \log L}{\partial q_i^{(k)}} \right) = 0 \) we get \( E(r_i^{(k)}) \) for \( i, k = 1,2 \).

Differentiating (6.7.3) with respect to \( q_i^{(k)} \) and \( q_j^{(k)} \) we get

\[
\frac{\partial^2 \log L}{\partial q_i^{(k)} q_j^{(k)}} = -r_i^{(k)} - A^{(k)} - r_i^{(k)} (q_j^{(k)})^2 + \frac{r_j^{(k)} (1-2q_j^{(k)})^2}{(1-q_i^{(k)})^2 (1-q_j^{(k)})^2 (q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)} q_j^{(k)})^2}
\]

(6.8.1)

and

\[
\frac{\partial^2 \log L}{\partial q_i^{(k)} q_j^{(k)}} = -r_i^{(k)} q_i^{(k)} q_j^{(k)} - r_j^{(k)} (1-2q_i^{(k)})(1-2q_j^{(k)}) + \frac{2r_j^{(k)}}{(q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)} q_j^{(k)})^2}
\]

for \( i, k = 1,2 \).

(6.8.2)

Taking expectations on both the sides and using \( E(r_i^{(k)}) \) we can obtain the asymptotic variance-covariance matrix of ML estimates as

\[
\Sigma = \begin{pmatrix}
V(q_1^{(k)}) & \text{Cov}(q_1^{(k)}, q_2^{(k)}) \\
\text{Cov}(q_1^{(k)}, q_2^{(k)}) & V(q_2^{(k)})
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-E\left( \frac{\partial^2 \log L}{\partial q_1^{(k)} q_1^{(k)}} \right) & -E\left( \frac{\partial^2 \log L}{\partial q_1^{(k)} q_2^{(k)}} \right) \\
-E\left( \frac{\partial^2 \log L}{\partial q_1^{(k)} q_2^{(k)}} \right) & -E\left( \frac{\partial^2 \log L}{\partial q_2^{(k)} q_2^{(k)}} \right)
\end{pmatrix}\begin{pmatrix} -1 \end{pmatrix}
\]

for \( i, k = 1,2 \).

(6.8.3)
6.9 ESTIMATION OF PARAMETERS OF MODEL (6.6.2) FROM PROGRESSIVELY GROUP CENSORED SAMPLES

In this section, we shall consider the maximum likelihood estimation of parameters of model (6.6.2) using group censoring. Life tests are sometimes, performed in group censoring so that only the total number of failures rather than times of failures are observed during each stage \((N_i + 1, N_i)\) where \(N_i\) \((i = 1, 2)\) are the fixed points of censoring. The number of failures observed due to each cause at each stage is, of course, a random variable. The likelihood function for Type-I progressively group censored sample of the model (6.6.2) is

\[
L \propto \prod_{j=1}^{2} \left( \sum_{i=1}^{N_1} g_j^{(1)}(t) \right)^{y_j^{(1)}} \prod_{j=1}^{2} \left( \sum_{i=N_1+1}^{N_2} g_j^{(2)}(t) \right)^{y_j^{(2)}} \left( l - H^{(1)}(N_1) \right)^{r_1^{(1)}} \left( l - H^{(2)}(N_2) \right)^{r_2^{(2)}}
\]

where \(n^{(1)} = n, n^{(2)} = n - m^{(1)} - r^{(1)}, m^{(2)} = n^{(2)} - r^{(2)}\).

