Chapter 3

\( n \)-normal operators and their spectral theory

3.1 Introduction

In [93], H. Weyl proved, \( \sigma(T) \setminus \sigma_{\omega}(T) = \Pi_{00}(T) \) for all compact perturbation of hermitian operator on Hilbert space which is called Weyl’s theorem. Later, the property established by Weyl for hermitian operators has been observed for several other classes of operators like normal operator, hyponormal operator,... etc [27,16,17]. M. Berkani, J. J. Koliha [21], introduced the concepts generalized Weyl’s theorem. An operator \( T \in \mathcal{B}(H) \) satisfies generalized Weyl’s theorem if \( \sigma(T) \setminus \sigma_{B\omega}(T) = E(T) \). Also he has introduced a concept of generalized Browder’s theorem. An operator \( T \in \mathcal{B}(H) \) satisfies generalized Browder’s theorem if \( \sigma(T) \setminus \sigma_{B\omega}(T) = \Pi(T) \). And they showed that \( T \) satisfies the generalized Weyl’s theorem whenever \( T \) is normal operator on a Hilbert space. In this chapter we shall introduce the class of \( n \)-normal operators and study some basic properties for this class and we shall prove that
Weyl’s theorem, Browder’s theorem, generalized Weyl’s theorem and generalized Browder’s theorem hold for \( n \)-normal operator. Also the continuity of the spectrum, approximate point spectrum, the Browder spectrum, the Weyl spectrum and the Weyl approximate point spectrum of \( n \)-normal operators on a Hilbert space are proved.

### 3.2 On \( n \)-normal operators

#### Definition 3.2.1.

Let \( n \in \mathbb{N} \). \( T \in \mathbb{B}(H) \) is called an \( n \)-normal operator if

\[
T^n T^* = T^* T^n.
\]

#### Proposition 3.2.1.

Let \( T \in \mathbb{B}(H) \). Then \( T \) is \( n \)-normal if and only if \( T^n \) is a normal operator.

**Proof.** Let \( T \) is \( n \)-normal, \( T^n T^* = T^* T^n \). Then \( T^n (T^*)^n = T^* T^n (T^*)^{n-1} \)

\[
= T^* (T^n T^*) (T^*)^{n-2} = (T^*)^2 T^n (T^*)^{n-2} = (T^*)^n T^n.
\]

Therefore \( T^n \) is normal.

Now let \( T^n \) is a normal operator. Since \( T^n T = T T^n \), by Fuglede theorem [87], \( T^* T^n = T^n T^* \). So \( T \) is \( n \)-normal. \( \square \)

It is clear that a bounded normal operator is \( n \)-normal for any \( n \). The converse is not true. Indeed if

\[
T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix},
\]

then \( T \) is 2-normal which is not normal. All nonzero nilpotent operators are \( n \)-normal operators, for \( n \geq k \) where \( k \) the index of nilpotance, but they are not normal. It is well known that if \( T \) is normal, then it is hyponormal. And if \( T \) is normal and \( T^k \) is compact for some \( k \), then \( T \) is compact by [48].

20
The following example shows that these need not be true in case of \( n \)-normal operator.

**Example 3.2.1.** Let \( H = \ell^2 \) and \( e_1, e_2, \ldots \) be standard orthogonal basis for \( \ell^2 \).

Define \( T \) on \( H \) by

\[
Te_i = \begin{cases} 
    e_1, & i=1 \\
    e_{i+1}, & i=2j, j = 1, 2, \ldots \\
    0, & i=2j+1
\end{cases}
\]

Then \( T^2 = P \), where \( P \) is the orthogonal projection on the space span by \( e_1 \).

So \( T \) is 2-normal but neither \( T \) nor \( T^* \) is hyponormal.

Now, since \( T^2 \) is a projection on one-dimensional space, it is compact. However, since range of \( T \) contains an infinite orthonormal set \( \{e_i, i = 1, 3, 5, \ldots \} \), \( T \) is not compact.

The following example shows that there exists an operator which is subnormal but not \( n \)-normal for any \( n \in \mathbb{N} \).

**Example 3.2.2.** Let \( U \) be unilateral shift on \( \ell^2 \), \( U(\alpha_0, \alpha_1, \cdots) = (0, \alpha_0, \alpha_1, \cdots) \).

Then \( U \) is subnormal but for any \( n \in \mathbb{N} \), \( U^n \) is not normal.

It is well known that if \( T \) is hyponormal and compact, then \( T \) is normal. But we note that the nilpotent operator \( T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) is an 2-normal operator, which is compact but not normal.
Theorem 3.2.2. The set of all $n$-normal operators on $H$ is a closed subset of $\mathbb{B}(H)$ which is closed under scalar multiplication.

Proof. First if $T$ is $n$-normal, and $\alpha$ is scalar, then $(\alpha T)^n(\alpha T)^* = \alpha^n(\alpha T)^n = \bar{\alpha} \alpha^n(T^*T^n)$. So $(\alpha T^*)(\alpha^n T^n) = (\alpha T)^*(\alpha T)^n$, and hence $\alpha T$ is $n$-normal. 

Now suppose that $\{T_k\}$ is sequence of $n$-normal operators converging to $T$ in $\mathbb{B}(H)$. Now $\|T^nT^* - T_k^nT_k^*\| \leq \|T^nT^* - T^nT_k^*\| + \|T_k^nT_k^* - T^nT_k^*\| \rightarrow 0$ as $k \rightarrow \infty$. Hence $T^*T^n = T^nT^*$. Thus $T$ is $n$-normal. \qed

Proposition 3.2.3. Let $T \in \mathbb{B}(H)$ be $n$-normal. Then the following hold.

1. $T^*$ is $n$-normal.

2. If $T^{-1}$ exists, then $(T^{-1})$ is $n$-normal.

3. If $S \in \mathbb{B}(H)$ is unitary equivalent to $T$, then $S$ is $n$-normal.

4. If $M$ is a closed subspace of $H$ such that $M$ reduces $T$, then $S = T/M$ is an $n$-normal operator.

Proof. (1) Since $T$ is $n$-normal, $T^n$ is normal. So $(T^n)^* = (T^*)^n$ is normal, $T^*$ is an $n$-normal operator.

(2) Since $T$ is $n$-normal, $T^n$ is normal. Since $(T^n)^{-1} = (T^{-1})^n$ is normal, $T^{-1}$ is an $n$-normal operator.

(3) Let $T$ be an $n$-normal operator and $S$ be unitary equivalent of $T$. Then there exists an unitary operator $U \in \mathbb{B}(H)$ such that $S = UTU^*$. So $S^n = UT^nU^*$. Since $T^n$ is normal, $S^n$ is normal. Therefore $S$ is $n$-normal.

(4) Since $T$ is $n$-normal, $T^n$ is normal. So $T^n/M$ is normal. And since $M$ is invariant under $T$, $T^n/M = (T/M)^n$. Thus $(T/M)^n$ is normal. So $T/M$ is $n$-normal. \qed
**Theorem 3.2.4.** If $S, T \in \mathcal{B}(H)$ are commuting $n$-normal operators, then $ST$ is an $n$-normal operator.

