Chapter 4

Generalized projections and generalized skew projections

4.1 Introduction

In the theory of operators acting on a complex Hilbert space, the class of normal operators finds at a central place. The spectral theorem is the most important statement about this class. The spectral theorem expresses a normal operator as an integral with respect to a projection valued measure (i.e., resolution of the identity). Thus projections play a central role in the study of normal operators. Following J. Groß, G.Trenkler [50], an operator $T \in \mathcal{B}(H)$ is a generalized projection if $T^2 = T^*$. It is clear if $P \in \mathcal{B}(H)$ is an orthogonal projection, then $P$ is a generalized projection. The following example shows the converse of the last statement need not be true.

**Example 4.1.1.** Let $T = \begin{pmatrix} -\frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{pmatrix}$, acting on a two-dimensional complex Hilbert space. Then one can easily show that $T^2 = T^*$ but $T \neq T^*$.
It well-known that if $T$ is a generalized projection, then $T$ is a normal operator [50] but the converse need not be true.

**Example 4.1.2.** Let $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $T$ is a normal operator on a two-dimensional Hilbert space. But

\[
T^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \neq T^*.
\]

In this chapter we shall introduce various subclasses of normal operators.

### 4.2 Square generalized projections.

We shall start with basic properties of generalized projections.

**Lemma 4.2.1.** If $0 \neq T \in \mathcal{B}(H)$ is a generalized projection, then $T$ is a partial isometry.

**Proof.** Since $T^2 = T^*$, $T$ is normal. So we have $(T^*T)^2 = (T^*T)(T^*T) = (T^*)^2T^2 = (T^2)^*T^* = (T^*)^*T^* = TT^* = T^*T$. Hence $T^*T$ is a projection. So $T$ is a partial isometry. \hfill $\Box$

**Definition 4.2.1.** An operator $X \in \mathcal{B}(H)$ is called generalized inverse of $T$ if $XTX = X$.

If $T \in \mathcal{B}(H)$ is generalized projection, then $T^*$ is generalized inverse of $T$. Because $T^*TT^* = T^{*2}T = (-T)T, = -T^2 = T^*$.

The following basic properties we mention without proof.

**Proposition 4.2.2.** If $T \in \mathcal{B}(H)$ is a generalized projection, then the following hold:

1. $T^*$ is a generalized projection.
2. If $T^{-1}$ exists, then $T^{-1}$ is a generalized projection and $T$ is unitary.

3. If $T \neq 0$, then $\|T\| = 1$.

4. If $T \neq 0$ and $(\alpha T)$ is a generalized projection for real $\alpha$, then $\alpha \in \{0, 1\}$.

5. If $S \in \mathbb{B}(H)$ and $S$, $T$ are unitarily equivalent, then $S$ is a generalized projection.

6. If $M$ a closed subspace of $H$ such that $M$ reduces $T$, then $S = T/M$ is a generalized projection.

**Proposition 4.2.3.** The class of generalized projections is strongly closed, hence uniformly closed, in $\mathbb{B}(H)$.

**Proof.** Let $\{T_n\}$ be a sequence of $k$-generalized projections such that $\{T_n\}$ converges strongly to $T \in \mathbb{B}(H)$. Now, since $T_n$’s are normal, for $x \in H$,

$$\|T^*_n x - T^*_m x\| = \|T_n x - T_m x\|.$$ 

Therefore $\{T^*_n x\}$ is a Cauchy sequence in $H$. Hence there is $y \in H$ such that $T^*_n x \to y$ as $n \to \infty$. Now, for $z \in H$

$$\langle y, z \rangle = \lim_{n \to 0} \langle T^*_n x, z \rangle,$$

$$= \lim_{n \to 0} \langle x, T_n z \rangle,$$

$$= \langle x, Tz \rangle,$$

$$= \langle T^* x, z \rangle.$$ 

So $y = T^* x$. Thus $T^*_n x \to T^* x$ as $n \to \infty$ for each $x \in H$. Since $\{T_n\}$ converges to $T$ strongly, by [52, Problem.93], $T^*_n$ converges strongly to $T^*$.

Thus $T^*_n$ converges strongly to $T^*$. So $T^* = T^*$. Thus $T$ is a $k$-generalized projection, which implies that the class of $k$-generalized projections is strongly closed. \qed

**Proposition 4.2.4.** Let $T_1, \ldots, T_l \in \mathbb{B}(H)$ be generalized projections. Then $(T_1 \oplus \ldots \oplus T_l)$ and $(T_1 \otimes \ldots \otimes T_l)$ are generalized projections.

**Proof.** Let $T_1, \ldots, T_l \in \mathbb{B}(H)$ and $T_i^2 = T_i^*$ for $i = 1, \ldots, l$, and let

$$x = (x_1 \oplus \ldots \oplus x_l) \in (H \oplus \ldots \oplus H).$$
Then \((T_1 \oplus \ldots \oplus T_l)^2 x = (T_1 \oplus \ldots \oplus T_l)^2(x_1 \oplus \ldots \oplus x_l) = T_1^2 x_1 \oplus \ldots \oplus T_l^2 x_l\)

\[= T_1^2 x_1 \oplus \ldots \oplus T_l^2 x_l = (T_1^* \oplus \ldots \oplus T_l^*) (x_1 \oplus \ldots \oplus x_l) = (T_1^* \oplus \ldots \oplus T_l^*)^* x. \]

Thus \((T_1 \oplus \ldots \oplus T_l)\) is a generalized projection.

Now assume \(x = (x_1 \otimes \ldots \otimes x_l) \in \left( \bigotimes^{l\text{-times}} H \right)\).

Then \((T_1 \otimes \ldots \otimes T_l)^2 x = (T_1 \otimes \ldots \otimes T_l)^2(x_1 \otimes \ldots \otimes x_l) = T_1^2 x_1 \otimes \ldots \otimes T_l^2 x_l\)

\[= T_1^* x_1 \otimes \ldots \otimes T_l^* x_l = (T_1^* \otimes \ldots \otimes T_l^*)(x_1 \otimes \ldots \otimes x_l) = (T_1^* \otimes \ldots \otimes T_l^*)^* x. \]

Hence \((T_1 \otimes \ldots \otimes T_l)\) is a generalized projection.

**Proposition 4.2.5.** Let \(T \in \mathbb{B}(H)\) be a generalized projection. Then the following hold.

1. If \((I - T)\) is a generalized projection, then \(T\) is an orthogonal projection.

2. If \(T = F + iG\) is the Cartesian decomposition of \(T\) such that \(FG = -GF\), then \(T\) is an orthogonal projection.

3. If \(T = F + iG\) is the Cartesian decomposition of \(T\) such that \(F\) is an orthogonal projection, then \(T\) is an orthogonal projection.

4. If \(T^3 = T\), then \(T\) is an orthogonal projection.

**Proof.** (1) Since \(T\) and \((I - T)\) are generalized projections, \((I - T^*) = (I - T)^2 = I - 2T + T^2 = I - 2T + T^*\). Thus \(2T = 2T^*, \) which implies that \(T = T^*\). Hence \(T\) is an orthogonal projection.

(2) Since \(T = F + iG\) and \(FG = -GF\), \(T^2\) is selfadjoint. Since \(T\) is a generalized projection, \(T^2 = T^*\). Hence \(T^*\) is selfadjoint, which implies that \(T\) is an orthogonal projection.

(3) Since \(T\) is a generalized projection, \(T^2 = T^*\). Hence

\[(F^2 - G^2) + i(FG + GF) = F - iG.\]

So \((F^2 - G^2) = F\). Since \(F\) is an orthogonal projection, \(G^2 = 0\). Thus \(T\) is an orthogonal projection.

(4) Since \(T = T^3\), \(TT^2 = TT^*, \) \(T\) is selfadjoint. Thus \(T\) is an orthogonal projection.

**Definition 4.2.2.** \(T \in \mathbb{B}(H)\) is called a square generalized projection if \(T^2\) is a generalized projection i.e., \(T^4 = (T^*)^2\).
Lemma 4.2.6. Let $T \in \mathcal{B}(H)$ be a generalized projection. Then $T$ is a square generalized projection.

Proof. Since $T$ is a generalized projection, $T^2 = T^*$. Hence

$$T^4 = T^2 T^* = (T^*)^2.$$

Thus $T$ is a square generalized projection. \qed

Let $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Hence $T^4 = (T^*)^2$, $T^2 \neq T^*$. Thus $T$ is a square generalized projection but it is not a generalized projection.

Proposition 4.2.7. Let $T \in \mathcal{B}(H)$ be a square generalized projection. Then $T$ is 2-normal.

Proof. Since $T$ is a square generalized projection, $T^4 = (T^*)^2$. Hence $T^6 = (T^*)^2 T^2 = T^2 (T^*)^2$. Thus $T^2$ is normal. So $T$ is 2-normal. \qed

The following example shows that the converse in general need not be true.

Example 4.2.1. Let $T = \begin{pmatrix} 1 & -i \\ i & 2 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 2 & 3i \\ -3i & 5 \end{pmatrix}$.

$(T^*)^2 = \begin{pmatrix} 2 & 3i \\ -3i & 5 \end{pmatrix}$. So $T^2$ is a selfadjoint operator. Hence $T$ is 2-normal but it is not a square generalized projection as $T^4 = \begin{pmatrix} 13 & 21i \\ -21i & 34 \end{pmatrix}$.

Example 4.2.2. Let $T = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Hence $T^4 = (T^*)^2$ but $T$ is not normal.