Then

\[
\log L = \text{const.} + \sum_{i=1}^{2} \left\{ \log q_i^{(1)} + \log(1 - q_i^{(1)}) \right\} + \sum_{i=1}^{2} \left\{ \log(1 - q_i^{(2)}) + \log q_i^{(2)} \right\} +
\]

\[
r_1^{(2)} \log\left( 1 - q_1^{(2)} \right) + r_2^{(2)} \log\left( 1 - q_2^{(2)} \right) + \log(1 - q_1^{(1)}) \right\} +
\]

\[
q_1^{(2)} \log\left( 1 - q_1^{(2)} \right) + q_2^{(2)} \log\left( 1 - q_2^{(2)} \right) + q_1^{(1)} \log\left( 1 - q_1^{(1)} \right) \right\} +
\]

\[
N_1 (n - r^{(1)}) \log q_1^{(1)} + \log q_2^{(1)} \} + (N_2 - N_1) m^{(2)} \left\{ \log q_1^{(2)} + \log q_2^{(2)} \right\} -
\]

\[
r^{(2)} \log q_1^{(2)} + q_2^{(2)} - 2q_1^{(1)} q_2^{(1)} \right\} - r^{(2)} \log q_1^{(2)} + q_2^{(2)} - 2q_1^{(1)} q_2^{(2)} \right\}.
\]

Differentiating \(\log L\) with respect to \(q_1^{(k)}, q_2^{(k)}\), \(k = 1, 2\) and equating to zero we can write in general as
\[
\frac{\partial \log L}{\partial q_i^{(k)}} = -r_i^{(k)} + \frac{r_j^{(k)} (N_k - N_{k-1}) (q_i^{(k)})^{N_k - N_{k-1}} (q_j^{(k)})^{N_{k-1} - N_k}}{q_i^{(k)}} + \frac{r_j^{(k)} (N_k - N_{k-1}) (q_i^{(k)})^{N_k - N_{k-1}} (q_j^{(k)})^{N_{k-1} - N_k}}{1 - (q_i^{(k)} q_j^{(k)})^{N_k - N_{k-1}}}
\]

\[
\frac{(N_k - N_{k-1}) (n^{(i)} - r^{(i)})}{q_i^{(k)}} - \frac{r^{(i)} (1 - 2q_i^{(k)})}{q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)} q_j^{(k)}} = 0
\]

for \(i, k = 1, 2; N_0 = 0, j = 1, 2\) when \(i = 2, 1\).

(6.9.2)

Taking \(E(\frac{\partial \log L}{\partial q_i^{(k)}}) = 0\) and solving them we can get \(E(r_i^{(k)})\) for \(i, k = 1, 2\).

Again differentiating (6.9.2) with respect to \(q_i^{(k)}\) and \(q_j^{(k)}\) independently we get

\[
\frac{\partial^2 \log L}{\partial q_i^{(k)} \partial q_j^{(k)}} = \frac{-r_i^{(k)} - r_j^{(k)} (N_k - N_{k-1})^2 (q_i^{(k)})^{N_k - N_{k-1}} (q_j^{(k)})^{N_{k-1} - N_k}}{(1 - q_i^{(k)})^2 (q_i^{(k)})^2} - \frac{r^{(i)} (N_k - N_{k-1}) (q_i^{(k)})^{N_k - N_{k-1}} (q_j^{(k)})^{N_{k-1} - N_k}}{\{1 - (q_i^{(k)} q_j^{(k)})^{N_k - N_{k-1}}\}^2}
\]

\[
- r^{(i)} (N_k - N_{k-1}) (N_k - N_{k-1} - 1) (q_i^{(k)})^{N_k - N_{k-1} - 2} (q_j^{(k)})^{N_{k-1} - N_k} - \frac{(N_k - N_{k-1}) (n^{(i)} - r^{(i)})}{(q_i^{(k)})^2} + \frac{r^{(i)} (1 - 2q_i^{(k)})^2}{\{q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)} q_j^{(k)}\}^2}
\]

(6.9.3)

and

\[
\frac{\partial^2 \log L}{\partial q_i^{(k)} \partial q_j^{(k)}} = \frac{r^{(i)} (N_k - N_{k-1})^2 (q_i^{(k)})^{2(N_k - N_{k-1})}}{q_i^{(k)} q_j^{(k)} \{1 - (q_i^{(k)} q_j^{(k)})^{N_k - N_{k-1}}\}^2} + \frac{r^{(i)} (1 - 2q_i^{(k)})(1 - 2q_j^{(k)})}{\{q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)} q_j^{(k)}\}^2}
\]

\[
+ \frac{2r^{(i)}}{q_i^{(k)} + q_j^{(k)} - 2q_i^{(k)} q_j^{(k)}}
\]

for \(i, k = 1, 2\).

