CHAPTER 5*

ESTIMATION OF PARAMETERS OF MIXED GEOMETRIC FAILURE MODELS FROM TYPE-I PROGRESSIVELY CENSORED AND GROUP CENSORED SAMPLES

5.1 INTRODUCTION

Statistical distributions, which can be expressed as super positions of (Usually simpler) component distributions are termed as mixture distributions. Mixtures with a finite number of components are termed as finite mixtures. The mixture of probability distributions can be applied in many fields such as ecology, fishery, plant and animal breeding. Medgyessi (1961) analyses absorption spectra in terms of normal mixtures, Gregor (1969) applied a mixture of normal distribution to data arising from measuring the content of DNA in the nuclei of liver cells of rats. The mixed normal distribution considered by Cohen (1965) is claimed to be applicable to the study of wind velocities and physical dimensions of mass-produced items. The mixed Weibull distribution discussed by Kao (1959) and Lee and Sinha (1976) are useful in reliability studies especially those involving electron tubes.

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From theoretical point of view mixtures of distributions present interesting problems, for e.g. Barlow and Proschan (1965) prove that the decreasing hazard rate (DHR) property is closed under mixtures. Also, by Tallis and Light (1968), Padget and Tsokos (1978) and others considered the problem of estimation for mixture data. Discrete mixture models arise when individuals in a population are each one of K distinct types, with a proportion $p_i$ of the population being of $i^{th}$ type; the $p_i$'s satisfy

$$0 < p_i < 1 \text{ and } \sum_i p_i = 1.$$ 

Individuals of type $i$ are assumed to have a lifetime distribution with survivor function $S_i(t)$. An individual randomly selected from this population then has survivor function

$$S(t) = p_1 S_1(t) + \ldots + p_K S_K(t) \quad (5.1.1)$$

Models of this kind are termed 'discrete mixture models' and are useful in situations where the population is homogeneous but it is not possible to distinguish between individuals of different types. Often the $S_i(t)$'s in (5.1.1) are taken to be from the same parametric family, though this is of course unnecessary. The properties of mixture model are easily derived from the properties of the K distributions or components; involved in the mixture. Models with K larger than 2 or 3 are rarely used, unless one is in a situation where the $S_i(t)$'s are completely known.

Mixtures of distributions are also important in life testing and reliability analysis. Here the observations are the times of failure of a sample of items. Often failure can occur for more than one reason and the failure distribution for each reason can be density function. The overall failure distribution is then a mixture. For details of mixtures of distribution, we refer to a survey paper by Gupta and Huang (1981). Generally in life testing failure data, the observed samples are often censored or truncated. Mendenhall and Hader (1958) studied this kind of problems for estimating the parameters of mixed exponentially distributed failure time distributions from Type-I censored life test data. The extension of the model considered by Mendenhall and Hader (1958) was studied by Patel (1998) for two stage progressive Type-I censoring with changing parameters.
In this chapter we have considered a mixture of two geometric distributions and we have obtained the maximum likelihood estimators of the parameters of the mixed geometric failure model from Type-I two stage progressively censored and group censored samples. Here we have assumed that the probability of failure of the components as well as mixing proportion change at each stage of censoring. Also, the asymptotic standard errors of the estimates are derived for both the cases. Numerical examples are given to exemplify the estimation procedure in last section.

5.2 MIXED FAILURE MODEL UNDER TYPE-I PROGRESSIVELY CENSORED SAMPLE

Here we consider mixed failure model with two modes of failure. A population is postulated as

\[ f(x) = \alpha f_1(x) + (1 - \alpha) f_2(x), \quad 0 \leq \alpha \leq 1. \]  

(5.2.1)

Where \( \alpha \) represents mixing proportion of units belonging sub population \( i=1 \) with probability mass \( f_i(x) \) which is given by

\[ f_i(x) = q_i p_i^{x-1}; \quad x = 1, 2, \quad i = 1, 2. \]  

(5.2.2)

Thus according to Patel and Gajjar (1990) the composite probability mass function of \( x \) for two stage Type-I progressive censoring is given by

\[ f(x) = \begin{cases} \alpha^{(1)} q_1^{(1)} p_1^{(1)x-1} + (1 - \alpha) q_2^{(1)} p_2^{(1)x-1}, & x = 1, 2, \ldots, N_1 \\ \frac{C}{C} \alpha^{(2)} q_1^{(2)} p_1^{(2)x-1} + (1 - \alpha^{(2)}) q_2^{(2)} p_2^{(2)x-1}, & x = N_1 + 1, N_1 + 2, \ldots \end{cases} \]  

(5.2.3)

where \( N_1 \) is specified and \( C \) can be determined from the condition

\[ P(1 \leq X \leq N_1) + CP(N_1 + 1 \leq X \leq \infty) = 1. \]

The cumulative failure probability distribution defined by

\[ F_j^{(i)}(x) = 1 - p_j^{(i)x-1} \quad \text{Where } i = 1, 2; \quad j = 1, 2. \]  

(5.2.4)
Using (5.2.3), constant C can be obtained as,

\[ C = \frac{\alpha^{(1)} p_1^{(1)N_1} + (1-\alpha^{(1)}) p_2^{(1)N_1}}{\alpha^{(2)} p_1^{(2)N_1} + (1-\alpha^{(2)}) p_2^{(2)N_1}} \]

and hence the probability mass function (pmf) of \( X \) can be written as

\[
f(x) = \begin{cases} 
\alpha^{(1)} q_1^{(1)} p_1^{(1)x-1} + (1-\alpha^{(1)}) q_2^{(1)} p_2^{(1)x-1}, & x=1,2,\ldots,N_1 \\
\alpha^{(2)} q_1^{(2)} p_1^{(2)x-1} + (1-\alpha^{(2)}) q_2^{(2)} p_2^{(2)x-1} & 
\end{cases}
\]

\[ , x = N_1 + 1, N_1 + 2, \ldots. \] (5.2.5)