We list some simple results pertaining to square generalized projections in the following.

Lemma 4.2.8. Let $T \in \mathcal{B}(H)$ be a square generalized projection. Then the following hold

1. $T^*$ is a square generalized projection.

2. If $T^{-1}$ exists, then $T^{-1}$ is a square generalized projection.
3. If $S \in \mathcal{B}(H)$ such that $S, T$ are unitarily equivalent, then $S$ is a square generalized projection.

4. If $M$ be a closed subspace of $H$ such that reduces $T$, then $T/M$ is a square generalized projection.

**Proposition 4.2.9.** The class of square generalized projections is strongly closed, hence uniformly closed on $\mathcal{B}(H)$.

**Proof.** Let $\{T_n\}$ be a sequence of generalized projections such that $\{T_n\}$ converges strongly to $T \in \mathcal{B}(H)$. Now, since $T_n^2$’s are normal, for $x \in H$,

$$\|T_n^2 x - T_m^2 x\| = \|T_n^2 x - T_m^2 x\|.$$

So $\{T_n^2 x\}$ is a Cauchy sequence in $H$. Hence there is $y \in H$ such that $T_n^2 x \rightarrow y$ as $n \rightarrow \infty$. Now, for $z \in H$

$$\langle y, z \rangle = \lim_{n \rightarrow 0} \langle T_n^2 x, z \rangle,$$

$$= \lim_{n \rightarrow 0} \langle x, T_n^2 z \rangle,$$

$$= \langle x, T^2 z \rangle,$$

$$= \langle T^* x, z \rangle.$$

So $y = T^* x$. Thus $T_n^2 x \rightarrow T^* x$ as $n \rightarrow \infty$ for each $x \in H$. Since $\{T_n\}$ converges to $T$ strongly, by [52, problem.93], $T_n^4$ converges strongly to $T^4$. Thus $T_n^2$ converges strongly to $T^2$ which implies that $T^4 = T^2$. Thus $T$ is a square generalized projection, which implies that the class of square generalized projections is strongly closed. $\square$

**Proposition 4.2.10.** Let $0 \neq T \in \mathcal{B}(H)$ be a square generalized projection. Then the following hold.

1. If $(I - T)$ is a square generalized projection, then $T$ is normal.

2. If $T^{-1}$ exists, then $T^2$ is unitary.

3. If $S$ is an idempotent such that $S$ is similar to $T$, then $T$ is a generalized projection.

**Proof.** (1) Since $(I - T)$ is a square generalized projection,

$$(I - T)^4 = (I - T^*)^2.$$
Hence

\[ I - 4T + 6T^2 - 4T^3 + T^4 = I - 2T^* + T^*^2. \]

Since \( T \) is a square generalized projection,

\[ T^* = 2T - 3T^2 + 2T^3. \]

By premultiplying and postmultiplying the last equation by \( T \) we get \( T^*T = TT^* \). Thus \( T \) is normal.

(2) Since \( T \) is a square generalized projection, \( T^2 \) is a generalized projection and invertible. Hence by Proposition 4.2.2 (2), \( T^* \) is unitary.

(3) Since \( S, T \) are similar, there is an invertible operator \( V \in \mathbb{B}(H) \) such that \( S = V^{-1}TV \). So \( S^2 = V^{-1}T^2V \). Since \( S^2 = S, V^{-1}T^2V = V^{-1}TV \). So \( T^2 = T \). Since \( T \) is a square generalized projection, \( T^2 = T^4 = (T^2)^* = T^* \). Thus \( T \) is a generalized projection.

**Proposition 4.2.11.** Let \( S, T \in \mathbb{B}(H) \) be commuting square generalized projections. Then \( ST \) is a square generalized projection.

**Proof.** Since \( (ST)^4 = S^4T^4 = S^*^2T^*^2 = ((TS)^*)^2 = (ST)^*^2 \), \( ST \) is a square generalized projection. \( \square \)

**Corollary 4.2.12.** If \( 0 \neq T \in \mathbb{B}(H) \) is a square generalized projection, then \( T^m \) is a square generalized projection for any positive integer \( m \).

Now we will be given enough conditions for the sum two square generalized projections to be a square generalized projection.

**Proposition 4.2.13.** Let \( S, T \in \mathbb{B}(H) \) be square generalized projections such that \( ST = TS = 0 \). Then \( (S + T) \) is a square generalized projection.

**Proof.** Since \( (S + T)^4 = ((S + T)^2)^2 = (S^2 + ST + TS + T^2)^2 = (S^2 + T^2)^2 = S^4 + S^2T^2 + T^2S^2 + T^4 = S^4 + T^4 = S^*^2 + T^*^2 = (S^2 + T^2)^* = (S + T)^*^2 \). Then \( S + T \) is a square generalized projection. \( \square \)

Only commutativity condition in the Proposition 4.2.13, is not enough for the sum to be a square generalized projection. For if \( 0 \neq T \in \mathbb{B}(H) \) be a square generalized projection, then \( (T + T)^4 = (2T)^4 = 16T^4 \) while \( ((T + T)^*)^2 = 4T^*^2 \).
Proposition 4.2.14. Let $T_1, \ldots, T_l \in B(H)$ be square generalized projections. Then the following hold.

1. $(T_1 \oplus \ldots \oplus T_l)$ is square generalized projections.
2. $(T_1 \otimes \ldots \otimes T_l)$ is square generalized projections.

Proof. 1. Let $T_1, \ldots, T_l \in B(H)$ and $T_i^4 = T_i^2$, for $i = 1, \ldots, l$, and

$$x = (x_1 \oplus \ldots \oplus x_l) \in (H \oplus \ldots \oplus H)_{l\text{-times}}.$$

Then $(T_1 \oplus \ldots \oplus T_l)^4 x = (T_1 \oplus \ldots \oplus T_l)(x_1 \oplus \ldots \oplus x_l) = T_1^4 x_1 \oplus \ldots \oplus T_l^4 x_l = T_l^2 x_1 \oplus \ldots \oplus T_l^2 x_l = ((T_1^2 \oplus \ldots \oplus T_l^2)(x_1 \oplus \ldots \oplus x_l) = (T_1^* \oplus \ldots \oplus T_l^*)^2(x_1 \oplus \ldots \oplus x_l) = ((T_1 \oplus \ldots \oplus T_l)^*)^2 x$. Thus $(T_1 \oplus \ldots \oplus T_l)$ is a square generalized projection.

2. Let $x = (x_1 \otimes \ldots \otimes x_l) \in (H \otimes \ldots \otimes H)_{l\text{-times}}$. Then

$$(T_1 \otimes \ldots \otimes T_l)^4 x = (T_1 \otimes \ldots \otimes T_l)^4(x_1 \otimes \ldots \otimes x_l) = T_1^4 x_1 \otimes \ldots \otimes T_l^4 x_l = T_l^2 x_1 \otimes \ldots \otimes T_l^2 x_l = ((T_1^2 \otimes \ldots \otimes T_l^2)(x_1 \otimes \ldots \otimes x_l) = (T_1^* \otimes \ldots \otimes T_l^*)^2(x_1 \otimes \ldots \otimes x_l) = ((T_1 \otimes \ldots \otimes T_l)^*)^2 x$. Hence $(T_1 \otimes \ldots \otimes T_l)$ is a square generalized projection. \(\Box\)

Proposition 4.2.15. Let $T \in B(H)$ be such that $T^{2^n} = T^{*2^{n-1}}$. Then $T$ is $2^{n-1}$-normal.

Proof. Since $T^{2^n} = T^{*2^{n-1}},$

$$T^{2^n}T^{2^{n-1}} = T^{2^{n-1}}T^{*2^{n-1}}$$

and

$$T^{2^{n-1}}T^{2^n} = T^{*2^{n-1}}T^{2^{n-1}}.$$

Hence $T^{2^{n-1}}T^{*2^{n-1}} = T^{*2^{n-1}}T^{2^{n-1}}$. Thus $T$ is $2^{n-1}$-normal. \(\Box\)

The following example shows that the converse in general need not be true.

Example 4.2.3. If $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, then $T^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a normal operator. So $T$ is 2-normal, but $T^* \neq T^2$.

By similar arguments given in the proof of the Proposition 4.2.15, one can prove the following.

Proposition 4.2.16. Let $T \in B(H)$ be such that $T^{3^n} = T^{*3^{n-1}}$. Then $T$ is $3^{n-1}$-normal.
4.3 Generalized skew projections

In this section we shall study the class of generalized skew projections.

Definition 4.3.1. Let $T \in \mathcal{B}(H)$ is called a $k$-generalized skew projection if $T^k = -T^*$. For $k = 2$, $T$ is called generalized skew projection.

Example 4.3.1. 
1. Let $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where $a, b \in \{\frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\}$. Then $T$ is a generalized skew projection.
2. Let $S = \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}$. Then $S^2 = \frac{1}{2} \begin{pmatrix} -1 & \sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = -S^*$. Hence $S$ is a generalized skew projection.

Lemma 4.3.1. If $T \in \mathcal{B}(H)$ is a $k$-generalized skew projection, then $T$ is a normal operator.

Proof. Since $T^k = -T^*$, $T^*T = -T^kT = -T^{k+1} = -TT^k = TT^*$. Hence $T$ is normal.

The converse of Lemma 4.3.1 need not be true.

Example 4.3.2. Let $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then $T$ is a normal operator acting on a two-dimensional Hilbert space but $T^2 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix} \neq -T^*$.

Proposition 4.3.2. Let $T \in \mathcal{B}(H)$ be a $k$-generalized skew projection. Then $(T^*T)^k = T^*T$.