(6.9.4)

Hence, by the method described in section-6.8 it is possible to find the asymptotic variance- covariance matrix of ML estimates in case of group censoring.
6.10 ILLUSTRATIVE EXAMPLES

Example-1:

Nelson (1970) presents data from life tests carried out during the development of a small electrical appliance. Appliances were essentially operated repeatedly until failure. Here we consider a portion of the data with 20 appliances only. Failures were classified into two modes as,

Cause-I: Failures under code other than '9' and

Cause-II: Failures under code '9' only.

Data consist of an ordered failure time (in completed cycles to failure) and a failure mode, with censoring time of 2560 cycles.

Failure due to cause-I: 11, 35, 49, 170, 329, 381, 708, 1062, 1594, 2327, 2451
Failure due to cause-II: 1167, 1925, 1990, 2223, 2400, 2551, 2471.

N = 2560, n = 20, r_1 = 11, r_2 = 7, n - r = 2

Upon solving the equation (6.3.4) and (6.3.5) we get ML estimates as

\[ \hat{q}_1 = 0.9996201, \quad \hat{q}_2 = 0.9997587 \]

\[ E(r_1) = 8.0040778, \quad E(r_2) = 3.4626587, \quad E(A) = 32790.633 \]

Thus equation (6.4.7) gives

\[ SE(\hat{q}_1) = 1.342594E-04 \quad and \quad SE(\hat{q}_2) = 1.2961355E-04 \]

\[ \text{cov}(\hat{q}_1, \hat{q}_2) = 0. \]

Example-2:

The following example relates to two types of failure namely electrical degradation of certain parts (cause-I) and faulty bonding of the leads (cause-II) leading to failures of transistors under accelerating testing considered by Peck (1966), the data is censored at time 629 hours and 44 out of 369 items were censored.
No. of failures due to

<table>
<thead>
<tr>
<th>Time in hours</th>
<th>Cause –I</th>
<th>Cause –II</th>
</tr>
</thead>
<tbody>
<tr>
<td>0 – 49</td>
<td>26</td>
<td>15</td>
</tr>
<tr>
<td>50 – 99</td>
<td>29</td>
<td>15</td>
</tr>
<tr>
<td>100 – 149</td>
<td>28</td>
<td>22</td>
</tr>
<tr>
<td>150 – 199</td>
<td>35</td>
<td>13</td>
</tr>
<tr>
<td>200 – 249</td>
<td>17</td>
<td>11</td>
</tr>
<tr>
<td>250 – 299</td>
<td>21</td>
<td>08</td>
</tr>
<tr>
<td>300 – 349</td>
<td>11</td>
<td>07</td>
</tr>
<tr>
<td>350 – 399</td>
<td>11</td>
<td>05</td>
</tr>
<tr>
<td>400 – 449</td>
<td>12</td>
<td>03</td>
</tr>
<tr>
<td>450 – 499</td>
<td>07</td>
<td>04</td>
</tr>
<tr>
<td>500 – 549</td>
<td>06</td>
<td>01</td>
</tr>
<tr>
<td>550 – 599</td>
<td>09</td>
<td>02</td>
</tr>
<tr>
<td>600 – 629</td>
<td>06</td>
<td>01</td>
</tr>
<tr>
<td>Total</td>
<td>218</td>
<td>107</td>
</tr>
</tbody>
</table>

Assuming one hour as one cycle and considering geometric lifetime model for number of cycles with pmf (6.2.8), we have

\[ n = 369, \quad N = 629, \quad r_1 = 218, \quad r_2 = 107, \quad r = 325, \quad n - r = 44 \]

Using the equations (6.5.2) and (6.5.3) we get ML estimates as

\[ \hat{q}_1 = 0.9977363, \quad \hat{q}_2 = 0.9988878. \]

and \( E(r_1) = 217.97538, \quad E(r_2) = 106.9721, \quad E(r) = 324.94748. \)

Finally, equations (6.5.6) – (6.5.8) produce

\[ SE(\hat{q}_1) = 1.7416063E-04, \quad SE(\hat{q}_2) = 1.1491642E-04 \]

\[ \text{cov}(\hat{q}_1, \hat{q}_2) = -3.3995926E-09. \]
**Example: 3**
Here we consider the complete data given by Nelson (1970). The failure modes are same as discussed in example: 1.