Using (5.2.4) the cumulative distribution function (cdf) of \( X \) is given by

\[
1 - F(x) = \begin{cases} 
\alpha^{(1)} (1 - F_1^{(1)}(x)) + (1-\alpha^{(1)}) (1 - F_2^{(1)}(x)); & x=1,2,\ldots,N_1 \\
\alpha^{(2)} (1 - F_1^{(2)}(N_1)) + (1-\alpha^{(2)}) (1 - F_2^{(2)}(N_1)) & 
\end{cases}
\]

\[ \times \left\{ \begin{array}{c}
\alpha^{(2)} (1 - F_1^{(2)}(x)) + (1-\alpha^{(2)}) (1 - F_2^{(2)}(x)) \\
\end{array} \right\} \]

\[ , x = N_1 + 1, N_1 + 2, \ldots. \]

We consider the sampling with Type-I progressive censoring with two stages of censoring. First \( n \) items are put on test. We terminate the test procedure at trial \( N_1 \).

Let us suppose that \( n_1 \) items fail during this stage. Suppose \( r_1 \) items have failed due to cause \( C_1 \) and the remaining \( n_1 - r_1 \) are due to cause \( C_2 \). Let \( m^{(1)} \) be the number of surviving items or withdrawn items from the test after \( N_1 \text{th} \) trial.

Similarly, for stage two suppose \( n_2 \) items fail during this stage. Let \( r_2 \) and \( n_2 - r_2 \) be the number of items that have failed due to cause \( C_1 \) and \( C_2 \) respectively. Thus, we have number of surviving items as \( m^{(2)} = n - n_1 - n_2 - m^{(1)} \) say. Let \( X_{j1}^{(i)} \) and \( X_{j2}^{(i)} \) be the time (trials) of failure of the \( j \text{th} \) item in the \( i \text{th} \) stage of censoring due to cause \( C_1 \) and \( C_2 \) respectively. Hence, for two-stage progressive censoring the likelihood is given by
Thus, L can be written as $L = L_1 \times L_2$

Where

$$L_1 = \left\{ \alpha^{(i)} p_1^{(i)N_1} + (1 - \alpha^{(i)}) p_2^{(i)N_1} \right\}^{n_{i-1}} \left( \sum_{j=1}^{r_i} x_j^{(i)} \right)^{r_i} \left( \alpha^{(i)} p_1^{(i)N_1} + (1 - \alpha^{(i)}) p_2^{(i)N_1} \right)^{n_{i-1}} \times \left( 1 - p_1^{(i)} \right)^{r_i} \left( 1 - \alpha^{(i)} \right)^{n_{i-1}} \left( 1 - p_2^{(i)} \right)^{n_{i-1}} \times$$

$$\left( \sum_{j=1}^{r_i} x_j^{(i)} \right)^{r_i} \left( \alpha^{(i)} p_1^{(i)N_1} + (1 - \alpha^{(i)}) p_2^{(i)N_1} \right)^{n_{i-1}} \times \left( 1 - p_1^{(i)} \right)^{r_i} \left( 1 - \alpha^{(i)} \right)^{n_{i-1}} \left( 1 - p_2^{(i)} \right)^{n_{i-1}} \times$$

(5.2.6)

and

$$L_2 = \left\{ \alpha^{(2)} p_1^{(2)N_2} + (1 - \alpha^{(2)}) p_2^{(2)N_2} \right\}^{m_{(2)}} \left( \sum_{j=1}^{r_2} x_j^{(2)} \right)^{r_2} \left( \alpha^{(2)} p_1^{(2)N_2} + (1 - \alpha^{(2)}) p_2^{(2)N_2} \right)^{m_{(2)}} \times \left( 1 - p_1^{(2)} \right)^{r_2} \left( 1 - \alpha^{(2)} \right)^{n_{2-r_2}} \left( 1 - p_2^{(2)} \right)^{n_{2-r_2}} \times$$

(5.2.7)

Where $m_{(2)} = n - n_1 - n_2 - m_{(1)}$ and $n_{(2)} = n - n_1 - m_{(1)}$. (5.2.8)

5.3. MAXIMUM LIKELIHOOD ESTIMATION

The method of maximum likelihood is used to estimate the parameters at each stage of censoring.

Let $K^{(i)} = \frac{\alpha^{(i)} p_1^{(i)N_1}}{\alpha^{(i)} p_1^{(i)N_1} + (1 - \alpha^{(i)}) p_2^{(i)N_1}}$. (5.3.1)
Using (5.2.6) and (5.3.1) we get maximum likelihood estimates of $p_1^{(1)}$, $p_2^{(1)}$ and $\alpha^{(1)}$ denoted by $\hat{p}_1^{(1)}$, $\hat{p}_2^{(1)}$ and $\hat{\alpha}^{(1)}$ respectively as

$$\hat{\alpha}^{(1)} = \frac{r_1 + k^{(1)} \left( \frac{n-n_1}{n} \right)}{n}, \quad (5.3.2)$$

$$\hat{p}_1^{(1)} = 1 - \left[ \frac{\bar{x}_{j1}^{(1)}}{r_1} + \frac{(n-n_1)N_1k^{(1)}}{n_1} \right]^{-1}, \quad (5.3.3)$$

$$\hat{p}_2^{(1)} = 1 - \left[ \frac{\bar{x}_{j2}^{(1)}}{n_1-r_1} + \frac{(n-n_1)N_1(1-k^{(1)})}{n-n_1} \right]^{-1}, \quad (5.3.4)$$

where $\bar{x}_{j1}^{(1)} = \sum_{j=1}^{n} x_{j1}^{(1)} \frac{1}{r_1}$ and $\bar{x}_{j2}^{(1)} = \sum_{j=1}^{n-r_1} x_{j2}^{(1)} \frac{1}{n_1-r_1}$.