Proof. Since $T^k = -T^*$, by Lemma 4.3.1, $T$ is normal. So we have $(T^*T)^k = (T^*T)\ldots(T^*T) = (T^*)^kT^k = (T^k)^*(T^k) = (-T^*)^*(-T^*) = TT^*$

$= T^*T$. □

Corollary 4.3.3. Let $T \in \mathcal{B}(H)$ be a generalized skew projection. Then $T$ is a partial isometry.

Lemma 4.3.4. If $0 \neq T \in \mathcal{B}(H)$ is a $k$-generalized skew projection, then $\|T\| = 1$. 

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Proof. Since $T^k = -T^*$, $\|T^k\| = \|T^*\| = \|T\|$. Since $T$ is normal, $\|T^k\| = \|T\|^k$. Hence $\|T\|^k = \|T^k\| = \|T\|$. Thus $\|T\| = 1$. \hfill \Box

Proposition 4.3.5. Let $0 \neq T \in B(H)$ be a $k$-generalized skew projection. Then the following hold.

1. If $(\alpha T)^k = - (\alpha T)^*$ for some complex number $\alpha$, then $\alpha \in \{0, 1, \omega, \cdots, \omega^k\}$, where $\omega$ is the $(k+1)^{th}$ root of $1$.

2. If $(\alpha T)^k = (\alpha T)^*$ for some complex $\alpha$, then $\alpha \in \{0, -1, \omega, \cdots, \omega^k\}$, where $\omega$ is the $(k+1)^{th}$ root of $-1$.

Proof. (1) Since $(\alpha T)^k = - (\alpha T)^*$, $T^k = -T^*$. Hence $\alpha^k T^k = -\overline{\alpha} T^* = \overline{\alpha} T^k$. So $\alpha^k = \overline{\alpha}$. Thus $|\alpha|^k = |\alpha|^k = |\overline{\alpha}| = |\alpha|$. So $|\alpha| = 0$ or $|\alpha| = 1$. If $|\alpha| = 0$ it is clear. If $|\alpha| = 1$, $\overline{\alpha} = \overline{\alpha} = \frac{1}{\alpha}$. So $\alpha^{k+1} = 1$. Hence $\alpha \in \{0, 1, \omega, \cdots, \omega^k\}$.

By similar arguments, one can prove (2). \hfill \Box

Corollary 4.3.6. Let $0 \neq T \in B(H)$ be a $k$-generalized skew projection. Then the following hold.

1. If $(\alpha T)^k = - (\alpha T)^*$ for some real $\alpha$, then $\alpha = 0$ or $\alpha = 1$.

2. If $(\alpha T)^k = (\alpha T)^*$ for some real $\alpha$. Then $\alpha = 0$ or $\alpha = -1$.

The following theorem gives characterizations of the $k$-generalized skew projections by using Theorem 2.1.1.

Theorem 4.3.7. For an operator $T \in B(H)$ the following statement are equivalent.

1. $T$ is a $k$-generalized skew projection.

2. $T$ is a normal operator and $\sigma(T) \subseteq \{0, -1, \omega, \cdots, \omega^k\}$, where $\omega$ is the $(k+1)^{th}$ root of $(-1)$.

3. $T$ is normal and $T^{k+2} = -T$.

Proof. (1) $\Rightarrow$ (2): Since $T^k = -T^*$, $T$ is normal. Let $E$ be the the resolution of identity for $T$. Then

$$T^k = \int \lambda^k d(E_{\lambda})$$
and
\[ T^* = \int \bar{\lambda} d(E_{\lambda}). \]

Now, since \( T^k = -T^* \),
\[ \int \lambda^k d(E_{\lambda}) = - \int \bar{\lambda} d(E_{\lambda}). \]

This gives
\[ \int (\lambda^k + \bar{\lambda}) d(E_{\lambda}) = 0. \]

Hence if \( \lambda \in \sigma(T) \) and
\[ \lambda^k = -\bar{\lambda}. \]

Since \( |\lambda|^k = |\lambda| \), \( \lambda = 0 \) or \( |\lambda| = 1 \). Suppose \( \lambda \in \sigma(T) \). Then \( \lambda = 0 \) or
\[ \lambda^k = -\bar{\lambda}, \]
\[ = -\frac{1}{\lambda}. \]

So \( \lambda = 0 \) or \( \lambda^{k+1} = -1 \). Hence \( \lambda = 0 \) or \( \lambda \) is a \((k+1)^{th}\) root of \(-1\). Thus
\[ \sigma(T) \subseteq \{0, -1, \omega, ..., \omega^k\}, \]
where \( \omega \) is the \((k+1)^{th}\) root of \(-1\).

(2) \( \Rightarrow \) (3): Let \( T \) be normal and \( \sigma(T) \subseteq \{0, -1, \omega, ..., \omega^k\} \), where \( \omega \) is the \((k+1)^{th}\) root of \((-1)\). If \( \lambda \in \sigma(T) \), then \( \lambda = 0 \) or \( \lambda^{k+1} = -1 \). Hence \( \lambda^{k+2} = -\lambda \). Therefore
\[ T^{k+2} = \int \lambda^{k+2} d(E_{\lambda}), \]
\[ = - \int \lambda d(E_{\lambda}) \]
\[ = -T. \]

(3) \( \Rightarrow \) (1): Let \( T \) be a normal operator and \( E \) be the the resolution of identity for \( T \). since \( T^{k+2} = -T \),
\[ \int \lambda^{k+2} d(E_{\lambda}) = - \int \lambda d(E_{\lambda}). \]

Hence
\[ \lambda^{k+2} + \lambda = 0 \]
for all \( \lambda \in \sigma(T) \). Now it is easy to deduce \( \lambda^k = -\bar{\lambda} \) for all \( \lambda \in \sigma(T) \) and so \( T^k = -T^* \). \[ \square \]
The \((k+3) \times (k+3)\) diagonal matrix having diagonal entries \(0, \omega, \ldots, \omega^k, -1\), where \(\omega\) is the \((k + 1)\)th root of \(-1\), is a \(k\)-generalized skew projection with \(\sigma(T) = \{0, -1, \omega, \ldots, \omega^k\}\).

The following example shows that the normality conditions for the operator in statements (2) in Theorem 4.3.7 are necessary for the equivalence.

**Example 4.3.3.** Let \(T = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \). We get \(\sigma(T) \subseteq \{0, -1, \omega, \ldots, \omega^k\}\), where \(\omega\) is the \((k + 1)\)th root of \((-1)\) for \(k \in \mathbb{N}\). But \(T^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\), \(-T^* = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\) (i.e., \(T^2 \neq -T^*\)). So \(T\) is not a \(k\)-generalized skew projection.

**Corollary 4.3.8.** Let \(T \in \mathbb{B}(H)\) be a generalized skew projection. Then
\[
\sigma(T) \subseteq \{0, -1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\}.
\]

**Lemma 4.3.9.** Let \(T \in \mathbb{B}(H)\) be a \(k\)-generalized skew projection. Then the following hold

1. \(T^{k+1}\) is a generalized skew projection.

2. The range of \(T\) is closed.

**Proof.** (1) Since \(T\) is a \(k\)-generalized skew projection, by Theorem 4.3.7, \(T^{k+2} = -T\). Hence
\[
(T^{k+1})^2 = T^{k+2}T^k,
\]
\[
= -TT^k,
\]
\[
= -T^{k+1}.
\]
Moreover, \((T^{k+1})^* - T^{k+1} = T^{k+1}T^* - T^{k+1} = T^{k+1} - T^{k+1} = 0\), \(T^{k+1}\) is selfadjoint. Therefore
\[
(T^{k+1})^2 = -(T^{k+1})^*.
\]
(2) Since \(T\) is a \(k\)-generalized projection, \(T\) is normal and its spectrum
\[
\sigma(T) \subseteq \{0, -1, \omega, \ldots, \omega^k\}\), where \(\omega\) is the \((k + 1)\)th root of \(-1\).
Hence $\sigma(T)$ is finite, so $0$ is not limited point of $\sigma(T)$. Therefore the range of $T$ is closed by [38, p. 1954].

In the following proposition we shall give a necessary and sufficient condition for the sum of two generalized skew projections to be a generalized skew projection.

**Proposition 4.3.10.** Let $S, T \in \mathcal{B}(H)$ be generalized skew projections. Then $(S + T)^2 = -(S + T)^*$ if and only if $ST = TS = 0$.

**Proof.** Suppose $(S + T)^2 = -(S + T)^*$. Now we also have

$$(S + T)^2 = S^2 + ST + TS + T^2 = -S^* + ST + TS - T^*.$$  

Hence $ST + TS = 0$. So by multiplying on the right by $S$ we get $S^2T + STS = 0$ and by multiplying this on the left by $S$ we get $STS + TS^2 = 0$. From these we have $S^2T = TS^2$. Since $T$ is normal, by Fuglede’s theorem [87], $(S^2)^*T = T(S^2)^*$. Thus we have $(-S^*)^*T = T(-S^*)^*$ and so $ST = TS$. Since $ST + TS = 0$, $TS = ST = 0$.