**Proof.** Since $S, T$ are commuting $n$-normal operators, $S^n, T^n$ are commuting normal operator. So $S^nT^n$ is a normal operator. Since $S^nT^n = (ST)^n, (ST)^n$ is normal. Hence $ST$ is $n$-normal. $\Box$

The following example shows that Theorem 3.2.4 is not necessarily true if $S, T$ are not commuting.

**Example 3.2.3.** Let $S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $T = \begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ be operators on the Hilbert space $\mathbb{C}^2$. Then $S$ and $T$ are 2-normal. We note that

$$ST = \begin{pmatrix} i & 2 \\ 0 & i \end{pmatrix} \neq \begin{pmatrix} i & -2 \\ 0 & i \end{pmatrix} = TS.$$  

But as $(ST)^2 = \begin{pmatrix} -1 & 4i \\ 0 & -1 \end{pmatrix}$ is not normal, $ST$ is not 2-normal.

By theorem 3.2.4, we get the following corollary.

**Corollary 3.2.5.** If $T \in \mathcal{B}(H)$ is $n$-normal, then $T^m$ is $n$-normal for any positive integer $m$.

The following example shows that sum of two commuting $n$-normal operators need not be an $n$-normal operator.

**Example 3.2.4.** Let $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $S$ and $T$ are commuting 2-normal. But
\[
S + T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad (S + T)^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}
\]
is not normal. Thus \(S+T\) is not 2-normal. We note here that \(S\) is a selfadjoint operator.

**Proposition 3.2.6.** Let \(S, T \in \mathcal{B}(H)\) be commuting \(n\)-normal operators such that \((S + T)^*\) commutes with \(\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k\). Then \((S + T)\) is an \(n\)-normal operator.

**Proof.** Since \((S + T)^n(S + T)^* = \left(\sum_{k=0}^{n} \binom{n}{k} S^{n-k} T^k\right)(S^* + T^*)\)

\[
= S^n S^* + \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k (S + T)^* + T^n S^* + S^n T^* + T^n T^*.
\]

And since \((S + T)^*\) is commuting with \(\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k\),

\[
(S + T)^n(S + T)^* = S^* S^n + (S + T)^* \sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k + S^* T^n + T^* S^n + T^* T^n.
\]

So

\[
(S + T)^n(S + T)^* = (S + T)^*(S^n + T^n) + (S + T)^*\left(\sum_{k=1}^{n-1} \binom{n}{k} S^{n-k} T^k\right).
\]

Hence

\[
(S + T)^n(S + T)^* = (S + T)^*\left(\sum_{k=0}^{n} \binom{n}{k} S^{n-k} T^k\right) = (S + T)^*(S + T)^n.
\]

\(\square\)
Lemma 3.2.7. If $S, T \in \mathcal{B}(H)$ are 2-normal operators and $ST + TS = 0$, then $T + S$ and $ST$ are 2-normal.

Proof. Since $ST + TS = 0$, $S^2T^2 = T^2S^2$. So $(S + T)^2 = S^2 + T^2$ is normal. Thus $(S + T)$ is a normal operator.

Now since $ST + TS = 0$, $(ST)^2 = -S^2T^2 = -T^2S^2$. Since $S^2, T^2$ are commuting normal, $(ST)^2 = -S^2T^2$ is normal.

Now we state some well-known lemmas which we shall need.

Lemma 3.2.8. Let $P, Q$ be the projections on closed subspaces $M, N$ respectively. Then $M \perp N$ if and only if $PQ = 0$.

Lemma 3.2.9. If $T$ is normal, then $Tx = \lambda x$ if and only if $T^*x = \overline{\lambda}x$.

Lemma 3.2.10. If $P$ is the projection on a closed subspace $M$ of $H$, then $M$ reduces of $T$ if and only if $TP = PT$.

Now we shall prove the following theorem for a finite dimensional Hilbert space $H$.

Theorem 3.2.11. Let $T$ be an operator on a finite dimensional Hilbert space $H$, $\lambda_1, ..., \lambda_m$ be eigenvalues of $T$ and $M_1, ..., M_m$ the corresponding eigenspaces. Suppose $\lambda_i^n \neq \lambda_j^n, i \neq j$. Then $M_i$'s are pairwise orthogonal and they span $H$ if and only if $T$ is an $n$-normal operator.

Proof. Assume $M_i$'s are pairwise orthogonal and they span $H$. Then for
\(x \in H, x = x_1 + x_2 + \ldots + x_m, x_i \in M_i,\)

\[T^n x = T^n x_1 + \ldots + T^n x_m,\]

\[= \lambda^n_1 x_1 + \ldots + \lambda^n_m x_m.\]

Since \(P_i\)'s are projection on eigenspace \(M_i\)'s which are pairwise orthogonal, by lemma 3.2.8, \(P_i x = x_i.\) Hence

\[Ix = x_1 + \ldots x_m = P_1 x + \ldots + P_m x = (P_1 + \ldots + P_m) x\] for every \(x \in H.\)

Thus \(I = \sum_{i=1}^{n} P_i.\) Since

\[T^n x = \lambda^n_1 x_1 + \ldots + \lambda^n_m x_m,\]

\[= \lambda^n_1 P_1 x + \ldots + \lambda^n_m P_m x,\]

\[= (\lambda^n_1 P_1 + \ldots + \lambda^n_m P_m) x\]

for all \(x \in H.\) So \(T^n = \sum_{i=1}^{m} \lambda^n_i P_i.\) Hence

\[T^*n = \bar{\lambda}_1^n P_1 + \ldots + \bar{\lambda}_m^n P_m.\]

Since \(M_i\)'s are pairwise orthogonal,

\[P_i P_j = \begin{cases} P_i, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}\]

So

\[T^n T^*n = |\lambda_1|^{2n} P_1 + \ldots + |\lambda_m|^{2n} P_m\]

and

\[T^*n T^n = |\lambda_1|^{2n} P_1 + \ldots + |\lambda_m|^{2n} P_m.\]
Thus $T^n$ is normal, i.e., $T$ is an $n$-normal operator.

Suppose $T$ is an $n$-normal operator. Then $T^n$ is a normal operator. We claim that $M_i$’s are pairwise orthogonal. Let $x_i, x_j$ be vectors in $M_i, M_j, (i \neq j)$ such that $T^n x_i = \lambda^n_i x_i$ and $T^n x_j = \lambda^n_j x_j$. Then

$$
\lambda^n_i \langle x_i, x_j \rangle = \langle \lambda^n_i x_i, x_j \rangle = \langle T^n x_i, x_j \rangle = \langle x_i, T^n x_j \rangle = \langle x_i, \lambda^n_j x_j \rangle = \lambda^n_j \langle x_i, x_j \rangle.
$$

So $(\lambda^n_i - \lambda^n_j) \langle x_i, x_j \rangle = 0$. Since $\lambda^n_i \neq \lambda^n_j$, $\langle x_i, x_j \rangle = 0$. This shows that $M_i$’s are pairwise orthogonal.