Using (5.3.2), (5.3.3) and (5.3.4) we can substitute their estimates in $k^{(1)}$ and hence we get an equation of the form $h^{(1)} = h(k^{(1)})$, a function of $k^{(1)}$. Thus, $g(k^{(1)}) = h(k^{(1)}) - k^{(1)} = 0$, $0 < k^{(1)} < 1$. It is easy to obtain $k^{(1)}$ by considering graph of $h(k^{(1)}) - k^{(1)}$ versus $k^{(1)}$ and solution of $g(k^{(1)}) = 0$. For second stage, we can find ML estimates using (5.2.7) as

$$\hat{\alpha}^{(2)} = \frac{k^{(2)}p_2^{(2)N_2}}{k^{(2)}p_2^{(2)N_2} + (1-k^{(2)})p_1^{(2)N_2}}, \quad (5.3.5)$$

$$\hat{p}_1^{(2)} = 1 - \left[ \frac{\bar{x}_{j1}^{(2)}}{r_2} + \frac{m^{(2)}N_2k^{(2)} - n^{(2)}N_1k^{(12)}}{r_2} \right]^{-1}, \quad (5.3.6)$$

$$\hat{p}_2^{(2)} = 1 - \left[ \frac{\bar{x}_{j2}^{(2)}}{n_2-r_2} + \frac{m^{(2)}N_2(1-k^{(2)}) - n^{(2)}N_1(1-k^{(12)})}{n_2-r_2} \right]^{-1}, \quad (5.3.7)$$
where

\[ K^{(2)} = \frac{\alpha^{(2)} p_1^{(2)N_2}}{\alpha^{(2)} p_2^{(2)N_2} + (1 - \alpha^{(2)}) p_1^{(2)N_2}} \]  

(5.3.8)

\[ K^{(12)} = \frac{\alpha^{(2)} p_1^{(2)N_1}}{\alpha^{(2)} p_1^{(2)N_1} + (1 - \alpha^{(2)}) p_2^{(2)N_1}} \]  

(5.3.9)

\[ r_2 + m^{(2)} k^{(2)} = n^{(2)} k^{(12)} \]  

(5.3.10)

and \[ \bar{x}_{j1}^{(2)} = \frac{\sum_{j=1}^{r_2} x_{j1}^{(2)}}{r_2}, \quad \bar{x}_{j2}^{(2)} = \frac{\sum_{j=1}^{n_2-r_2} x_{j2}^{(2)}}{n_2 - r_2}. \]

Using (5.3.6) and (5.3.7) in (5.3.5) we can obtain \( \hat{\alpha}^{(2)} \) in terms of \( k^{(2)} \) and hence by solving simultaneous equations of \( k^{(12)}, \hat{p}_1^{(2)}, \hat{p}_2^{(2)} \) we can obtain estimates of \( p_1^{(2)}, p_2^{(2)} \) and \( \alpha^{(2)} \). Substituting (5.3.5), (5.3.6), (5.3.7) in (5.3.8) we get single equation involving \( k^{(2)} \) only. That is \( k^{(2)} = h(\hat{k}^{(2)}) \) and hence by drawing a graph of \( g(\hat{k}^{(2)}) = h(\hat{k}^{(2)}) - k^{(2)} \) versus \( k^{(2)} \) we can obtain a solution of \( h(\hat{k}^{(2)}) - k^{(2)} = 0. \)

**5.4ASYMPTOTIC STANDARD ERRORS OF THE ESTIMATES**

In this section we obtain the asymptotic standard errors of the estimates of \( \alpha^{(i)}, p_j^{(i)} \) for \( i, j = 1,2 \). It is easy to verify that

\[ E(r_i) = \alpha^{(i)} E(n^{(i)}) \left[ \frac{F_i^{(i)}(N_i) - F_{i-1}^{(i)}(N_{i-1})}{1 - F_i^{(i)}(N_{i-1})} \right] \]
\[ E(n_i - r_i) = (1 - \alpha^{(i)}) E(n^{(i)}) \left[ \frac{F_2^{(i)}(N_i) - F^{(i)}(N_{i-1})}{1 - F^{(i)}(N_{i-1})} \right] \]

\[ E(n_i) = E(n^{(i)}) \left[ \frac{F^{(i)}(N_i) - F^{(i)}(N_{i-1})}{1 - F^{(i)}(N_{i-1})} \right] \]

\[ E(n^{(2)} - n_2) = E(n^{(2)}) \left[ \frac{1 - F^{(2)}(N_2)}{1 - F^{(2)}(N_1)} \right] \]

\[ E(n^{(2)}) = n[1 - F^{(1)}(N_1)] - m^{(1)} \] (5.4.1)

where \( F^{(i)}(x) = \alpha^{(i)} F_1^{(i)}(x) + (1 - \alpha^{(i)}) F_2^{(i)}(x) \), for \( i = 1, 2 \). (5.4.2)

and \( F_1^{(1)}(N_0) = F_2^{(1)}(N_0) = F^{(1)}(N_0) = N_0 = 0 \) (5.4.3)