Now suppose that $ST = TS = 0$. Then

$$(S + T)^2 = S^2 + ST + TS + T^2,$$

$$= S^2 + T^2,$$

$$= (-S^*) + (-T^*),$$

$$= -(S + T)^*.$$  

$\square$

Only commutativity in the above Proposition is not enough for the sum to be a k-generalized skew projection. Let $0 \neq T \in B(H)$ be a k-generalized skew projection and $S = T$. Then for $k > 1$, $(T + T)^k = (2T)^k = 2^kT^k = -2^kT^* \neq -2T^*$. Further, for k-generalized skew projections $S, T \in \mathcal{B}(H)$ with $ST = -TS$, $(ST)^k = -(ST)^*$, or $(ST)^k = (ST)^*$. Indeed, since

$$(ST)^k = \underbrace{(ST) \cdots (ST)}_{k\text{-times}}$$

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we have two cases:

(i) If $k$ is odd, then

$$(ST)^k = -S^kT^k = -(S)^*(-T)^* = -S^*T^* = -(TS)^* = (ST)^*.$$ 

(ii) If $k$ is even, then

$$(ST)^k = S^kT^k = -(S^*)*(-T^*) = (S^*T^*) = (TS)^* = -(ST)^*.$$ 

Further, for generalized skew projections $S,T \in \mathbb{B}(H)$ with $ST = TS$, $ST$ is a generalized projection.

The following discusses the behavior of $k$-generalized skew projections with respect to the direct sum and the tensor product.

**Proposition 4.3.11.** Let $T_1, \ldots, T_l \in \mathbb{B}(H)$ be $k$-generalized skew projections. Then $(T_1 \oplus \ldots \oplus T_l)$ and $(T_1 \otimes \ldots \otimes T_l)$ are $k$-generalized skew projections.

**Proof.** Let $T_1, \ldots, T_l \in \mathbb{B}(H)$ and $T_i^k = -T_i^*$ for $i = 1, \ldots, l$, and let $x = (x_1 \oplus \ldots \oplus x_l) \in \bigoplus H \oplus \ldots \oplus H$. Then

$$(T_1 \oplus \ldots \oplus T_l)^k x = (T_1 \oplus \ldots \oplus T_l)^k (x_1 \oplus \ldots \oplus x_l) = T_1^k x_1 \oplus \ldots \oplus T_l^k x_l$$

$= -T_1^* x_1 \oplus \ldots \oplus -T_l^* x_l = (-T_1^* \oplus \ldots \oplus -T_l^*) (x_1 \oplus \ldots \oplus x_l) = - (T_1 \oplus \ldots \oplus T_l)^* x.$

Thus $(T_1 \oplus \ldots \oplus T_l)$ is a generalized skew projection.

Now let $x = (x_1 \otimes \ldots \otimes x_l) \in \bigotimes H \otimes \ldots \otimes H$. Then

$$(T_1 \otimes \ldots \otimes T_l)^k x = (T_1 \otimes \ldots \otimes T_l)^k (x_1 \otimes \ldots \otimes x_l) = T_1^k x_1 \otimes \ldots \otimes T_l^k x_l$$

$= -T_1^* x_1 \otimes \ldots \otimes -T_l^* x_l = ((-T_1^*) \otimes \ldots \otimes (-T_l^*)) (x_1 \otimes \ldots \otimes x_l)$

$= (-1)^l (T_1^* \otimes \ldots \otimes T_l^*) (x_1 \otimes \ldots \otimes x_l) = (-1)^l (T_1 \otimes \ldots \otimes T_l)^* x.$

Hence $(T_1 \otimes \ldots \otimes T_l)$ is a generalized skew projection. \qed

**Proposition 4.3.12.** Let $T \in \mathbb{B}(H)$ be a $k$-generalized skew projection. Then the following hold.

1. $T^*$ is a $k$-generalized skew projection.

2. If $T^{-1}$ exists, then $T^{-1}$ is a $k$-generalized skew projection and $T$ is an unitary operator.
3. If $S \in \mathcal{B}(H)$, and $S$, $T$ are unitarily equivalent, then $S$ is a $k$-generalized skew projection.

4. If $M$ is a closed subspace of $H$ such that $M$ reduces $T$, then $S = T/M$ is a $k$-generalized skew projection.

**Proposition 4.3.13.** The class of $k$-generalized skew projections is strongly closed, hence uniformly closed, in $\mathcal{B}(H)$.

**Proof.** Let $\{T_n\}$ be a sequence of $k$-generalized skew projections such that $\{T_n\}$ converges strongly to $T \in \mathcal{B}(H)$. Now, since $T_n$’s are normal, for $x \in H$, $\|T_n^*x - T_m^*x\| = \|T_n x - T_m x\|$. So $\{T_n^*x\}$ is a Cauchy sequence in $H$. So there is $y \in H$ such that $T_n^*x \to y$ as $n \to \infty$. Now, for $z \in H$,

$$\langle y, z \rangle = \lim_{n \to 0} \langle T_n^*x, z \rangle,$$

$$= \lim_{n \to 0} \langle x, T_n z \rangle,$$

$$= \langle x, Tz \rangle,$$

$$= \langle T^*x, z \rangle.$$

So $y = T^*x$. Thus $T_n^*x \to T^*x$ as $n \to \infty$ for each $x \in H$. Since $\{T_n\}$ converges to $T$ strongly, by [52, Problem.93], $T_n^k$ converges strongly to $T^k$. Thus $T_n^*$ converges strongly to $-T^k$ which implies that $T^k = -T^*$. Thus $T$ is a $k$-generalized skew projection, which implies that the class of $k$-generalized skew projections is strongly closed. \qed

**Proposition 4.3.14.** Let $T \in \mathcal{B}(H)$ be a generalized skew projection. Then the following hold

1. If $(I + T)^2 = -(I + T)^*$, then $(I + T) = 0$.

2. If $(I - T)^2 = -(I - T)^*$, then $(I - T) = T^*$.

3. If $T = A + iB$ be the Cartesian decomposition of $T$ such that $AB = -BA,$ then $T^2 = -T$.

4. If $T = A + iB$ be the Cartesian decomposition of $T$ such that $A^2 = -A,$ then $T = A$. 

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Proof. (1) Since $-I - T^* = -(I + T)^* = (I + T)^2 = I + 2T + T^2 = I + 2T - T^*$, $2I + 2T = 0$. Therefore $I + T = 0$.

(2) Since $-I + T^* = -(I - T)^* = (I - T)^2 = I - 2T + T^2 = I - 2T - T^*$, $2I - 2T = 2T^*$. So $T^* = I - T$.

(3) Since $T = A + iB$, $AB = -BA$. So $T^2 = A^2 - B^2$ is a self-adjoint operator. Thus $-T^*$ is a self-adjoint operator, so is $T$. Hence $T^2 = -T$.

(4) Since $T^2 = -T^*$, $A^2 - B^2 + i(AB + BA) = -A + iB$. So $A^2 - B^2 = -A$. Since $A^2 = -A$, $B^2 = 0$. Since $B$ is self-adjoint, $B = 0$. Hence $T = A$. \(\Box\)

Proposition 4.3.15. Let $T \in \mathbb{B}(H)$ be a k-generalized skew projection. Then the following hold.

1. If $T^{k+1} = T$, then $T^k = -T$.

2. If $S \in \mathbb{B}(H)$ such that $S^k = S$ and $S, T$ are similar, then $T = -T^*$.

Proof. (1) Since $T = T^{k+1} = T^k(T) = -T^*T$, $T$ is a self-adjoint operator. Hence $T^k = -T$.

(2) Since $S, T$ are similar, there is an invertible operator $V \in \mathbb{B}(H)$ such that $T = V^{-1}SV$. Thus $-T^* = T^k = V^{-1}S^kV = V^{-1}SV = T$. \(\Box\)

Now a necessary and sufficient condition for difference of two generalized skew projections to be a generalized skew projection is given.

Definition 4.3.2. Let $S, T \in \mathbb{B}(H)$. We said $S$ is lower or equal to $T$ with respect to the $*$-order, which is denoted by $S \leq^* T$, if

1. $S^*S = S^*T$;

2. $SS^* = TS^*$.

In the sequel, we shall need following lemma.

Lemma 4.3.16. Let $S, T \in \mathbb{B}(H)$, where $S$ is normal. Then $S \leq^* T$ if and only if $SS = ST = TS$.

Proof. Since $S \leq^* T$ if and only if $S^*S = S^*T$ and $SS^* = TS^*$. Since $S$ is normal, $S \leq^* T$ if and only if $S^*S = S^*T = TS^*$. By Putnam Fuglede’s theorem [87] $ST = TS$ and $SS = ST$. Hence $S \leq^* T$ if and only if $SS = ST = TS$. \(\Box\)
Theorem 4.3.17. Let \( S, T \in \mathcal{B}(H) \) be two generalized skew projections. Then \( T - S \) is a generalized skew projection if and only if \( S \leq^* T \).