Let $M = M_1 + \ldots + M_m$. Then $M$ is a closed subspace of $H$. Let $P$ be associated projection onto $M$. Then $P = P_1 + \ldots + P_m$. Since $T^n$ is normal, each $M_i$ reduces $T^n$. It follows that $T^n P = PT^n$. Consequently $M^{\perp}$ is invariant under $T^n$. Suppose that $M^{\perp} \neq \{0\}$. Let $T_1 = T^n / M^{\perp}$. Then $T_1$ is an operator on non-trivial finite dimensional complex Hilbert space $M^{\perp}$ with empty point spectrum which is impossible. Therefore $M^{\perp} = \{0\}$. i.e., $M = H$.

**Theorem 3.2.12.** Let $T_1, \ldots, T_m$ be $n$-normal operators in $\mathbb{B}(H)$. Then $(T_1 \oplus \ldots \oplus T_m)$ and $(T_1 \otimes \ldots \otimes T_m)$ are $n$-normal operators.

**Proof.** Since $(T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^* = (T_1^* \oplus \ldots \oplus T_m^*) (T_1^* \oplus \ldots \oplus T_m^*)$

$$ = T_1^* T_1^* \oplus \ldots \oplus T_m^* T_m^* = T_1^* T_1^* \oplus \ldots \oplus T_m^* T_m^* = (T_1^* \oplus \ldots \oplus T_m^*) (T_1^* \oplus \ldots \oplus T_m^*)
$$

$$ = (T_1 \oplus \ldots \oplus T_m)^* (T_1 \oplus \ldots \oplus T_m)^n. \text{ Then } (T_1 \oplus \ldots \oplus T_m)^n \text{ is an } n\text{-normal operator.}
$$

Now let $x_1, \ldots, x_m \in H$. Then $(T_1 \otimes \ldots \otimes T_m)^* (x_1 \otimes \ldots \otimes x_m)$

$$ = (T_1^* \otimes \ldots \otimes T_m^*) (T_1^* \otimes \ldots \otimes T_m^*) (x_1 \otimes \ldots \otimes x_m)
$$

$$ = T_1^* T_1^* x_1 \otimes \ldots \otimes T_m^* T_m^* x_m
$$

27
\[ T_1^* T_1^n x_1 \otimes \ldots \otimes T_m^* T_m^n x_m = (T_1^* \otimes \ldots \otimes T_m^*) (T_1^n \otimes \ldots \otimes T_m^n) (x_1 \otimes \ldots \otimes x_m) \]
\[ = (T_1 \otimes \ldots \otimes T_m)^* (T_1 \otimes \ldots \otimes T_m)^n (x_1 \otimes \ldots \otimes x_m). \] So \((T_1 \otimes \ldots \otimes T_m)^n (T_1 \otimes \ldots \otimes T_m)^* = (T_1 \otimes \ldots \otimes T_m)^*(T_1 \otimes \ldots \otimes T_m)^n. \) Thus \((T_1 \otimes \ldots \otimes T_m)\) is an \(n\)-normal operator. \(\square\)

Now the following example shows that the class of 2-normal operators may not have the translation-invariant property.

**Example 3.2.5.** Let 
\[ T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]
We note here that 
\[ T - \lambda = \begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} \]
is not 2-normal. So 2-normality is not translation-invariant.

In fact we have the following theorem.

**Proposition 3.2.13.** \((T - \lambda)\) is an \(n\)-normal operator for every \(\lambda \in \mathbb{C}\) if and only if \(T\) is a normal operator.

**Proof.** Suppose \((T - \lambda)\) is an \(n\)-normal operator for every \(\lambda \in \mathbb{C}\). Then
\[ (T - \lambda)^*(T - \lambda)^n = (T - \lambda)^n (T - \lambda)^*. \]

Hence
\[ (T^* - \overline{\lambda})(\sum_{k=1}^{n} (-1)^k \binom{n}{k} T^{n-k} \lambda^k) = (\sum_{k=1}^{n} (-1)^k \binom{n}{k} T^{n-k} \lambda^k) (T^* - \overline{\lambda}). \]

So
\[ \sum_{k=1}^{n} (-1)^k \binom{n}{k} T^{n-k} \lambda^k - \sum_{k=1}^{n} (-1)^k \binom{n}{k} T^{n-k} \lambda^k \overline{\lambda} = \sum_{k=1}^{n} (-1)^k \binom{n}{k} T^{n-k} \lambda^k + \sum_{k=1}^{n} (-1)^k \binom{n}{k} T^{n-k} \lambda^k \overline{\lambda}. \]

Therefore
\[ \sum_{k=1}^{n} (-1)^k \binom{n}{k} (\lambda)^{k} (T^* T^{n-k} - T^{n-k} T^*) = 0. \]
From the left side of the last equation we get the term which $k = n$ is zero. Hence
\[ \sum_{k=1}^{n-1} (-1)^k \binom{n}{k} (\lambda)^k (T^*T^{n-k} - T^{n-k}T^*) = 0. \]
Thus
\[ (-1)^{n-1} n(\lambda)^{n-1} (T^*T - TT^*) + \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} (\lambda)^k (T^*T^{n-k} - T^{n-k}T^*) = 0. \]
Put \( \lambda = re^{i\theta}, \) \( 0 \leq \theta \leq 2\pi, \) \( r > 0, \) we get
\[ (-1)^{n-1} n(re^{i\theta})^{n-1} (T^*T - TT^*) + \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} (re^{i\theta})^k (T^*T^{n-k} - T^{n-k}T^*) \]
is vanish. So
\[ (-1)^{n-1} (T^*T - TT^*) + \frac{1}{n(re^{i\theta})^{n-1}} \left( \sum_{k=1}^{n-2} (-1)^k \binom{n}{k} (re^{i\theta})^k (T^*T^{n-k} - T^{n-k}T^*) \right) \]
also is vanish. Taking \( r \to \infty, \) we get \( T^*T - TT^* = 0. \) Hence \( T \) is normal.
The converse is trivial. \( \square \)

**Proposition 3.2.14.** Let \( T \in B(H) \) with the Cartesian decomposition
\( T = A + iB \) where \( A \) and \( B \) are selfadjoint operators. Then \( T \) is 2-normal operator if and only if \( B^2 \) commutes with \( A, \) and \( A^2 \) commutes with \( B. \)

**Proof.** Suppose that \( B^2A = AB^2 \) and \( A^2B = BA^2. \) Then
\[ T^2T^* = (A + iB)^2(A - iB), \]
\[ = (A^2 + iAB + iBA - B^2)(A - iB), \]
\[ = A^3 - iA^2B - B^2A + iB^3 + iABA + AB^2 + iBA^2 + BAB. \]
and
\[ T^*T^2 = A^3 - AB^2 + iA^2B + iABA - iBA^2 + iB^3 + BAB + B^2A. \]
Since $B^2 A = AB^2$ and $A^2 B = BA^2$, $T^2 T^* = T^* T^2$. Hence $T$ is 2-normal.