Using equations (5.2.6) and (5.2.7) we obtain

\[ E \left( -\frac{\partial^2 \log L}{\partial \alpha^{(i)^2}} \right) = \frac{E(r_i)}{\alpha^{(i)^2}} + \frac{E(n_i - r_i)}{(1 - \alpha^{(i)})^2} + E(n^{(i)}) - n_i \left( \frac{k^{(i)} - \alpha^{(i)}}{\alpha^{(i)}(1 - \alpha^{(i)})} \right)^2 - \]

\[ E(n^{(i)}) \left( \frac{k^{(i-1)} - \alpha^{(i)}}{\alpha^{(i)}(1 - \alpha^{(i)})} \right)^2 \] (5.4.4)

\[ E \left( -\frac{\partial^2 \log L}{\partial \alpha^{(i)^2} \partial p_i^{(i)}} \right) \left[ \frac{E(n^{(i)}) - n_i}{P_i^{(i)}} \right] = \left[ \frac{N_i k^{(i)}}{N_i \alpha^{(i)}} + \frac{N_i \alpha^{(i)}(1 - F^{(i)}(N_i))(F^{(i)}(N_i) - F_1^{(i)}(N_i))}{(1 - F^{(i)}(N_i))^2} \right] \]

\[ + \frac{1}{P_i^{(i)}} \left[ E(n^{(i)}) \left( \frac{N_i k^{(i-1)}}{N_i \alpha^{(i)}} - \frac{N_i \alpha^{(i)}(1 - F^{(i)}(N_{i-1}))(F^{(i)}(N_i) - F_1^{(i)}(N_{i-1}))}{(1 - F^{(i)}(N_{i-1}))^2} \right) \right] \] (5.4.5)
\[ E\left(-\frac{\partial^2 \log L}{\partial \alpha^{(i)} \partial p_2^{(i)}}\right) = \frac{1}{p_2^{(i)}} E(n^{(i)} - n_i) \begin{bmatrix} \frac{N_i(1-k^{(i)})}{1-\alpha^{(i)}} + \\
\frac{N_i(1-\alpha^{(i)})(1-F_2^{(i)}(N_i))(F_2^{(i)}(N_i)-F_1^{(i)}(N_i))}{(1-F^{(i)}(N_i))^2} \end{bmatrix} \]

\[ -\frac{1}{p_2^{(i)}} E(n^{(i)}) \begin{bmatrix} \frac{N_i-1(1-k^{(i-1)})}{1-\alpha^{(i)}} \\
\frac{N_i-1(1-\alpha^{(i)})(1-F_2^{(i)}(N_i-1))(F_2^{(i)}(N_i-1)-F_1^{(i)}(N_i-1))}{(1-F^{(i)}(N_{i-1}))^2} \end{bmatrix} \]  

(5.4.6)

\[ E\left(-\frac{\partial^2 \log L}{\partial p_1^{(i)} \partial p_2^{(i)}}\right) = \frac{1}{p_1^{(i)} p_2^{(i)}} E(n^{(i)} - n_i) N_i^2 k^{(i)} (1-k^{(i)}) - E(n^{(i)}) N_i k^{(i-1)} (1-k^{(i-1)}) \]

(5.4.7)

\[ E\left(-\frac{\partial^2 \log L}{\partial p_1^{(i)^2}}\right) = \frac{E(r_i)}{p_1^{(i)^2}} \left[ \frac{p_1^{(i)^{N_i-1}} (1-N_{i-1} q_1^{(i)}) - p_1^{(i)} (1+N_{i-1} q_1^{(i)})}{q_1^{(i)} (p_1^{(i)^{N_i-1}} - p_1^{(i)^{N_i}})} - 1 \right] \]

\[ + \frac{E(r_i)}{q_1^{(i)^2}} \left[ \frac{E(n^{(i)} - n_i) \alpha^{(i)} N_i (N_{i-1}) (1-F_1^{(i)}(N_i))}{p_1^{(i)^2} (1-F^{(i)}(N_{i-1}))} \right] \]

\[ + \frac{E(n^{(i)} - n_i)}{p_1^{(i)^2}} \left[ \frac{\alpha^{(i)} N_i (1-F_1^{(i)}(N_i))}{(1-F^{(i)}(N_{i-1}))} \right]^2 \]

\[ + \frac{E(n^{(i)}) \alpha^{(i)} N_{i-1} (N_{i-1}-1) (1-F_1^{(i)}(N_{i-1}))}{p_1^{(i)^2} (1-F^{(i)}(N_{i-1}))} \]

\[ - \frac{E(n^{(i)})}{p_1^{(i)^2}} \left[ \frac{\alpha^{(i)} N_{i-1} (1-F_1^{(i)}(N_{i-1}))}{(1-F^{(i)}(N_{i-1}))} \right]^2 \]  

(5.4.8)
\[
E\left(-\frac{\partial^2 \log L}{\partial p_2^{(i)}^2}\right) = \frac{E(n_i-r_i)}{p_2^{(i)}^2} \left[ \frac{p_2^{(i)}^{N_{i-1}} (1 - N_{i-1} q_2^{(i)}) - p_2^{(i)} (1 + N_{i-1} q_2^{(i)})}{q_2^{(i)} (p_2^{(i)})^{N_{i-1}} - p_2^{(i)} N_{i-1}} - 1 \right]
+ \frac{E(n_i-r_i)}{q_2^{(i)}^2} \frac{E(n_i-n_i) (1-\alpha^{(i)}) N_i (N_{i-1}) (1-F_2^{(i)}(N_i))}{p_2^{(i)}^2 (1-F^{(i)}(N_i))}
+ \frac{E(n_i-n_i)}{p_2^{(i)}^2} \left[ \frac{(1-\alpha^{(i)}) N_i (1-F_2^{(i)}(N_i))}{(1-F^{(i)}(N_i))} \right]^2
+ \frac{E(n_i)(1-\alpha^{(i)}) N_{i-1} (N_{i-1}-1) (1-F_2^{(i)}(N_{i-1}))}{p_2^{(i)}^2 (1-F^{(i)}(N_{i-1}))}
- \frac{E(n_i)}{p_2^{(i)}^2} \left[ \frac{(1-\alpha^{(i)}) N_{i-1} (1-F_2^{(i)}(N_{i-1}))}{(1-F^{(i)}(N_{i-1}))} \right]^2.
\]

(5.4.9)

5.5 NUMERICAL EXAMPLE

Consider the simulated sample of size 100 from a mixed geometric distribution with pmf given in (5.2.1) and parameters \(\alpha=0.3, p_1 = 0.2\) and \(p_2 = 0.4\). We represent the simulated data in the following way.