Proof. Suppose \( T - S \) is a generalized skew projection. Then

\[
(T - S)^2 = -(T - S)^*.
\]

Hence

\[
T^2 - ST - TS + S^2 = -T^* + S^*.
\]

So \(-2S^* = ST + TS\). Hence \(2S^2 = ST + TS\). Thus

\[
2S^3 = STS + S^2T
\]

\[
= STS + TS^2.
\]

So \(S^2T = TS^2\). Since \( S \) is skew generalized projection, \(-S^*T = -TS^*\). By Fuglede’s theorem [87], \( ST = TS \). Hence \( S^2 = ST = TS \). Hence by Lemma 4.3.16, \( S \leq^* T \). Suppose \( S \leq^* T \). Then by Lemma 4.3.16,

\[
-S^* = S^2,
\]

\[
= ST,
\]

\[
= TS.
\]

Hence \(-2S^* = ST + TS\). Then by premultiplying and postmultiplying the last equation by \( S^* \) we get

\[
STS^* + TSS^* = S^*ST + S^*TS.
\]

While by premultiplying and postmultiplying the last equation by \( S \) we get

\[
ST = -S^*TS^* + STSS^* - SS^*TS, TS = S^*STS - S^*TS^* - STS^*S
\]

respectively. Since \( S^*S = SS^* \),

\[
ST + TS = -2S^*TS^*,
\]

\[
= -2S^*.
\]

Hence \((T - S)^2 = -T^* + S^* = -(T - S)^*\). So \( T - S \) is a generalized skew projection. \(\square\)

In the following theorem, we have discussed, for generalized skew projections \( T_1, T_2 \), when \( \alpha T_1 + \beta T_2 \) is a generalized skew projection for \( \alpha, \beta \in \mathbb{C} \).
Theorem 4.3.18. Let $T_1, T_2 \in \mathbb{B}(H)$ be distinct nonzero generalized skew projections, $\alpha, \beta$ be nonzero scalars and $\rho = \{0, 1, \frac{-1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\}$. Then $T = \alpha T_1 + \beta T_2$ is a generalized skew projection if and only if any one of the following disjoint sets of conditions holds:

1. $T_1 T_2 = 0$ and $\alpha, \beta \in \rho$.

2. $T_1 T_2 = -T_2^*$ and $\alpha = 1$, $\beta \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\alpha = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, $\beta \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} + i\frac{\sqrt{3}}{2}\}$. 

3. $T_1 T_2 = (\frac{1}{2} - i\frac{\sqrt{3}}{2})T_2^*$ and $\alpha = 1$, $\beta \in \{-\sqrt{3}i, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} - i\frac{\sqrt{3}}{2}\}$ or $\alpha = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, $\beta \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\alpha = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\beta \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}$.

4. $T_1 T_2 = (\frac{1}{2} + i\frac{\sqrt{3}}{2})T_2^*$ and $\alpha = 1$, $\beta \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{-3}{2} - i\frac{\sqrt{3}}{2}\}$ or $\alpha = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, $\beta \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\alpha = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\beta \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$.

5. $T_1 T_2 = -T_1^*$ and $\beta = 1$, $\alpha \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\beta = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, $\alpha \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\beta = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\alpha \in \{-\sqrt{3}i, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} - i\frac{\sqrt{3}}{2}\}$.

6. $T_1 T_2 = (\frac{1}{2} - i\frac{\sqrt{3}}{2})T_1^*$ and $\beta = 1$, $\alpha \in \{-\sqrt{3}i, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\beta = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, $\alpha \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\beta = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\alpha \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}$.

7. $T_1 T_2 = (\frac{1}{2} + i\frac{\sqrt{3}}{2})T_1^*$ and $\beta = 1$, $\alpha \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{-3}{2} - i\frac{\sqrt{3}}{2}\}$ or $\beta = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, $\alpha \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$ or $\beta = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, $\alpha \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}$.

8. $T_1 T_2 = T_1^* + T_2^*$ and $\alpha, \beta$ are any nonzero solutions to the equations $\bar{\alpha} - \alpha^2 = 2\alpha\beta = \beta - \beta^2$.

9. $T_1 T_2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2})T_1^* + (\frac{-1}{2} - i\frac{\sqrt{3}}{2})T_2^*$ and $\alpha, \beta$ are any nonzero solutions to the equations $(-1 - \sqrt{3}i)(\bar{\alpha} - \alpha^2) = 4\alpha\beta = (-1 + \sqrt{3}i)(\beta - \beta^2)$.

10. $T_1 T_2 = (\frac{-1}{2} - i\frac{\sqrt{3}}{2})T_1^* + (\frac{-1}{2} + i\frac{\sqrt{3}}{2})T_2^*$ and $\alpha, \beta$ are any nonzero solutions to the equations $(-1 + \sqrt{3}i)(\bar{\alpha} - \alpha^2) = 4\alpha\beta = (-1 - \sqrt{3}i)(\beta - \beta^2)$. 

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11. \( T_1 T_2 = \frac{1}{2}(\gamma_1 T_1^* + \gamma_2 T_2^*) \) and \( \alpha, \beta \notin \rho \), where \( \gamma_1 = -\frac{\sigma - \alpha^2}{\alpha \beta}, \gamma_2 = -\frac{\beta - \beta^2}{\alpha \beta} \)
are nonzero solutions of the equation \( (\gamma_1^2 + 2\gamma_2) = (\frac{-1}{2} - i\frac{\sqrt{3}}{2})(\gamma_1^2 + 2\gamma_1) \)
or to the equation \( \gamma_1^2 + 2\gamma_2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2})(\gamma_2^2 + 2\gamma_1) \),
and \( \gamma_1^2 + 2\gamma_2 \neq 0 \) and (consequently) \( \gamma_2^2 + 2\gamma_1 \neq 0 \).

12. \( T_1 T_2 \neq T_2 T_1 \) and \( \alpha \beta (T_1 T_2 + T_2 T_1) = (\alpha - \alpha^2)T_1^* + (\beta - \beta^2)T_2^* \), where \( \alpha, \beta \) satisfy the equation \( (\alpha - \alpha^2)(\beta - \beta^2) = \alpha^2 \beta^2 \).

**Proof.** Suppose \( \alpha, \beta \) are nonzero complex numbers and \( T_1, T_2 \in \mathbb{B}(H) \) are
generalized skew projections. Then \( T = \alpha T_1 + \beta T_2 \) is a generalized skew projection if and only if

\[
(\alpha T_1 + \beta T_2)^2 = -(\alpha T_1 + \beta T_2)^*
\]

if and only if

\[
(\alpha T_1)^2 + \alpha \beta T_1 T_2 + \alpha \beta T_2 T_1 + (\beta T_2)^2 = -(\alpha T_1^* + \beta T_2^*)
\]

if and only if

\[
\alpha^2 T_1^2 + \alpha \beta (T_1 T_2 + T_2 T_1) + \beta^2 T_2^2 = -\alpha T_1^* - \beta T_2^*,
\]

\[
\alpha^2 T_1^2 + \alpha \beta (T_1 T_2 + T_2 T_1) + \beta^2 T_2^* = -\alpha T_1^* - \beta T_2^*
\]

if and only if

\[
(\alpha - \alpha^2)T_1^* + (\beta - \beta^2)T_2^* = -\alpha \beta (T_1 T_2 + T_2 T_1).
\]
Let \( \gamma_1 = -\frac{\sigma - \alpha^2}{\alpha \beta} \) and \( \gamma_2 = -\frac{\beta - \beta^2}{\alpha \beta} \). Then the last equation is equivalent to

\[
\gamma_1 T_1^* + \gamma_2 T_2^* = T_1 T_2 + T_2 T_1. \tag{4.3.1}
\]

Now, premultiplying 4.3.1 by \( T_1 \) yields

\[
\gamma_1 T_1 T_1^* + \gamma_2 T_1 T_2^* = T_1^2 T_2 + T_1 T_2 T_1 = -T_1^* T_2 + T_1 T_2 T_1.
\]
Postmultiplying 4.3.1 by \( T_1 \) yields

\[
\gamma_1 T_1^* T_1 + \gamma_2 T_2^* T_1 = T_1 T_2 T_1 + T_2 T_1^2 = T_1 T_2 T_1 - T_2 T_1^*.
\]
Since \( T_1, T_2 \) are normal,

\[
\gamma_2 (T_1 T_2^* - T_2^* T_1) = T_2 T_1^* - T_1^* T_2.
\]
Since 4.3.1 is invariant with respect to interchanging the subscripts “1” and “2”, it also follows that
\[
\gamma_1(T_1^*T_2 - T_2^*T_1^*) = T_2^*T_1 - T_1^*T_2.
\]

Hence
\[
\gamma_2\gamma_1(T_1^*T_2 - T_2^*T_1^*) = \gamma_2(T_2^*T_1 - T_1T_2^*) = (T_1^*T_2 - T_2^*T_1^*).
\]

Thus if 4.3.1 holds, then
\[
T_1^*T_2 = T_2^*T_1^* \text{ or } \gamma_1\gamma_2 = 1. \quad (4.3.2)
\]

Now, if \(T_1^*T_2 = T_2^*T_1^*\), then by Fuglede’s theorem \(T_1T_2 = T_2T_1\). Thus if \(T^2 = -T^*\) and \(\gamma_1\gamma_2 \neq 1\),

then by 4.3.1
\[
\gamma_1T_1^* + \gamma_2T_2^* = 2T_1T_2. \quad (4.3.3)
\]

Or
\[
T_1T_2 \neq T_2T_1 \text{ and } \gamma_1\gamma_2 = 1.
\]

In fact if \(T_1T_2 \neq T_2T_1\), then 4.3.1 is equivalent to that \(\gamma_1\gamma_2 = 1\) which is nothing but (12) of the theorem. Suppose 4.3.3 holds. Then
\[
\gamma_1T_1^* + \gamma_2T_2^* = 2T_2T_1.
\]

So that
\[
\gamma_1T_2T_2^*T_1^* + \gamma_2T_2T_2^*T_1^* = 2T_2T_2^*T_2T_1^*.
\]

Since \(T_2\) is a partial isometry, \(T_2^*T_2^*T_2 = T_2\) and \(T_2T_2^*T_2^* = T_2^*\) [31, p.250].