Now, let $T$ be 2-normal. So $T^2 T^* = T^* T^2$. Hence

$$-B^2 A + iBA^2 - iA^2 B + AB^2 = -AB^2 + iA^2 B - iBA^2 + B^2 A,$$

$$(AB^2 - B^2 A) + i(BA^2 - A^2 B) = 0.$$ 

Let $T_1 = AB^2 - B^2 A$, $T_2 = BA^2 - A^2 B$. Then $T_1^* = -T_1$, $T_2^* = -T_2$ (i.e., $T_1$, $T_2$ are skew hermitian) and $T_1 + iT_2 = 0$. So $-T_1 + iT_2 = 0$. This gives $T_1 = AB^2 - B^2 A = 0$. Similarly, $B^2 A = AB^2$.

It is clear that a 2-normal operator is a $2k$-normal operator and a 3-normal operator is a $3k$-normal operator. The following examples show that a 2-normal operator need not be 3-normal and vice versa.

**Example 3.2.6.** Let $T = \begin{pmatrix} 2 & 1 \\ 0 & -2 \end{pmatrix}$. Then $T^2 = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ is a normal operator. But $T^3 = \begin{pmatrix} 8 & 4 \\ 0 & -8 \end{pmatrix}$ is not normal. So $T$ is 2-normal but it is not 3-normal.

**Example 3.2.7.** Let $T = \begin{pmatrix} 2 & 2 \\ -2 & 0 \end{pmatrix}$. Then $T^3 = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$ is a normal operator. But $T^2 = \begin{pmatrix} 0 & 4 \\ -4 & -4 \end{pmatrix}$ is not normal. So $T$ is 3-normal but it is not 2-normal.

**Proposition 3.2.15.** Suppose that $T \in \mathbb{B}(H)$ is a $k$-normal operator and it is also a $(k+1)$-normal operator for some positive integer $k$. Then $T$ is $(k+2)$-normal. And hence $T$ is $n$-normal for all $n \geq k$. 

30
Proof. Since $T$ is $k$-normal, $T^kT^* = T^*T^k$. Hence $TT^kT^* = TT^*T^kT$. So $T^{k+1}T^*T = TT^*T^{k+1}$. Since $T$ is $(k + 1)$-normal, $T^*T^{k+2} = T^{k+2}T^*$. Thus $T$ is $(k + 2)$-normal. \qed

**Corollary 3.2.16.** If $T$ is 2-normal and 3-normal, then $T$ is an $n$-normal operator for all $n \geq 2$.

The following example shows an operator which is both 2-normal and a 3-normal, may not be normal.

**Example 3.2.8.** Let $T = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}$, $a \neq 0$ be an operator acting in two-dimensional complex Hilbert space. Then $T$ is 2-normal, and 3-normal. Hence it is $n$-normal for all $n \geq 2$ but it is not normal.

**Proposition 3.2.17.** Suppose $T \in \mathbb{B}(H)$ is a $k$-normal operator for a positive integer $k$ and it is a partial isometry. Then $T$ is a $(k+1)$-normal operator. And hence $T$ is $n$-normal for all $n \geq k$.

Proof. Since $T$ is a partial isometry, by [31, p.250], $TT^*T = T$. Hence $TT^*T^k = T^k$ and $T^kT^*T = T^k$. Since $T$ is $k$-normal, $T^{k+1}T^* = T^k$ and $TT^{k+1}T^* = T^{k+1}$. Therefore $T$ is $(k + 1)$-normal. And hence by Proposition 3.2.15, $T$ is $n$-normal for all $n \geq k$. \qed

**Corollary 3.2.18.** If $T \in \mathbb{B}(H)$ is 2-normal and a partial isometry, then $T$ is $n$-normal for all integer $n \geq 2$.

We note that, in Example 3.2.8 if $a$ equal to 1, then $T$ is a 2-normal operator and a partial isometry but not normal.
Lemma 3.2.19.

Let $T \in B(H)$ be $k$-normal and $(k + 1)$-normal. If either $T$ or $T^*$ is injective, then $T$ is normal.

Proof. Since $T$ is $(k + 1)$-normal, $T^{k+1}T^* = T^*T^{k+1}$. Since $T$ is $k$-normal, $T^{k+1}T^* = T^k T^*T$. Hence $T^k(TT^* - T^*T) = 0$. Since $T$ is injective, $TT^* - T^*T = 0$. Thus $T$ is normal. In case $T^*$ is injective, since $T^*$ is $k$-normal and $(k + 1)$-normal, $T^*$ is normal. Hence $T$ is normal. \hfill \qedsymbol

Proposition 3.2.20. Let $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $a, b, c, d \in \mathbb{C}$. Then $T$ is 2-normal if and only if $(a + d) = 0$ and $(|b| = |c| \text{ or } b(\overline{d} - \overline{a}) = \overline{c}(d - a))$.

Proof. Suppose $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is 2-normal. Then $T^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + bd & cb + d^2 \end{pmatrix}$ is normal. Hence $|ab + bc| = |ac + dc|$ and

$$(ab + bd)((cd + d^2) - (a^2 + bc)) = (ac + dc)((cb + d^2) - (a^2 + bc)).$$

Since $|b(a + d)| = |c(a + d)|$ and

$$b(a + d)(\overline{c}b + \overline{d}^2 - \overline{a}^2 - \overline{b}c) = \overline{c}(a + d)(cb + d^2 - a^2 - bc),$$

$$|b||a + d| = |c||a + d| \text{ and } b(a + d)(\overline{d}^2 - \overline{a}^2) = \overline{c}(\overline{a} - \overline{d})(d^2 - a^2).$$

Hence

$$|b||a + d| = |c||a + d| \text{ and } b(a + d)(\overline{d} - \overline{a})(\overline{d} + \overline{a}) = \overline{c}(\overline{a} - \overline{d})(d - a)(d + a).$$

So

$$|b||a + d| = |c||a + d| \text{ and } b(\overline{d} - \overline{a})|a + d|^2 = \overline{c}(d - a)|a + d|^2.$$
Thus

$$|b| = |c| \text{ or } |a + d| = 0 \text{ and } b(\overline{d} - \overline{a}) = \overline{c}(d - a) \text{ or } |a + d|^2 = 0.$$  

\[\square\]

By giving arguments similar to those in the Proposition 3.2.20, one can prove the following.

**Proposition 3.2.21.** Let \( T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \), where \( a, b, c, d \in \mathbb{C} \). Then \( T \) is 3-normal if and only if \((a^2 + bc + ad + d^2) = 0 \) and \(|b| = |c| \text{ or } \overline{c}(d - a) = b(\overline{d} - \overline{a})\).

Next, we characterize when a two-dimensional upper triangular complex matrix is \( n \)-normal.

**Proposition 3.2.22.** For \( n \geq 2 \) we have \( T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \) is \( n \)-normal if and only if \( b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) = 0 \).