<table>
<thead>
<tr>
<th>Trials</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of failures due to cause C_1</td>
<td>13</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Number of failures due to cause C_2</td>
<td>8</td>
<td>18</td>
<td>18</td>
<td>3</td>
<td>6</td>
<td>4</td>
<td>7</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Let us assume that \(N_1=4, N_2 = 8\) and \(m^{(i)} = 2\) so that we have \(r_1 = 26, r_2 = 3,\)

\(n_1=73, m_1 - r_1 = 47, n_2 = 21, n_2 - r_2 =18, m^{(2)} = 4\) and \(n =100.\)

Using above data and the equation \(g(\hat{k}^{(i)}) = h(\hat{k}^{(i)}) - \hat{k}^{(i)}\) we get \(\hat{k}^{(i)} = 0.0001.\)

Substituting \(\hat{k}^{(i)}\) in (5.3.2), (5.3.3) and (5.3.4) we get \(\hat{\alpha}^{(i)} = 0.260027,\)
\(\hat{p}_1^{(i)} = 0.10094543\) and \(\hat{p}_2^{(i)} = 0.777188.\)
Similarly from $g(k^{(2)}) = h(k^{(2)}) - k^{(2)}$ we get $\hat{k}^{(2)} = 0.001$ and thus $\hat{\alpha}^{(2)} = 0.9490648$, $\hat{\rho}_1^{(2)} = 0.1910249$ and $\hat{\rho}_2^{(2)} = 0.6528605$.

Now using the results of section-4 the variance-co variance matrix $\Sigma_1$ of the estimates for the first stage can be obtained as follows:

\[
\Sigma_1 = \begin{pmatrix}
2.735179E-07 & 2.926926E-10 & 3.734740E-11 \\
1.564896E-06 & 2.898070E-10 & 6.472788E-07 \\
\end{pmatrix}
\]

Similarly, for second stage of censoring the variance-co variance matrix $\Sigma_2$ is given by

\[
\Sigma_2 = \begin{pmatrix}
0.013187 & -0.011813 & 1.225935E-03 \\
0.0116857 & -1.071397E-05 & 4.112210E-03 \\
\end{pmatrix}
\]

5.6 MIXED FAILURE MODEL UNDER TYPE-I PROGRESSIVELY GROUP CENSORED SAMPLE

According to Patel and Gajjar (1990) the composite probability mass function of $X$ for two stage progressively group censoring is given by,

\[
f(x) = \begin{cases} 
\alpha^{(1)} q_1^{(1)} p_1^{(1)x-1} + (1-\alpha^{(1)}) q_2^{(1)} p_2^{(1)x-1} ; x = 1,2,\ldots,N_i \\
C\left(\alpha^{(2)} q_1^{(2)} p_1^{(2)x-1} + (1-\alpha^{(2)}) q_2^{(2)} p_2^{(2)x-1}\right) ; x = N_1+1,N_2+1,\ldots
\end{cases}
\]

(5.6.1)
The commutative failure probability distribution defined by

\[ F_j^{(i)}(x) = 1 - p_j^{(i)} x, \quad i = 1, 2; \quad j = 1, 2. \]  

(5.6.2)

Using equation (5.6.1) the constant can be obtained as,

\[ C = \frac{\alpha^{(i)} p_1^{(i)N_1} + (1 - \alpha^{(i)}) p_2^{(i)N_1}}{\alpha^{(2)} p_1^{(2)N_1} + (1 - \alpha^{(2)}) p_2^{(2)N_1}}. \]

Hence, the p.m.f. of \( x \) can be written as

\[
f(x) = \begin{cases} 
\frac{\alpha^{(i)} p_1^{(i)x} + (1 - \alpha^{(i)}) p_2^{(i)x}}{\alpha^{(2)} p_1^{(2)x} + (1 - \alpha^{(2)}) p_2^{(2)x}}, & x = 1, 2, 3 \ldots N_1 \\
1 - \frac{\alpha^{(i)} (1 - F_1^{(i)}(x)) + (1 - \alpha^{(i)})(1 - F_2^{(i)}(x))}{\alpha^{(2)} (1 - F_1^{(2)}(x)) + (1 - \alpha^{(2)})(1 - F_2^{(2)}(x))}, & x = N_1 + 1, N_1 + 2, \ldots 
\end{cases}
\]

(5.6.3)

And the c.d.f. of \( X \) can be given as

\[
1 - F(x) = \begin{cases} 
\frac{\alpha^{(i)} (1 - F_1^{(i)}(x)) + (1 - \alpha^{(i)})(1 - F_2^{(i)}(x))}{\alpha^{(2)} (1 - F_1^{(2)}(x)) + (1 - \alpha^{(2)})(1 - F_2^{(2)}(x))}, & x = N_1 + 1, N_1 + 2, \ldots 
\end{cases}
\]

(5.6.4)