Hence
\[
\gamma_1T_2T_2^*T_1^* + \gamma_2T_2^* = 2T_2T_1^*.
\]

Therefore
\[
\gamma_1T_2T_2^*T_1^* + \gamma_2T_2^* = 2T_2T_1 = \gamma_1T_1^* + \gamma_2T_2^*.
\]

Thus \(\gamma_1(T_1^* - T_2^*T_1^*) = 0\) which is equivalent to
\[
\alpha - \alpha^2 = 0 \text{ or } T_2^*T_1^*T_1^* = T_1^*, \quad (4.3.4)
\]

similarly, if 4.3.3 holds, then we have also
\[
\beta - \beta^2 = 0 \text{ or } T_1^*T_2^*T_2^* = T_2^*, \quad (4.3.5)
\]

Since \(c \in \mathbb{C}\) is a nonzero solution to the equation \(\overline{c} - c^2 = 0\) if and only if
The expressions 4.3.4 and 4.3.5 show that if $T_1T_2 = T_2T_1$, then 4.3.3 is divided into the following four disjoint cases:

(i) $\alpha \in \rho$, and $\beta \in \rho$

(ii) $\alpha \in \rho$, $\beta \notin \rho$ and $T_1T_1^*T_2^* = T_2^*$ or

(iii) $\alpha \notin \rho$, $\beta \in \rho$ and $T_2T_2^*T_1^* = T_1^*$ or

(iv) $\alpha \notin \rho$, $\beta \notin \rho$, $T_2T_2^*T_1^* = T_2^*$ and $T_1T_1^*T_2^* = T_2^*$.

If (i) above holds, then $\gamma_1 = \gamma_2 = 0$. So $T_1T_2 = 0$ which is (1) of the theorem. Suppose (ii) above holds, then $T_1T_1^*T_2^* = T_2^*$. So $T_1^*T_1T_2 = T_2$. Now, as $\alpha \in \rho$, $\gamma_1 = 0$. So from 4.3.3, we have $T_2^* = 2\gamma_1^{-1}T_1T_2$. So

$$T_2 = 2\gamma_1^{-1}T_2^*T_1^*.$$  \hspace{1cm} (4.3.6)

Thus

$$(2\gamma_2^{-1} + 4\gamma_2^{-2})T_2^*T_1^* = 0.$$  

Since $T_2 \neq 0$, from above we can conclude that $T_1T_2 \neq 0$. Hence

$$(\gamma_2^{-1} + 2\gamma_2^{-2}) = 0$$

which is equivalent to

$$\frac{1}{2}\gamma_2 \in \{-1, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{1}{2} - i\frac{\sqrt{3}}{2}\},$$

we recall that

$$\frac{1}{2}\gamma_2 = -\frac{\beta - \beta^2}{2\alpha\beta}. \hspace{1cm} (4.3.7)$$

In 4.3.7 if we take $\gamma_2 = -2$, then from 4.3.6 we have $T_1T_2 = -T_2^*$ and $\bar{\beta} - \beta^2 = 2\beta\alpha$. Now, if $\alpha = 1$, then $\bar{\beta} - \beta^2 = 2\beta$. So

$$\beta \in \{-1, \frac{-3}{2} - i\frac{\sqrt{3}}{2}, \frac{-3}{2} + i\frac{\sqrt{3}}{2}\}.$$  

If $\alpha = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$, then $\bar{\beta} - \beta^2 = (-1 - \sqrt{3}i)\beta$. So

$$\beta \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} + i\frac{\sqrt{3}}{2}\}.$$  

If $\alpha = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$, then $\bar{\beta} - \beta^2 = (-1 + \sqrt{3}i)\beta$. So

$$\beta \in \{-\sqrt{3}i, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}.$$
This completes the proof of (2) of the theorem.

Now, if \( \gamma_2 = 1 - i\sqrt{3} \), then

\[ T_1T_2 = (\frac{1}{2} - i\frac{\sqrt{3}}{2})T_2^* \]

and

\[ \overline{\beta} - \beta^2 = (-1 + \sqrt{3}i)\beta \alpha. \]

Now, if \( \alpha = 1 \), then

\[ \beta \in \{-\sqrt{3}i, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}. \]

If \( \alpha = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \), then

\[ \beta \in \{-1, -\frac{3}{2} - i\frac{\sqrt{3}}{2}, \frac{3}{2} + i\frac{\sqrt{3}}{2}\}. \]

If \( \alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \), then

\[ \beta \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}. \]

This completes the proof of (3) of the theorem.

Similarly if \( \gamma_2 = 1 + i\frac{\sqrt{3}}{2} \), then

\[ T_1T_2 = (\frac{1}{2} + i\frac{\sqrt{3}}{2})T_2^* \]

and

\[ \overline{\beta} - \beta^2 = (-1 - \sqrt{3}i)\beta \alpha. \]

Now if \( \alpha = 1 \), then

\[ \beta \in \{\sqrt{3}i, \frac{1}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}. \]

If \( \alpha = -\frac{1}{2} - i\frac{\sqrt{3}}{2} \), then

\[ \beta \in \{-\sqrt{3}i, \frac{1}{2} - i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2}\}. \]

If \( \alpha = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \), then

\[ \beta \in \{-1, -\frac{3}{2} - i\frac{\sqrt{3}}{2}, \frac{3}{2} + i\frac{\sqrt{3}}{2}\}. \]

This complete the proof of (4) of the theorem.

Proofs (5),(6),(7) follow by replacing subscripts “1” by “2”, “2” by “1”, and \( \alpha \) by \( \beta \), \( \beta \) by \( \alpha \) in (2),(3), (4) respectively.

Suppose (iv) holds. On Premultiplying 4.3.3 by \( \gamma_1T_1^* \) we get

\[ \gamma_1^2T_1^2 + \gamma_1\gamma_2T_1T_2^* = 2\gamma_1T_1T_2, \]
which is equivalent to
\[-\gamma_1^2 T_1 + \gamma_1 \gamma_2 T_1^* T_2^* = 2\gamma_1 T_2.\]
So
\[\gamma_1 \gamma_2 T_1^* T_2^* = 2\gamma_1 T_2 + \gamma_1^2 T_1.\]
Now, on postmultiplying 4.3.3 by $\gamma_2 T_2^*$ we get
\[\gamma_1 \gamma_2 T_1^* T_2^* + \gamma_2^2 T_2^* T_2 = 2\gamma_2 T_1 T_2 T_2^*,\]
which is equivalent to
\[\gamma_1 \gamma_2 T_1^* T_2^* - \gamma_2^2 T_2 = 2\gamma_2 T_1.\]
So
\[\gamma_1 \gamma_2 T_1^* T_2^* = 2\gamma_2 T_1 + \gamma_2^2 T_2.\]
Hence
\[2\gamma_1 T_2 + \gamma_1^2 T_1 = 2\gamma_2 T_1 + \gamma_2^2 T_2.\]
So
\[(\gamma_1^2 - 2\gamma_2) T_1 = (\gamma_2^2 - 2\gamma_1) T_2.\]  
(4.3.8)
Since $T_1, T_2$ are assumed nonzero, $(\gamma_1^2 - 2\gamma_2) = 0$ if and only if $(\gamma_2^2 - 2\gamma_1) = 0$.
In this case
\[\gamma_1, \gamma_2 \in \{0, 2, -1 + i\sqrt{3}, -1 - i\sqrt{3}\}\]
(v) If $\gamma_2 = 0$, then $\gamma_1 = 0$,
(vi) If $\gamma_2 = 2$, then $\gamma_1 = 2$,
(vii) If $\gamma_2 = -1 + \sqrt{3}i$, then $\gamma_1 = -1 - \sqrt{3}i$,
(viii) If $\gamma_2 = -1 - \sqrt{3}i$, then $\gamma_1 = -1 + \sqrt{3}i$.
Since $\alpha, \beta \notin \rho$, $\gamma_1, \gamma_2$ are nonzero. Now, if $\gamma_1 = \gamma_2 = 2$, then by 4.3.3, 
$T_1 T_2 = T_1^* + T_2^*$ and $\alpha, \beta$ satisfy
\[\overline{\alpha} - \alpha^2 = 2\alpha \beta,\]
\[= \overline{\beta} - \beta^2,\]
which proves (8) of the theorem. If $\gamma_2 = -1 + i\sqrt{3}$, then $\gamma_1 = -1 - i\sqrt{3}$. So by 4.3.3,
\[T_1 T_2 = (\frac{-1}{2} + i\frac{\sqrt{3}}{2}) T_1^* + (\frac{-1}{2} - i\frac{\sqrt{3}}{2}) T_2^*.\]
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and $\alpha, \beta$ satisfy

$$(-1 - \sqrt{3}i)(\overline{\alpha} - \alpha^2) = 4\alpha\beta,$$

$$= (-1 + \sqrt{3}i)(\overline{\beta} - \beta^2),$$

which proves (9) of the theorem. If $\gamma_2 = -1 - i\sqrt{3}$, then $\gamma_1 = -1 + i\sqrt{3}$. So by 4.3.3,

$$T_1T_2 = (\frac{-1}{2} - i\frac{\sqrt{3}}{2})T_1^* + (\frac{-1}{2} + i\frac{\sqrt{3}}{2})T_2^*$$

and $\alpha, \beta$ satisfy

$$(-1 + \sqrt{3}i)(\overline{\alpha} - \alpha^2) = 4\alpha\beta,$$

$$= (-1 - \sqrt{3}i)(\overline{\beta} - \beta^2),$$

which proves (10) of the theorem.