**Proof.** Let \( T = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \). Then \( T \) is \( n \)-normal if and only if

\[T^n = \begin{pmatrix} a^n & b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) \\ 0 & c^n \end{pmatrix},\]

is normal if and only if

\[|b(a^{n-1} + a^{n-2}c + ... + c^{n-1})| = 0\]

if and only if

\[b(a^{n-1} + a^{n-2}c + ... + c^{n-1}) = 0.\]

\[\square\]
We note that by using the Proposition 3.2.22, we get a 3-normal operator which is not normal.

**Example 3.2.9.** Consider $n = 3$ in the Proposition 3.2.22. Then $T$ is a 3-normal operator if and only if $b(a^2 + ac + c^2) = 0$. Take $a = 2$, $b = 1$, and $c = -1 + \sqrt{3}i$. Then $T = \begin{pmatrix} 2 & 1 \\ 0 & -1 + \sqrt{3}i \end{pmatrix}$ is 3-normal. Note that

$$T^3 = \begin{pmatrix} 8 & 0 \\ 0 & 8 \end{pmatrix}$$

is normal. Thus $T$ is 3-normal.

**Proposition 3.2.23.** Let $T \in \mathbb{B}(H)$, $F = T^n + T^*$, and $G = T^n - T^*$. Then $T$ is an $n$-normal operator if and only if $G$ commutes with $F$.

**Proof.** $FG = GF$ if and only if

$$(T^n + T^*)(T^n - T^*) = (T^n - T^*)(T^n + T^*)$$

if and only if

$$T^{2n} - T^nT^* + T^*T^n - T^{*2} = T^{2n} + T^nT^* - T^*T^n - T^{*2}$$

if and only if $T^nT^* - T^*T^n = 0$ if and only if $T$ is an $n$-normal operator. \(\square\)

**Proposition 3.2.24.** Let $T \in \mathbb{B}(H)$, $B = T^nT^*$, $F = T^n + T^*$, and $G = T^n - T^*$. If $T$ is an $n$-normal, then $B$ commutes with $F$ and $G$.

**Proof.** Since $T$ is $n$-normal, $BF = T^nT^*(T^n + T^*) = T^nT^*T^n + T^nT^*T^*$

$= T^nT^nT^* + T^*T^nT^* = (T^n + T^*)T^nT^* = FB$. By similar way we can prove that $BG = GB$. \(\square\)
Proposition 3.2.25. Let $T$ be a weighted shift with nonzero weights $\{\alpha_k\}_{k=0}^\infty$. Then $T$ is $n$-normal if and only if $|\alpha_{k-n}| \ldots |\alpha_{k-1}| = |\alpha_k| \ldots |\alpha_{k+n-1}|,$ for $k = n, n+1, \ldots$.

Proof. Let $\{e_k\}_{k=0}^\infty$ be an orthogonal basis of Hilbert space $H$. Since $T^n e_k = \alpha_k \ldots \alpha_{k+n-1} e_{k+n}$ and $T^n e_k = \overline{\alpha_{k-1} \ldots \alpha_{k-n}} e_{k-n},$ $T^n T^n e_k = |\alpha_{k-1}|^2 \ldots |\alpha_{k-n}|^2 e_k$ and $T^n T^n e_k = |\alpha_k|^2 \ldots |\alpha_{k+n-1}|^2 e_k.$ Thus $T^n$ is normal if and only if $|\alpha_k|^2 \ldots |\alpha_{k+n-1}|^2 = |\alpha_{k-1}|^2 \ldots |\alpha_{k-n}|^2$ for $k = n, n+1, \ldots$. \hfill $\square$

Recall that for $T \in \mathbb{B}(H)$ and $x \in H$ The sequence $\{T^n x\}_{n=0}^\infty$ is called orbit of $x$ under $T$, and is denoted by $\text{orb}(T, x)$. And a vector $x$ is called a hypercyclic vector for $T$ if $\text{orb}(T, x)$ is dense in $H$.

C. Kitai in [60], showed that hyponormal operators are not hypercyclic.

Proposition 3.2.26. Let $T \in \mathbb{B}(H)$ be an $n$-normal operator and invertible. Then $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace.

Proof. Since $T$ is $n$-normal and invertible, by Proposition 3.2.1 and Proposition 3.2.3, $T^n$ and $(T^{-1})^n$ are normal respectively. Hence by [60, Corollary 4.5] $T^n$ and $(T^{-1})^n$ both have no hypercyclic vector. Thus by [16], $T$ and $T^{-1}$ both have no hypercyclic vector. Therefore by [56], $T$ and $T^{-1}$ have a common nontrivial closed invariant subspace. \hfill $\square$

Lemma 3.2.27. If $T \in \mathbb{B}(H)$ is an $n$-normal operator, then $T$ has SVEP.
Proof. Since $T$ is $n$-normal, by Proposition 3.2.1, $T^n$ is normal. Hence $T^n$ has SVEP. Thus by [68, Theorem 3.3.9], $T$ has SVEP. \hfill \Box

Definition 3.2.2. Let $T \in \mathcal{B}H$. If $\lambda \in \text{iso}\sigma(T)$, the Riesz idempotent $E_\lambda$ of $T$ with respect to $\lambda$ is defined by

$$E_\lambda = \frac{1}{2\pi i} \int_{\partial D} (z - T)^{-1} dz,$$

where $\lambda \in D$ is an open disk which is far from the rest of $\sigma(T)$.

It is well known that Riesz idempotent satisfies $E^2_\lambda = E_\lambda$, $E_\lambda T = TE_\lambda$, $\sigma(T/E_\lambda H) = \lambda$ and $\ker(\lambda - T) \subseteq E_\lambda H$.

Lemma 3.2.28. Let $T \in \mathcal{B}(H)$ be an $n$-normal operator and $\lambda \in \sigma(T)$ be an isolated point of $\sigma(T)$. Then $H_T(\{\lambda\}) = E_\lambda H$, where $E_\lambda$ denotes the Riesz idempotent for $\lambda$.

Proof. Since $T$ is $n$-normal, by Lemma 3.2.27, $T$ has SVEP. Hence by [67, Corollary 2.4], $H_T(\{\lambda\}) = \{x \in H : \|(T - \lambda)^k x\|^{\frac{1}{k}} \to 0\}$. Also by [85, p.424], $E_\lambda H = \{x \in H : \|(T - \lambda)^k x\|^{\frac{1}{k}} \to 0\}$. Thus $H_T(\{\lambda\}) = E_\lambda H$ for Riesz idempotent $E_\lambda$. \hfill \Box

Theorem 3.2.29. Let $D$ be an arbitrary bounded disk in $\mathbb{C}$. If $T \in \mathcal{B}(H)$ is 2-normal with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$, then the operator

$$\lambda - T : W^2(D, H) \to L^2(D, H)$$

is one to one, for each $\lambda \in D$.