We consider the sampling with progressively group censoring so that only the total number of failures rather than times of failure are observed during each stage \([N_{i-1}+1, N_i]\), \( i = 1, 2 \) where \( N_0 = 0 \). First \( n \) times are put on test. Let us suppose that \( n_1 \) items fail in the interval \([1, N_1]\) due to cause \( C_1 \) and the remaining \( n_1 - r_1 \) have failed due to cause \( C_2 \). At this stage suppose we withdraw some items say \( m^{(1)} \). Now at second stage suppose \( n_2 \) items have failed in interval \([N_1+1, N_2]\).
Out of these, suppose \( r_2 \) items have failed due to cause \( C_1 \) and \( n_2 - r_2 \) failed due to cause \( C_2 \). Thus, we have number of surviving items as

\[
n - n_1 - n_2 - m^{(1)} = m^{(2)} \quad \text{(say)}.
\]

Hence, the likelihood function is given by

\[
L = L_1 \times L_2
\]

where,

\[
L_1 = (\alpha^{(1)})^{r_1} (1 - P_1^{(N_1)})^{r_1} (1 - \alpha^{(1)})^{n_1 - r_1} \times \left\{ \alpha^{(1)} P_1^{(N_1)} + (1 - \alpha^{(1)}) P_2^{(N_1)} \right\}^{n - n_1}
\]

(5.6.5)

\[
L_2 = (\alpha^{(2)})^{r_2} (1 - \alpha^{(1)})^{n_2 - r_2} \left( P_1^{(N_1)} - P_2^{(N_1)} \right)^{r_2} \left( P_1^{(N_2)} - P_2^{(N_2)} \right)^{n_2 - r_2} \times \frac{\left\{ \alpha^{(2)} P_1^{(N_2)} + (1 - \alpha^{(2)}) P_2^{(N_2)} \right\}^{m^{(2)}}}{\left\{ \alpha^{(2)} P_1^{(N_1)} + (1 - \alpha^{(2)}) P_2^{(N_1)} \right\}^{n^{(2)}}}
\]

(5.6.6)

where, \( m^{(2)} = n - n_1 - n_2 - m^{(1)} \) and \( n^{(2)} = n - n_1 - m^{(1)} \).

### 5.7 Maximum Likelihood Estimation for Group Censored Sample

Here we estimate the parameters by using the method of maximum likelihood.

Let

\[
K^{(i)} = \frac{\alpha^{(i)} P_1^{(i)N_1}}{\alpha^{(i)} P_1^{(i)N_1} + (1 - \alpha^{(i)}) P_2^{(i)N_1}}
\]

(5.7.1)
Using likelihood function (5.6.5) and equation (5.7.1), we get
\[ \hat{p}_1^{(1)}, \hat{p}_2^{(1)} \text{ and } \hat{\alpha}^{(1)} \text{ as,} \]
\[ \hat{\alpha}^{(1)} = \frac{r_1 + m^{(1)}k^{(1)}}{n} \] (5.7.2)
\[ \hat{p}_1^{(1)} = \left[ 1 - \frac{r_1}{r_1 + m^{(1)}k^{(1)}} \right]^{N_1} \] (5.7.3)
\[ \hat{p}_2^{(1)} = \left[ 1 - \frac{n_1 - r_1}{(n_1 - r_1) + m^{(1)}(1 - k^{(1)})} \right]^{N_1} \] (5.7.4)

Substituting (5.7.2), (5.7.3) and (5.7.4) we get R.H.S. of (5.7.1) as some function of 
\[ k^{(1)} \text{ say } h(k^{(1)}), 0 < k^{(1)} < 1. \]

It is easy to obtain \[ k^{(1)} \] by considering graph of \[ h(\hat{k}^{(1)}) - \hat{k}^{(1)} \] versus 
\[ \hat{k}^{(1)} \] and solution of \[ h(\hat{k}^{(1)}) - \hat{k}^{(1)} = 0. \]

For second stage, let
\[ K^{(2)} = \frac{\alpha^{(2)}p_1^{(2)N_2}}{\alpha^{(2)}p_1^{(2)N_2} + (1 - \alpha^{(2)})p_2^{(2)N_2}} \] (5.7.5)
\[ K^{(12)} = \frac{\alpha^{(2)}p_1^{(2)N_1}}{\alpha^{(2)}p_1^{(2)N_1} + (1 - \alpha^{(2)})p_2^{(2)N_1}} \] (5.7.6)

Using likelihood function (5.6.6) and equations (5.7.5) & (5.7.6), we get
\[ \hat{\alpha}^{(2)}, \hat{p}_1^{(2)} \text{ and } \hat{p}_2^{(2)} \text{ as} \]
\[ \hat{\alpha}^{(2)} = \frac{r_2 + m^{(2)}k^{(2)}}{r_2 + m^{(2)}k^{(2)} + (n^{(2)} - r_2 - m^{(2)}k^{(2)}) \left( \frac{1 - F_1^{(2)}(N_1)}{1 - F_2^{(2)}(N_1)} \right)} \] (5.7.7)
\[ \hat{p}_1^{(2)} = \left[ 1 + \frac{r_2}{m^{(2)}k^{(2)}} \right]^{(N_1-N_2)} \] (5.7.8)
\[ \hat{p}_2^{(2)} = \left[ 1 + \frac{n_2 - r_2}{m^{(2)}(1 - k^{(2)})} \right]^{(N_1-N_2)} \] (5.7.9)
Using (5.7.8) and (5.7.9) in (5.7.5) we can obtain $\hat{\alpha}(2)$ in terms of $k(2)$ and hence by solving simultaneous equations in $\hat{k}(2)$, $\hat{p}_1(2)$ and $\hat{p}_2(2)$ we can obtain estimates of $\hat{p}_1(1), \hat{p}_2(2)$ and $\hat{\alpha}(2)$. Substituting these in $k(2)$ we get single equation involving $k(2)$ only, say $h(k(2)) = k(2)$. Hence by drawing a graph of $h(k(2)) - \hat{k}(2)$ versus $k(2)$ we can obtain $\hat{k}(2)$ as solution of $h(k(2)) - \hat{k}(2) = 0$.