Now, if $\gamma_1^2 - 2\gamma_2 \neq 0$, $\gamma_2^2 - 2\gamma_1 \neq 0$, let

$$\delta_{12} = \frac{\gamma_1^2 - 2\gamma_2}{\gamma_2^2 - 2\gamma_1}.$$ 

Then by 4.3.8, $T_2 = \delta_{12}T_1$. Hence $R(T_1) = R(T_2)$ and therefore the orthogonal projection $T_1T_1^* = -T_1^3$ and $T_2T_2^* = -T_2^3$ are identical. Since $T_2 = \delta_{12}T_1$,

$$T_2^3 = \delta_{12}^3T_1^3.$$ 

Since $T_1^3 = T_2^3 \neq 0$, $(1 - \delta_{12}^3) = 0$. Since $T_1 \neq T_2$, $\delta_{12}^3 \neq 1$. So

$$\delta_{12} = \frac{-1}{2} - i\frac{\sqrt{3}}{2}$$

or

$$\delta_{12} = \frac{-1}{2} + i\frac{\sqrt{3}}{2}$$

this completes the proof of (11) of the theorem.

If any of 1-12 of the theorem is satisfies, then 4.3.1 holds which is equivalent to

$$T = \alpha T_1 + \beta T_2$$

is a generalized skew projection.
Suppose the scalars $\alpha, \beta$ in Theorem 4.3.18 are real numbers. Then
In (1) of the Theorem 4.3.18, the only choice $\alpha = \beta = 1$.
In (2) the only choice is $\alpha = 1, \beta = -1$.
In (5) of the Theorem 4.3.18, the only choice is $\alpha = -1, \beta = 1$.
In (8) of the Theorem 4.3.18, the only choice is $\alpha = \beta = \frac{1}{3}$
In (12) $\alpha$ and $\beta$ should satisfy $\alpha + \beta = 1$, i.e., $\alpha = 1 - \beta$ for all real $\beta$.
In other cases we do not have any choice of real $\alpha$ and $\beta$.

4.4 Generalized skew projections in Banach algebras

In this section we discuss generalizations of some result for generalized skew projection in a Hilbert space to element of complex Banach algebra.
Throughout this section, $\mathcal{A}$ will denote a complex Banach algebra with unit 1. If $a \in \mathcal{A}$, we denote the spectrum and the spectral radius of $a$ by $\sigma(a)$ and $r(a)$ respectively. An element $h \in \mathcal{A}$ is said to be hermitian if $\|e^{ht}\| = 1$ for all $t \in \mathbb{R}$ [24]. $\mathcal{H}(\mathcal{A})$ denotes the set of hermitian elements of $\mathcal{A}$. It is well known that if $h \in \mathcal{H}(\mathcal{A})$, then it need not follows that $h^2 \in \mathcal{H}(\mathcal{A})$ [24]; $\mathcal{H}(\mathcal{A})$ is a closed real subspace of $\mathcal{A}$; $\mathcal{H}(\mathcal{A}) \cap i\mathcal{H}(\mathcal{A}) = \{0\}$; and if $h, k \in \mathcal{H}(\mathcal{A})$, then $i(hk - kh) \in \mathcal{H}(\mathcal{A})$, $\sigma(h) \subset \mathbb{R}$ and $r(h) = \|h\|$ [24, p. 47, 53, 54]. As discussed in [24, §5], we consider
$$
\mathcal{J}(\mathcal{A}) = \{h + ik : h, k \in \mathcal{H}(\mathcal{A})\}.
$$
Since $\mathcal{H}(\mathcal{A}) \cap i\mathcal{H}(\mathcal{A}) = \{0\}$, each element of $\mathcal{J}(\mathcal{A})$ has a unique representation of the form $h + ik$ with $h, k \in \mathcal{H}(\mathcal{A})$, and the linear involution $^*$ on $\mathcal{J}(\mathcal{A})$ defined by $(h + ik)^* = h - ik$. An element $a \in \mathcal{J}(\mathcal{A})$ is normal if $aa^* = a^*a$. 

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If \( a = h + ik \in \mathcal{J}(\mathcal{A}) \) it is clear that \( a \) is normal if and only if \( kh = hk \).

We say that \( \rho \subset \mathcal{A} \) is commutative if \( ab = ba \) for all \( a, b \in \rho \). If \( \rho \) is a commutative subset of \( \mathcal{A} \), then the centralizer of \( \rho \) is given by
\[
\Gamma(\rho) = \{ x \in \mathcal{A} : xs = sx \text{ for every } s \in \rho \}.
\]
We have \( \rho \subseteq \Gamma(\Gamma(\rho)) \) and \( \Gamma(\Gamma(\rho)) \) is a commutative Banach algebra (with unit 1) [87, §11.12].

For a normal element \( a = h + ik \in \mathcal{J}(\mathcal{A}) \), let \( B(a) = \Gamma(\Gamma(\{h,k\})) \) and let \( \Delta_a \) denotes the set of all nontrivial complex homomorphisms of \( B(a) \). It is well known that if \( a = h + ik \in \mathcal{J}(\mathcal{A}) \) is normal, then

1. \( \sigma(x) = \{ \phi(x) : \phi \in \Delta_a \} \) for all \( x \in B(a) \); by [87, Theorems 11.9 and 11.22]

2. \( \phi(h), \phi(k) \in \mathbb{R} \) for all \( \phi \in \Delta_a \);

3. \( \phi(a^*) = \overline{\phi(a)} \) for all \( \phi \in \Delta_a \) and \( \sigma(a^*) = \{ \overline{\lambda} : \lambda \in \sigma(a) \} \);

4. if \( x, y \in B(a) \), then \( \sigma(x + y) \subseteq \sigma(x) + \sigma(y) \) and \( \sigma(xy) \subseteq \sigma(x)\sigma(y) \) [87, Theorem 11.23].

An element \( a = h + ik \in \mathcal{J}(\mathcal{A}) \) is a partial isometry if \( a = aa^*a \).

**Definition 4.4.1.** An element \( a = h + ik \in \mathcal{J}(\mathcal{A}) \) is called generalized skew projection if \( a^2 = -a^* \).

**Theorem 4.4.1.** Suppose \( a = h + ik \in \mathcal{J}(\mathcal{A}) \) with \( hk, h^2, k^2 \in \mathcal{H}(\mathcal{A}) \). Then \( a \) is a generalized skew projection if and only if \( a \) is normal and
\[
\sigma(a) \subseteq \{0\} \cup \{ \lambda \in \mathbb{C} : \lambda^3 = -1 \}.
\]

**Proof.** Let \( a \) be a generalized skew projection. Then \( a^2 = -a^* \). Hence \( -aa^* = a^3 = -a^*a \). Thus \( aa^* = a^*a \).

Now, let \( \lambda \in \sigma(a) \). Then \( \lambda = \phi(a) \) for some \( \phi \in \Delta_a \). Hence \( \overline{\lambda} = \phi(a^*) = -a^*a = a^3 = -a^*a \).
\( \phi(-a^2) = -\phi(a)^2 = -\lambda^2 \). If \( \lambda = 0 \) it is clear. If \( \lambda \neq 0 \), then \( |\lambda| = 1 \). Hence \( \lambda = \frac{1}{i} \). So \( \lambda^3 = -1 \).

Conversely, let \( a \) be normal and \( \sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = -1\} \).

Let \( b = a^2 + a^* \). Since \( a \) is normal , \( hk = kh \). Thus \( b \in \mathcal{B}(a) \). Let \( \lambda \in \sigma(b) \). Then \( \lambda = \phi(b) = \phi(a)^2 + \bar{\phi}(a) \) for some \( \phi \in \Delta_a \).

Case 1: If \( \phi(a) = 0 \), then \( \lambda = 0 \)

Case 2: If \( \phi(a) \neq 0 \), then \( \phi(a)^3 = -1 \) as \( \phi(a) \in \sigma(a) \).

It follows that \( \lambda \phi(a) = \phi(a)^3 + \overline{\phi(a)} \phi(a) = -1 + 1 = 0 \). Therefore \( \lambda = 0 \). Hence \( \sigma(b) = \{0\} \). So \( r(b) = 0 \). Since \( b = h^2 + 2ikh - k^2 + h - ik \),

\[
(2hk - k) = b - (h^2 + h - k^2).
\]

Since \( 2hk - k \in \mathcal{H}(A) \), \( \sigma(2hk - k) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\} \). Hence \( 2hk - k = 0 \) and \( 2hk = k \). Thus \( b = h^2 + h - k^2 \in \mathcal{H}(A) \). Since \( r(b) = 0 \), \( b = 0 \). Hence \( a^2 = -a^* \). \( \square \)

**Proposition 4.4.2.** Suppose \( a = h + ik \in J(A) \) is a generalized skew projection and \( h^2, k^2, hk \in \mathcal{H}(A) \). Then \( (a^*)^2 = -a \) and hence \( a^4 = -a \).

**Proof.** Since \( a^2 = -a^* \), \( h^2 + h - k^2 = i(k - 2hk) \). Hence by [10] show that \( \sigma(h^2 + h - k^2) \subseteq i\mathbb{R} \), \( \sigma(k - hk) \subseteq \mathbb{R} \). Since \( \sigma(h^2 + h - k^2) = \sigma(i(k - 2hk)) \subseteq \mathbb{R} \), \( \sigma(k - 2hk) \subseteq \mathbb{R} \cap i\mathbb{R} = \{0\} \). Hence \( r(k - 2hk) = 0 \). Since \( hk \in \mathcal{H}(A) \), \( k - 2hk \in \mathcal{H}(A) \). Hence \( k = 2hk \). Since \( h^2 + h - k^2 \in \mathcal{H}(A) \), \( r(h^2 + h - k^2) = r(i(k - 2hk)) = 0 \). Thus \( h^2 - k^2 = -h \). Therefore \( (a^*)^2 = h^2 - 2ikh - k^2 = -h - ik = -(h + ik) = -a \), and hence \( a^4 = (a^*)^2 = -a \). \( \square \)

**Corollary 4.4.3.** For \( a = h + ik \in J(A) \) with \( hk, h^2, k^2 \in \mathcal{H}(A) \). The following are equivalent

1. \( a \) is normal, partial isometry and \( a^4 = -a \);
2. \( a \) is normal and \( a^4 = -a \);
3. \( \sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = -1\} \);
4. \( a^2 = -a^* \).