Proof. Let $f \in W^2(D, H)$ such that $(\lambda - T)f = 0$ i.e.,

$$\|(\lambda - T)f\|_{W^2} = 0. \quad (3.2.1)$$
Then, for $i = 1, 2$, we have

$$\| (\lambda - T) \overline{\partial} f \|_{2,D} = 0. \quad (3.2.2)$$

Hence for $i = 1, 2$, we get $\| (\lambda^2 - T^2) \overline{\partial} f \|_{2,D} = 0$. Since $T^2$ is normal, for $i = 1, 2$

$$\| (\overline{\lambda}^2 - T^*^2) \overline{\partial} f \|_{2,D} = 0. \quad (3.2.3)$$

Since $\lambda - T$ is invertible for $\lambda \in D \setminus \sigma(T)$, the equation 3.2.2, implies that $\| \overline{\partial} f \|_{2,D \setminus \sigma(T)} = 0$. Therefore

$$\| (\overline{\lambda} - T^*) \overline{\partial} f \|_{2,D \setminus \sigma(T)} = 0. \quad (3.2.4)$$

Since $\sigma(T) \cap (-\sigma(T)) = \emptyset$, $\overline{\lambda} + T^*$ is invertible for $\lambda \in \sigma(T)$. Therefore, from equation 3.2.3, we have

$$\| (\overline{\lambda} - T^*) \overline{\partial} f \|_{2,\sigma(T)} = 0. \quad (3.2.5)$$

Hence from 3.2.4 and 3.2.5, we get

$$\| (\overline{\lambda} - T^*) \overline{\partial} f \|_{2,D} = 0. \quad (3.2.6)$$

By Proposition 2.1.2, we obtain

$$\| (I - P)f \|_{2,D} = 0, \quad (3.2.7)$$

where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Hence $(\lambda - T)Pf = (\lambda - T)f = 0$. Since by Lemma 3.2.27, $T$ has SVEP, $f = Pf = 0$. Hence $\lambda - T$ is one to one. \qed
Lemma 3.2.30. Suppose that $T, N \in \mathcal{B}(H)$ such that $T$ is 2-normal with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$, $N^m = 0$, and $T, N$ are commute. If $S = T + N$, then $\lambda - S$ is one-to-one on $W^2(D, H)$, where $D$ be an arbitrary bounded disk in $\mathbb{C}$.

Proof. Assume that $f \in W^2(D, H)$ such that $(\lambda - S)f = 0$. Then

$$(\lambda - T)f = Nf. \quad (3.2.8)$$

Hence $(\lambda - T)N^{j-1}f = N^j f$ for $j = 1, 2, \ldots, m$. Now we have to prove that $N^j f = 0$ for $j = 1, 2, \ldots, m - 1$. Since $N^m = 0$, $(\lambda - T)N^{m-1}f = N^m f = 0$. Since by Theorem 3.2.29, $\lambda - T$ is one-to-one, $N^{m-1}f = 0$. By continuously, we get $f = 0$. Hence $\lambda - S$ is one-to-one. \qed

Lemma 3.2.31. Let $T \in \mathcal{B}(H)$ be a 2-normal operator with the property that $\sigma(T) \cap (-\sigma(T)) = \emptyset$. If $V$ is an isometry, then the operator

$$\lambda - VTV^* : W^2(D, H) \longrightarrow L^2(D, H)$$

is one to one.

Proof. Let $f \in W^2(D, H)$ such that $(\lambda - VTV^*)f = 0$. Then $(\lambda - T)V^*f = 0$. Hence for $i = 0, 1, 2$ $(\lambda - T)V^*\hat{\partial}^i f = 0$. By Theorem 3.2.29, for $i = 0, 1, 2$, $V^*\hat{\partial}^i f = 0$. Hence for $i = 0, 1, 2$, $VTV^*\hat{\partial}^i f = 0$. Thus $\lambda \hat{\partial}^i f = 0$ for $i = 0, 1, 2$. By Proposition 2.1.2 with $T = 0$, we get $\| (I - P)f \|_{2, D} = 0$, where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Hence $\lambda f = \lambda Pf = 0$. By [30, Corollary 10.7], there exists a constant $c > 0$ such that $c\| Pf \|_{2, D} \leq \| \lambda Pf \|_{2, D} = 0$. So $f = Pf = 0$. Thus $\lambda - VTV^*$ is one to one. \qed

Recall that an operator $T \in \mathcal{B}(H)$ is called scalar of order $m$ if there is a continuous unital homomorphism $\phi : C_0^m(\mathbb{C}) \longrightarrow \mathcal{B}(H)$ such that $\phi(z) = T$.
where \( z \) stands for the identity function on \( \mathbb{C} \), where \( C_0^m(\mathbb{C}) \) is the algebra of compactly supported functions on \( \mathbb{C} \), continuously differentiable of order \( m \), \( 0 \leq m \leq \infty \). An operator \( T \in \mathcal{B}(H) \) is called subscalar if it is similar to the restriction of a scalar operator to an invariant subspace.

**Proposition 3.2.32.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. If \( T \) is quasinilpotent, then \( T \) is nilpotent, and hence \( T \) is subscalar.

*Proof.* Since \( T \) is quasinilpotent, \( \sigma(T) = \{0\} \). Hence by the spectral mapping theorem we get \( \sigma(T^n) = \sigma(T)^n = \{0\} \). Thus \( T^n \) is quasinilpotent and normal. So \( T^n = 0 \) i.e., \( T \) is nilpotent and \( T \) is algebraic operator and hence by [61], \( T \) is subscalar. \( \square \)

**Proposition 3.2.33.** Let \( T \in \mathcal{B}(H) \) be a 2-normal Operator with the property that \( \sigma(T) \cap (-\sigma(T)) = \emptyset \). Then \( T \) is subscalar of order 2.

*Proof.* Consider an arbitrary bounded open disk \( D \subset \mathbb{C} \) which contains \( \sigma(T) \) and the quotient space \( \tilde{H}(D) = W^2(D, H)/(\lambda - T)W^2(D, H) \) endowed with the Hilbert space norm. The classes of a vector \( f \in W^2(D, H) \), an operator \( A : W^2(D, H) \rightarrow W^2(D, H) \) on \( \tilde{H}(D) \) will be denoted respectively by \( \tilde{f} \), \( \tilde{A} \). Let \( M \) be the operator of multiplication by \( \lambda \) on \( W^2(D, H) \). Then \( M \) is a scalar operator of order 2 and has a spectral distribution \( \phi \). Let \( S = \tilde{M} \). Since \( (\lambda - T)W^2(D, H) \) is invariant under every operator \( M_f \), \( f \in C_0^2(\mathbb{C}) \), we infer that \( S \) is a scalar operator of order 2 with spectral distribution \( \tilde{\phi} \).

Consider the natural map \( V : H \rightarrow \tilde{H}(D) \) denoted by \( Vh = 1 \otimes h \), for \( h \in H \), where \( 1 \otimes h \) denotes the constant function sending \( \lambda \in D \) to \( h \). Then \( R(V) \) is closed and \( VTh = 1 \otimes Th = \lambda \otimes h = S(1 \otimes h) = SVh \). In particular
$R(V)$ is an invariant subspace for $S$.