### 5.8 Asymptotic Standard Errors of the Estimates

In this section we obtain the asymptotic standard errors of the estimate of $\alpha^{(i)}, p_{ji}^{(i)}$ for $i, j = 1, 2$. It is easy to verify that

\[
E(n_i - r_i) = (1 - \alpha^{(i)}) E(n^{(i)}) \left\{ \frac{F_i^{(i)}(N_i) - F_i^{(i)}(N_{i-1})}{1 - F^{(i)}(N_{i-1})} \right\}
\]

\[
E(r_i) = \alpha^{(i)} E(n^{(i)}) \left\{ \frac{F_i^{(i)}(N_i) - F_i^{(i)}(N_{i-1})}{1 - F^{(i)}(N_{i-1})} \right\}
\]

\[
E(n_i) = E(n^{(i)}) \left\{ \frac{F^{(i)}(N_i) - F^{(i)}(N_{i-1})}{1 - F^{(i)}(N_{i-1})} \right\}
\]

\[
E(n^{(2)} - n_2) = E(n^{(2)}) \left\{ \frac{1 - F^{(2)}(N_2)}{1 - F^{(2)}(N_1)} \right\}
\]

\[
E(n^{(2)}) = n(1 - F^{(1)}(N_1)) - m^{(1)}
\]  

Where, $F^{(i)}(x) = \alpha^{(i)} F_1^{(i)}(x) + (1 - \alpha^{(i)}) F_2^{(i)}(x)$; for $i = 1, 2$.  

and $F_1^{(1)}(N_0) = F_2^{(1)}(N_0) = F^{(1)}(N_0) = 0$.
Using the equations (5.6.5) and (5.6.6), we obtain

\[
E\left( \frac{\partial^2 \log L}{\partial \alpha^{(i)}^2} \right) = \frac{E(r_1) + E(n_i - r_i) + E(n^{(i)} - n_i)}{\alpha^{(i)}^2} \left[ \frac{k^{(i)} - \alpha^{(i)}}{\alpha^{(i)}(1 - \alpha^{(i)})} \right]^2 \\
- \frac{E(n^{(i)} - n_i)}{\alpha^{(i)}(1 - \alpha^{(i)})}
\]

(5.8.4)

\[
E\left( \frac{\partial^2 \log L}{\partial p_1^{(i)}^2} \right) = \frac{E(r_1)}{p_1^{(i)}^2} \left[ \frac{(1 - F_1^{(i)}(N_i - 1))(1 - F_1^{(i)}(N_i))}{(F_1^{(i)}(N_i) - F_1^{(i)}(N_i - 1))^2} \right] + \\
\frac{E(n^{(i)} - n_i)}{p_1^{(i)}^2} \left[ (F_1^{(i)}(N_i) - F_1^{(i)}(N_i - 1))(N_i - 1 - F_1^{(i)}(N_i)) + N_i(1 - F_1^{(i)}(N_i)) \right] + \\
\frac{E(n^{(i)} - n_i)}{p_1^{(i)}^2} [N_i k^{(i)}(N_i k^{(i)} - N_i + 1)] - \\
\frac{E(n^{(i)} - n_i)}{p_1^{(i)}^2} [N_i - k^{(i)}(N_i - 1)]
\]

(5.8.5)

\[
E\left( \frac{\partial^2 \log L}{\partial p_2^{(i)}^2} \right) = \frac{E(n_i - r_i)}{p_2^{(i)}^2} \left[ \frac{(1 - F_2^{(i)}(N_i - 1))(1 - F_2^{(i)}(N_i))}{(F_2^{(i)}(N_i) - F_2^{(i)}(N_i - 1))^2} \right] + \\
\frac{E(n^{(i)} - n_i)}{p_2^{(i)}^2} \left[ (F_2^{(i)}(N_i) - F_2^{(i)}(N_i - 1))(N_i - 1 - F_2^{(i)}(N_i)) + N_i(1 - F_2^{(i)}(N_i)) \right] + \\
\frac{E(n^{(i)} - n_i)}{p_2^{(i)}^2} [N_i(1 - k^{(i)})] - \\
\frac{E(n^{(i)} - n_i)}{p_2^{(i)}^2} [N_i - k^{(i)}]
\]

(5.8.6)

\[
E\left( \frac{\partial^2 \log L}{\partial p_1^{(i)} \partial p_2^{(i)}^2} \right) = \frac{1}{p_1^{(i)} p_2^{(i)}} \left[ E(n^{(i)} - n_i)N_i^2 k^{(i)}(1 - k^{(i)}) - E(n^{(i)} - n_i)N_i^2 k^{(i-1)}(1 - k^{(i-1)}) \right]
\]

(5.8.7)
\[ E\left( -\frac{\partial^2 \log L}{\partial \alpha^{(i)} \partial p_i^{(i)}} \right) = \frac{E(n-n_i)}{p_i^{(i)}} \left[ \frac{N_i k^{(i)}(1-k^{(i)})}{\alpha^{(i)}(1-\alpha^{(i)})} \right] \]  

\[ E\left( -\frac{\partial^2 \log L}{\partial \alpha^{(2)} \partial p_1^{(2)}} \right) = \frac{E(m^{(2)})}{p_1^{(2)}} \left[ \frac{N_2 k^{(2)}}{\alpha^{(2)}} + \frac{(F_2^{(2)}(N_2) - F_1^{(2)}(N_2))\alpha^{(2)}N_2(1-F_1^{(2)}(N_2))}{(1-F_2^{(2)}(N_2))^2} \right] + \frac{E(n^{(2)})}{p_1^{(2)}} \left[ \frac{N_1 k^{(12)}}{\alpha^{(2)}} - \frac{(F_1^{(2)}(N_1) - F_1^{(2)}(N_1))\alpha^{(2)}N_1(1-F_1^{(2)}(N_2))}{(1-F_2^{(2)}(N_1))^2} \right] \]  

\[ \frac{E(n^{(2)})}{p_1^{(2)}} \left[ \frac{N_1 k^{(12)}}{\alpha^{(2)}} - \frac{(F_1^{(2)}(N_1) - F_1^{(2)}(N_1))\alpha^{(2)}N_1(1-F_1^{(2)}(N_2))}{(1-F_2^{(2)}(N_1))^2} \right] \]  