**Proof.** \( (1) \implies (2) \): it is clear.

\( (2) \implies (3) \): use spectral mapping theorem [87, Theorem 10.28].

\( (3) \implies (4) \): By Proposition 4.4.2.

\( (4) \implies (1) \): Clearly \( a \) is normal. Now as \( a^4 = -a \), \( -aa^*a = a^4 = -a \). Hence \( aa^*a = a \). Thus \( a \) is a partial isometry. \( \square \)
Example 4.4.1. [24, §6] Let $A = \mathbb{C}^3$ with pointwise multiplication and $p : A \rightarrow [0, \infty)$ be defined by

$$p(\alpha, \beta, \gamma) = \sup\{|\lambda^{-1}\alpha + \beta + \lambda \gamma| : \lambda \in \mathbb{C}, |\lambda| = 1\}$$

for $(\alpha, \beta, \gamma) \in A$. Define the norm $\|\cdot\|$ on $A$ by

$$\|a\| = \sup\{p(xa) : x \in A, p(x)=1\}.$$ 

Then $(A, \|\cdot\|)$ is a complex (commutative) Banach algebra with unit $1 = (1, 1, 1)$. It is clear

$$\sigma(\alpha, \beta, \gamma) = \{\alpha, \beta, \gamma\} \text{ for } (\alpha, \beta, \gamma) \in A.$$ 

Hence for $a \in A$, $\sigma(a) = \{0\}$ if and only if $a = 0$, $a \in A$.

Let $h_0 = (-1, 0, 1)$. Then $h_0$ is hermitian but $h_0^2$ is not hermitian [24, §6]. By [24, p.58]

$$A = \{\alpha + \beta h_0 + \gamma h_0^2 : \alpha, \beta, \gamma \in \mathbb{C}\}, \mathcal{H}(A) = \{\alpha + \beta h_0 : \alpha, \beta \in \mathbb{R}\}.$$ 

Hence we have

$$\mathcal{J}(A) = \{\xi + \eta h_0 : \xi, \eta \in \mathbb{C}\}$$

and it is clear that each element of $\mathcal{J}(A)$ is normal.

Lemma 4.4.4. Let $A$ be as in Example 4.4.1. Then for $a = \xi + \eta h_0 \in \mathcal{J}(A)$ the following are equivalent.

1. $\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = -1\}$;
2. $a^2 = -a^*$;
3. $a \in \{((\lambda, \lambda, \lambda) : \lambda = 0 \text{ or } \lambda^3 = -1\}$,
4. $a^4 = -a$.

Proof. $(1) \Rightarrow (2)$: Since $a$ is normal, by Theorem 4.4.1, we have $\sigma(a^2 - a^*) = \{0\}$. Thus $a^2 = a^*$.
$(2) \Rightarrow (3)$: Let $a \in \mathcal{J}(A)$. Since $a^2 = -a^*$,

$$((\xi - \eta)^2, \xi^2, (\xi + \eta)^2) = -((\xi - \eta), \xi, (\xi + \eta)).$$
If $\xi = 0$, then $\eta^2 = \bar{\eta} = -\eta$, and so $\eta = 0$, hence $a = (0,0,0)$.

If $\xi \neq 0$, then since $\xi^2 = -\bar{\xi}$,

\[-(\xi - \bar{\eta}) = (\xi - \eta)^2 = \xi^2 - 2\xi\eta + \eta^2 = -\bar{\xi} - 2\xi\eta + \eta^2.\]

And

\[-(\xi + \eta) = -\bar{\xi} + 2\xi\eta + \eta^2.\]

Hence

\[\eta^2 = \bar{\eta} + 2\xi\eta = -\eta^2.\]

Therefore $\eta = 0$. Since $\xi^2 = -\bar{\xi}$,

$|\xi| = 1$ and $\xi^3 = -1$.

(3) $\implies$ (4): It is clear.

(4) $\implies$ (1): By using spectral mapping theorem we get

$\sigma(a) \subseteq \{0\} \cup \{\lambda \in \mathbb{C} : \lambda^3 = -1\}$. \hfill $\Box$

**Lemma 4.4.5.** Let $a, b \in J(A)$ such that $a$ is a generalized skew projection. Then $ab + ba = 0$ if and only if $ab = ba = 0$.

**Proof.** Since $ab + ba = 0$, $a^2b + aba = 0 = aba + ba^2$. So $-a^*b + aba = 0 = aba - ba^*$. Hence $a^*b = ba^*$. Since $a$ is normal, by [88, Proposition 2.1], we get $ab = ba$. \hfill $\Box$

**Proposition 4.4.6.** Let $a, b \in J(A)$ be generalized skew projections. Then $a + b$ is a generalized skew projection if and only if $ab + ba = 0$.

**Proof.** Assume $ab + ba = 0$. Then

\[(a + b)^2 = a^2 + ab + ba + b^2 = -a^* - b^* = -(a + b)^*.\]

Hence $a + b$ is a generalized skew projection.

Conversely, let $(a + b)^2 = -(a + b)^*$. Then $a^2 + ab + ba + b^2 = -a^* - b^*$. Hence $ab + ba = 0$. \hfill $\Box$
Theorem 4.4.7. Let \( a = h + ik, b \in \mathcal{J}(\mathcal{A}) \) be generalized skew projections such that \( h^2, k^2, hk \in \mathcal{H}(\mathcal{A}). \) Then \( b - a \) is a generalized skew projection if and only if \( ab = ba = -a^* \).

Proof. Assume \( ab = ba = -a^* \). Then

\[
(b - a)^2 = b^2 - ab - ba + a^2
= b^* + a^* + a^* - a^*
= -(b - a)^*.
\]

Hence \( b - a \) is a generalized skew projection.

Conversely, assume \( (b - a)^2 = -(b - a)^* \). Then \( b^2 - ab - ba + a^2 = -b^* + a^* \). Since \( a, b \) are generalized skew projections, \( -b^* - ab - ba - a^* = -b^* + a^* \). Thus

\[
2a^* = -ab - ba = -(ab + ba). \tag{4.4.1}
\]

Premultiplying 4.4.1 by \( a^* \), we get \(-2a = 2(a^*)^2 = -a^*ab - a^*ba \). Hence

\[
2a = a^*ab + a^*ba. \tag{4.4.4}
\]

Postmultiplying 4.4.4 by \( a^* \), we get \(-2a = 2(a^*)^2 = -(aba^* + baa^*) \). Hence \( a^*ab + a^*ba = aba^* + baa^* \). Since \( aa^* = -a^3 = a^*a, \)

\[
-a^3b + a^*ba = aba^* - ba^3. \tag{4.4.2}
\]

Postmultiplying 4.4.2 by \( a \), yields \(-a^3ba + a^*ba^2 = aba^*a - ba^4 \). Hence \(-a^3ba - a^*ba^* = aba^*a + ba^4 \). Thus

\[
ba = -a^3ba - a^*ba^* - aba^*a. \tag{4.4.3}
\]

Now, by premultiplying 4.4.2 by \( a \), yields \(-a^4b + aa^*ba = a^2ba^* - aba^3 \). Hence

\[
ab = -a^*ba^* - aa^*ba - aba^3. \tag{4.4.4}
\]

From 4.4.3 and 4.4.4 we get \( ba = -a^3ba - a^*ba^* + aba^3 \) and \( ab = -a^*ba^* + a^3ba - aba^3 \) respectively. Hence \( ab + ba = -2a^*ba^* \). Thus from 4.4.1, we get \( a^* = a^*ba^* \). Thus \( aba = (a^*)^2b(a^*)^2 = a^*a^*ba^*a^* = (a^*)^3 \). Therefore \( aba = -a^6 = -a^4a^2 = a^3 = -a^*a = -aa^* \). So by pre and post multiplying by \( a \) on both sides we get \( a^*ba = a \) and \( aba^* = a \). Hence from 4.4.3 and 4.4.4, we get

\[
ba = -a^* \text{ and } ab = -a^*. \]

\( \square \)
4.5 Concluding Remarks

In this chapter we have investigated characterizations for generalized projections on a Hilbert space and we have given some conditions under which a generalized projection becomes an orthogonal projection. And motivated by the study of generalized projections we have introduced and investigated square generalized projections and shown that every generalized projection is a square generalized projection and if $T$ is a square generalized projection then $T$ is a 2-normal operator. Motivated by the study of generalized projections we have introduced and investigated $k$-generalized skew projections and shown that $T$ is a $k$-generalized skew projection if and only if it is normal and its spectrum is contained in the set of $(k+1)^{th}$ roots of $-1$ together with 0. Elementary properties of these operators are investigated. Also, necessary and sufficient conditions for a linear combination of two generalized skew projections to be a generalized skew projection are developed. Finally, we have generalized some of results for generalized skew projections to element of a complex Banach algebra.