In addition, we assume that out of 36 appliances two items are withdrawn from the test at the end of $N_1 = 2500$ cycles and the test is continued with the remaining items and the test is finally terminated at the end of $N_2 = 6000$ cycles. Also, assume that the cycles follow geometric lifetime model given in (6.6.9).

Thus we have $n = 36$, $r_1 = 16$, $r_2 = 14$, $r = r_1 + r_2 = 30$, $r_1^{(1)} = 12$, $r_2^{(1)} = 6$,

$$m^{(1)} = 2, n^{(2)} = 16, m^{(2)} = 4, r_1^{(2)} = 4, r_2^{(2)} = 8.$$  

Using the results (6.7.3), (6.8.1) and (6.8.2) the maximum likelihood estimates

(i) For 1st stage are

$$\hat{q}_1^{(1)} = 0.9998218, \hat{q}_2^{(1)} = 0.999911$$

with $SE(\hat{q}_1^{(1)}) = 5.2092399E-05$ and $SE(\hat{q}_2^{(1)}) = 3.6817492E-05$

$$Cov(\hat{q}_1^{(1)}, \hat{q}_2^{(1)}) = 2.75142454E-17$$

(ii) For second stage

$$\hat{q}_1^{(2)} = 0.9998123, \hat{q}_2^{(2)} = 0.9996244$$

with $SE(\hat{q}_1^{(2)}) = 1.049985E-04$ and $SE(\hat{q}_2^{(2)}) = 1.4850194E-04$

$$Cov(\hat{q}_1^{(2)}, \hat{q}_2^{(2)}) = 2.2141436E-15$$

**Example: 4**

Again, we consider the data given in example-2 as follows:

We further assume that out of 369 transistors 6 transistors are withdrawn from the test at the end of 299 hours and then test is continued up to time 629 hours with the remaining survival transistors.

Thus we have $N_1 = 299$, $N_2 = 629$, $n = 369$.  

105
Using results [6.9.2] to [6.9.4], the maximum likelihood estimates are as follows:

(i) For the first stage

\[ \hat{q}_1^{(1)} = 0.9977819, \quad \hat{q}_2^{(1)} = 0.9987707, \quad \text{SE}(\hat{q}_1^{(1)}) = 1.5584029\times 10^{-4}, \]
\[ \text{SE}(\hat{q}_2^{(1)}) = 1.2239442\times 10^{-4}, \quad \text{Cov}(\hat{q}_1^{(1)}, \hat{q}_2^{(1)}) = 6.2880699\times 10^{-9}. \]

(ii) For second stage

\[ \hat{q}_1^{(2)} = 0.9976774, \quad \hat{q}_2^{(2)} = 0.9992117, \quad \text{SE}(\hat{q}_1^{(2)}) = 3.996981\times 10^{-4}, \]
\[ \text{SE}(\hat{q}_2^{(2)}) = 2.8163808\times 10^{-4}, \quad \text{Cov}(\hat{q}_1^{(2)}, \hat{q}_2^{(2)}) = 7.2391509\times 10^{-8}. \]

**APPENDICES**

(1) In order to solve the equations (6.3.4) and (6.3.5) simultaneously for obtaining the estimates of \( q_1 \) and \( q_2 \), the standard iterative methods may be employed. Newton’s method for instance, will usually be satisfactory. If \( q_1^0 \) and \( q_2^0 \) are the approximate solutions, let

\[ \hat{q}_1 = q_1^0 + h \quad \text{and} \quad \hat{q}_2 = q_2^0 + k \]

where \( h \) and \( k \) are the corrections to be determined by the iterative process.