(5.8.9)

\[ \frac{E(n^{(2)})}{p_2^{(2)}} \left[ \frac{N_1(1-k^{(12)})}{1-\alpha^{(2)}} + \frac{(F_2^{(2)}(N_1) - F_1^{(2)}(N_1))(1-\alpha^{(2)})N_1(1-F_2^{(2)}(N_2))}{(1-F_2^{(2)}(N_1))^2} \right] \]

(5.8.10)

for \( i=1,2; j=1,2 \). where \( k^{(0i)} = \alpha^{(1)} \), \( n^{(1)} = n, N_0 = 0, F_1^{(1)}(N_0) = F_2^{(1)}(N_0) = 0 \).

Using (5.8.4) to (5.8.10) for \( i^{th} \) stage of censoring the asymptotic variance covariance matrix \( \Sigma_i \) of the estimates is obtained as

\[
\Sigma_i = \begin{pmatrix}
E\left( -\frac{\partial^2 \log L}{\partial \alpha^{(i)}^2} \right) & E\left( -\frac{\partial^2 \log L}{\partial \alpha^{(i)} \partial p_i^{(i)}} \right) & E\left( -\frac{\partial^2 \log L}{\partial \alpha^{(i)} \partial p_2^{(i)}} \right) \\
E\left( -\frac{\partial^2 \log L}{\partial p_1^{(i)} \partial p_2^{(i)}} \right) & E\left( -\frac{\partial^2 \log L}{\partial p_1^{(i)}^2} \right) & E\left( -\frac{\partial^2 \log L}{\partial p_2^{(i)}^2} \right)
\end{pmatrix}^{-1}
\]

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The asymptotic co-variance between triplets \((\alpha^{(i)}, p_1^{(i)}, p_2^{(i)})\) for \(i = 1, 2\) are zero.

5.9 NUMERICAL EXAMPLE

Consider the following simulated sample of size 115 from a mixed geometric distribution with p.m.f. given in (5.6.1) and parameters are \(\alpha^{(1)} = 0.3, p_1^{(1)} = 0.7, p_2^{(1)} = 0.6, \alpha^{(2)} = 0.5, p_1^{(2)} = 0.65\) and \(p_2^{(2)} = 0.8\)

<table>
<thead>
<tr>
<th>Trials</th>
<th>No. of failures due to causes (C_1)</th>
<th>No. of failures due to causes (C_2)</th>
<th>No. of withdrawn items</th>
</tr>
</thead>
<tbody>
<tr>
<td>1-4</td>
<td>25</td>
<td>58</td>
<td>5</td>
</tr>
<tr>
<td>5-8</td>
<td>12</td>
<td>12</td>
<td>3</td>
</tr>
</tbody>
</table>

Let \(N_1 = 4, N_2 = 8\) thus we get \(r_1 = 25, n_1 - r_1 = 58, m^{(1)} = 5, n_1 = 83, r_2 = 12, n^{(2)} = 27, n_2 - r_2 = 12, m^{(2)} = 3, n_2 = 24\) and \(n = 115\).

Using the equation \(g(k^{(1)}) = h(\hat{k}^{(1)}) - (\hat{k}^{(1)})\), we get \(\hat{k}^{(1)} = 0.4425\).

Substituting \((\hat{k}^{(1)})\) in (5.7.2), (5.7.3), (5.7.4) we get
\(\hat{\alpha}^{(1)} = 0.3405, \quad \hat{p}_1^{(1)} = 0.77545, \quad \hat{p}_2^{(1)} = 0.69642\).

Similarly, from the equation \(g(k^{(2)}) = h(\hat{k}^{(2)}) - \hat{k}^{(2)}\) we get
\(\hat{k}^{(2)} = 0.1\) and thus, \(\hat{\alpha}^{(2)} = 0.8631, \quad \hat{p}_1^{(2)} = 0.3951, \quad \hat{p}_2^{(2)} = 0.6546\).

Now, using (5.8.1) to (5.8.10) the asymptotic variance co-variance matrix for 1st stage can be obtained as,

\[
\Sigma_1 = \begin{pmatrix}
0.12786 & 50.178613 & -42.848 \\
0.136684 & 43.0699 & 0.183074
\end{pmatrix}
\]

Similarly, for the second stage of censoring we get variance – covariance matrix as,
The standard error of the estimates for both the stages will be

\[
\Sigma_2 = \begin{bmatrix}
0.503437 & 0.416719 & 0.036649 \\
0.350588 & -0.026030 & 0.010519 \\
\end{bmatrix}
\]

S.E (\(\hat{\alpha}^{(1)}\)) = 0.357575, S.E (\(\hat{\beta}_1^{(1)}\)) = .369707, S.E (\(\hat{\beta}_2^{(1)}\)) = 0.4278719

S.E (\(\hat{\alpha}^{(2)}\)) =0.70953, S.E (\(\hat{\beta}_1^{(2)}\)) =0.592104, S.E (\(\hat{\beta}_2^{(2)}\)) = 0.10075.