Indeed, if $h \in H$, then

$$(T - \lambda)(1 \otimes h) = 1 \otimes (\lambda - T)h = (\lambda - T)h \in (T - \lambda)W^2(D, H).$$

So $\lambda(1 \otimes h) - T(1 \otimes h) \in (T - \lambda)W^2(D, H)$. Thus $(\lambda \otimes h) - (1 \otimes Th) \in (T - \lambda)W^2(D, H)$. Hence $\lambda \otimes Th = \lambda \otimes h$.

Now we shall prove that $V$ is one to one.

Let $\{h_n\}, \{f_n\}$ be sequences respectively in $H, W^2(D, H)$ such that

$$\lim_{n \to \infty} \| (\lambda - T) f_n + 1 \otimes h \|_{W^2} = 0.$$  \hspace{1cm} (3.2.9)

It suffices to show that $\lim h = 0$.

By the definition of the norm of $W^2(D, H)$ 3.2.9 implies that

$$\lim_{n \to \infty} \| (\lambda - T) \partial_i f_n \|_{2,D} = 0 \text{ for } i = 1, 2.$$ \hspace{1cm} (3.2.10)

So $\lim_{n \to \infty} \| (\lambda - T) \partial_i f_n \|_{2,D} = 0 \text{ for } i = 1, 2$. Since $T^2$ is normal, for $i = 1, 2$

$$\lim_{n \to \infty} \| (\lambda^2 - T^* T) \partial_i f_n \|_{2,D} = 0.$$ \hspace{1cm} (3.2.11)

Since $\lambda - T$ invertible for $\lambda \in D \setminus \sigma(T)$, 3.2.10 implies that

$$\lim_{n \to \infty} \| \partial f_n \|_{2,D \setminus \sigma(T)} = 0.$$

Therefore

$$\lim_{n \to \infty} \| (\lambda - T^*) \partial f_n \|_{2,D \setminus \sigma(T)} = 0.$$ \hspace{1cm} (3.2.12)

Since $\sigma(T) \cap (-\sigma(T)) = \emptyset$ and $\sigma(T^*) = \sigma(T)^*$, $\lambda + T^*$ is invertible for $\lambda \in \sigma(T)$. Therefore from 3.2.11, we have

$$\lim_{n \to \infty} \| (\lambda - T^*) \partial f_n \|_{2, \sigma(T)} = 0.$$ \hspace{1cm} (3.2.13)
Hence by 3.2.12 and 3.2.13 we get
\[
\lim_{n \to \infty} \|(X - T^*)\partial_i f_n\|_{2,D} = 0. \tag{3.2.14}
\]
By Proposition 2.1.2, we obtain
\[
\lim_{n \to \infty} \|(I - P)f_i\|_{2,D} = 0, \tag{3.2.15}
\]
where $P$ denotes the orthogonal projection of $L^2(D, H)$ onto the Bergman space $A^2(D, H)$. Substituting 3.2.15 into 3.2.9, we get
\[
\lim_{n \to \infty} \|(\lambda - T)Pf_n + 1 \otimes h\|_{2,D} = 0.
\]
Let $\Gamma$ be a curve in $D$ Surrounding $\sigma(T)$. Then for $\lambda \in \Gamma$
\[
\lim_{n \to \infty} \|Pf_n(\lambda) + (\lambda - T)^{-1}(1 \otimes h)\| = 0
\]
uniformly. Hence by Riesz-Dunford functional calculus,
\[
\lim_{n \to \infty} \|\frac{1}{2\pi i} \int_{\Gamma} Pf_n(\lambda)d\lambda + h\| = 0.
\]
Since $\frac{1}{2\pi i} \int_{\Gamma} Pf_n(\lambda)d\lambda = 0$, by Cauchy’s theorem,
\[
h = 0. \text{ Thus } V \text{ is one to one. Hence } VTV^{-1} = S. \text{ Thus } T \text{ is subscalar.} \]

### 3.3 Continuity of the spectrum

The following well known lemma [55, 72] is required in the proof of Lemma 3.3.2.

**Lemma 3.3.1.** Let $S, T \in B(H)$ be commuting operators. Then the following hold.

1. $ST$ is Fredholm if and only if $S$ and $T$ are both Fredholm.
2. If $S$ and $T$ are both Fredholm, then $i(ST) = i(S) + i(T)$.  

41
Lemma 3.3.2. Suppose $S, T \in \mathcal{B}(H)$ are commuting $n$-normal operators. Then $ST$ is Weyl if and only if $S$ and $T$ are both Weyl.

Proof. Assume $ST$ is Weyl. Then $ST$ is Fredholm of index zero. Hence $(ST)^n$ is Fredholm and $i((ST)^n) = 0$. Since $(ST)^n = S^nT^n$, by Lemma 3.3.1 $S^n$ and $T^n$ are both Fredholm and normal. So $i(T^n) = i(S^n) = 0$. Hence by Lemma 3.3.1 $S$ and $T$ are Fredholm and $0 = i(T^n) = i(T) + \ldots + i(T)$. Thus $i(T) = 0$. Similarly, $i(S) = 0$. So $S$ and $T$ are Weyl.

Conversely, if $S$ and $T$ are both Weyl, then $S$ and $T$ are both Fredholm of index zero. Hence by Lemma 3.3.1, $ST$ is Fredholm and

$$i(ST) = i(S) + i(T) = 0.$$ 

Thus $ST$ is Weyl. \qed

The following example shows that the Lemma 3.3.2, need not be true with the condition of commuting operators only.

Example 3.3.1. Let $T = \begin{pmatrix} U & 0 \\ 0 & I \end{pmatrix}$ and $S = \begin{pmatrix} I & 0 \\ 0 & U^* \end{pmatrix}$, where $U$ is the unilateral shift on $\ell^2$. Evidently,

$$i(ST) = i(\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}) = i(U) + i(U^*) = 0,$$

but $S$ and $T$ are not Weyl.
For a normal operator \( T \in \mathcal{B}(H) \), if \( \sigma(T) = \{ \lambda \} \), then \( T = \lambda I \). Following is an analogous result for an \( n \)-normal operator.

**Lemma 3.3.3.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator such that \( \sigma(T) = \{ \lambda \} \). Then \( T - \lambda I \) is nilpotent.

**Proof.** Since \( T \) is an \( n \)-normal operator and \( \sigma(T) = \{ \lambda \} \), \( T^n \) is normal operator and \( \sigma(T^n) = \{ \lambda^n \} \). So \( T^n = \lambda^n \). Let

\[
T^n - \lambda^n = c(T - \lambda)^m(T - \alpha_1)...(T - \alpha_k), \text{ (where } m, k > 1)\]

Since \( T - \alpha_i \) is invertible for every \( \alpha_i \neq \lambda \), \((i = 1, ..., k)\). Hence \((T - \lambda)^m = 0\). Thus \( T - \lambda \) is nilpotent. \( \square \)

Now consider Weyl’s Theorem for \( n \)-normal operators through the local spectral theory.