Using Taylor’s Theorem and neglecting higher powers of \( h \) and \( k \) above the first, we get

\[ h \frac{\partial^2 \log L}{\partial q_1^0 \partial q_1^0} + k \frac{\partial^2 \log L}{\partial q_1^0 \partial q_2^0} = -\frac{\partial \log L}{\partial q_1^0} \]

(2)

\[ k \frac{\partial^2 \log L}{\partial q_2^0 \partial q_2^0} + h \frac{\partial^2 \log L}{\partial q_1^0 \partial q_2^0} = -\frac{\partial \log L}{\partial q_2^0} \]

(3)
The corrections \( h \) and \( k \) can be determined by solving the equations (2) and (3) simultaneously. (6.3.4) and (6.3.5) give the terms on the right. The number of repetitions of the iterative process for a given degree of accuracy will generally depend upon the initial approximations.

The use of order statistics and empirical distribution function may be made to obtain a better first approximation.

The failure time probability distribution of an item due to cause I is given by

\[
f_i(x) = (1 - q_i)q_i^{x-1}; i = 1, 2
\]

and

\[
F_i(N) = P(X_i \leq N) = 1 - q_i^N
\]

As \( r_i \) items fail up to time \( N \) due to the cause I, the consistent estimator of \( F_i(N) \) is given by,

\[
\hat{F}_i(N) = \frac{r_i}{n}
\]

Thus \( 1 - q_i^N = \frac{r_i}{n} \) gives first approximation as

\[
\hat{q}_i^0 = \left( \frac{n - r_i}{n} \right)^{\frac{1}{N}} ; \quad i = 1, 2.
\]

(2) The first stage of censoring probability that an item will fail due to cause I by time \( N_1 \) is given by

\[
P(X < N_1) = 1 - (q_i^{(1)})^{N_1}
\]

but \( \hat{P}(X \leq N_1) \approx \frac{r_i^{(1)}}{n} \), hence \( \frac{r_i^{(1)}}{n} \approx 1 - (q_i^{(1)})^{N_1} \) \( \Rightarrow \hat{q}_i^{(1)} \approx \left( 1 - \frac{r_i^{(1)}}{n} \right)^{\frac{1}{N_1}} \) and for the second stage of censoring

\[
P(N_1 + 1 \leq X \leq N_2) = 1 - (q_i^{(2)})^{N_2 - N_1}
\]

\[
= \frac{r_i^{(2)}}{n^{(2)}}
\]
which we can use as initial estimates to solve likelihood equations. Now we can apply the method of iteration as follow:

Let $q_{1}^{(j)} = q_{10}^{(j)} + h_{1}^{(j)}$ and $q_{2}^{(j)} = q_{20}^{(j)} + k_{1}^{(j)}$ where $h_{1}^{(j)}$ and $k_{1}^{(j)}$ are the corrections to determine maximum likelihood estimates by solving the following equations and repeat the process till the stable values of $q_{1}^{(j)}$ and $q_{2}^{(j)}$ are obtained.

\[
\begin{align*}
- \frac{\partial \log L}{\partial q_{1}^{(j)}} &= h_{1}^{(j)} \frac{\partial^2 \log L}{\partial q_{1}^{(j)} \partial q_{1}^{(j)}} + k_{1}^{(j)} \frac{\partial^2 \log L}{\partial q_{1}^{(j)} \partial q_{2}^{(j)}} \\
- \frac{\partial \log L}{\partial q_{2}^{(j)}} &= h_{1}^{(j)} \frac{\partial^2 \log L}{\partial q_{1}^{(j)} \partial q_{2}^{(j)}} + k_{1}^{(j)} \frac{\partial^2 \log L}{\partial q_{2}^{(j)} \partial q_{2}^{(j)}}
\end{align*}
\]