**Proposition 3.3.4.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then Weyl’s theorem holds for \( T \).

**Proof.** Let \( \lambda \in \sigma(T)\setminus\sigma_\omega(T) \). Then \( T - \lambda \) is Weyl and not invertible. Let \( \lambda \) is interior point of \( \sigma(T) \). Then there exists an open set \( U \) containing \( \lambda \) such that \( \dim N(T - \mu) > 0 \) for all \( \mu \in U \). Hence by [43, Theorems 8,9], \( T \) does not have SVEP. This is a contradiction with Lemma 3.2.27. Hence \( \lambda \) is a boundary point of \( \sigma(T) \) and hence by [31, Theorem XI 6.8], it is an isolated eigenvalue of finite multiplicity point of \( \sigma(T) \). Thus \( \sigma(T)\setminus\sigma_\omega(T) \subset \Pi_{00}(T) \).

Next, suppose \( \lambda \in \Pi_{00}(T) \), with the associated Riesz idempotent \( E_\lambda \). Then

\[
0 < \dim (T - \lambda) < \infty; \ T = T/E_\lambda H \oplus T/(I - E_\lambda)H
\]

and
\[\sigma(T/E_{\lambda}H) = \{\lambda\}, \sigma(T/(I - E_{\lambda})H) = \sigma(T)\setminus\{\lambda\}.\]

Since \(T^n = T^n/E_{\lambda}H \oplus T^n/(I - E_{\lambda})H\), by Proposition 3.2.3, an operator \(T^n/E_{\lambda}H = (T/E_{\lambda}H)^n\) is normal. Hence \(T/E_{\lambda}H\) is \(n\)-normal. Thus by Lemma 3.3.3, there exists a positive integer \(m\) such that \((T/E_{\lambda}H - \lambda)^m = 0\). Since \(T/(I - E_{\lambda})H - \lambda I\) invertible, \(T - \lambda\) is Weyl. Hence \(\lambda \in \sigma(T)\setminus\sigma_\omega(T)\). So \(\sigma(T)\setminus\sigma_\omega(T) = \Pi_{00}(T)\) i.e., \(\sigma_\omega(T) = \sigma(T)\setminus\Pi_{00}(T)\). Thus Weyl’s theorem holds for an \(n\)-normal operator. \(\square\)

**Proposition 3.3.5.** Let \(T \in \mathcal{B}(H)\) be an \(n\)-normal operator with the property that \(\sigma(T) \cap [-\sigma(T)] = \emptyset\) or \(\{0\}\). If \(\lambda \in \Pi_{00}(T)\), then \(\lambda^n \in \Pi_{00}(T^n)\).

**Proof.** Since \(T \in \mathcal{B}(H)\) is an \(n\)-normal operator with the property that
\[\sigma(T) \cap [-\sigma(T)] = \emptyset\) or \(\{0\}\),
\[\sigma(T^n) - \Pi_{00}(T^n) = \sigma_\omega(T^n) = \sigma_\omega(T)^n = (\sigma(T) - \Pi_{00}(T))^n.\]

Let \(\lambda \in \Pi_{00}(T)\). Suppose \(\lambda^n \notin \Pi_{00}(T^n)\). Then \(\lambda^n \in \sigma(T^n) - \Pi_{00}(T^n)\). So \(\lambda^n \in (\sigma(T) - \Pi_{00}(T))^n\). If \(\lambda = 0\), then it is clear that \(0 \in \sigma(T) - \Pi_{00}(T)\)
which is a contradiction. If \(\lambda \neq 0\), then either \(\lambda \in \sigma(T) - \Pi_{00}(T)\) or \(-\lambda \in \sigma(T) - \Pi_{00}(T)\). Since \(\lambda \in \Pi_{00}(T)\) and \(\sigma(T) \cap [-\sigma(T)] = \emptyset\), \(-\lambda \notin \sigma(T)\).

Hence \(\lambda \in \sigma(T) - \Pi_{00}(T)\) which is a contradiction. Thus \(\lambda^n \in \Pi_{00}(T^n)\). \(\square\)

**Lemma 3.3.6.** Let \(T \in \mathcal{B}(H)\) be an operator with that \(\sigma(T) \cap [-\sigma(T)] = \emptyset\)
or \(\{0\}\) and \(\lambda \in \sigma_\omega(T)\). If \(\lambda \notin \sigma(T) - \Pi_{00}(T)\), then \(\lambda^n \notin (\sigma(T) - \Pi_{00}(T))^n\).

**Proof.** If \(\lambda = 0\) it is clear. Let \(\lambda \neq 0\). Suppose \(\lambda \notin (\sigma(T) - \Pi_{00}(T))\) but \(\lambda^n \in (\sigma(T) - \Pi_{00}(T))^n\). If \(n\) is odd, then \(\lambda \in (\sigma(T) - \Pi_{00}(T))\) which is a contradiction.

If \(n\) is even, then either \(\lambda \in \sigma(T) - \Pi_{00}(T)\) or \(-\lambda \in \sigma(T) - \Pi_{00}(T)\). If
\[-\lambda \in (\sigma(T) - \Pi_{00}(T)), \text{ then } -\lambda \in \sigma_\omega(T). \text{ Hence } \lambda \in \sigma(T) \cap [-\sigma(T)] = \emptyset \text{ which is a contradiction. Thus } \lambda^n \notin (\sigma(T) - \Pi_{00}(T))^n. \]

**Lemma 3.3.7.** If \( T \in \mathbb{B}(H) \) is an \( n \)-normal operator and \( N \in \mathbb{B}(H) \) is a nilpotent operator commuting with \( T \), then Weyl’s theorem holds for \( T + N \).

*Proof.* Since \( T \) is \( n \)-normal, by Proposition 3.3.4, Weyl’s theorem holds for \( T \). Hence by [78], Weyl’s theorem holds for \( T + N \).

**Lemma 3.3.8.** Let \( T \in \mathbb{B}(H) \) be an \( n \)-normal operator and \( K \in \mathbb{K}(H) \) commutes with \( T \). If \( \Pi_{00}(T + K) = \Pi_{00}(T) \), then Weyl’s theorem holds for \( T + K \).

*Proof.* Since \( T \) is an \( n \)-normal operator, by Proposition 3.3.4, Weyl’s theorem holds for \( T \). Hence by [26], Weyl’s theorem holds for \( T + K \).

Recall that the reduced minimum modulus of \( T \in \mathbb{B}(H) \) defined as follows:

\[
\gamma_T(\lambda) = \inf \left\{ \frac{\| (T - \lambda)x \|}{\text{dist}(x, N(T - \lambda))} : x \in H \setminus \ker(T - \lambda) \right\}.
\]

**Lemma 3.3.9.** If \( T \in \mathbb{B}(H) \) is an \( n \)-normal operator, then \( \gamma_T(\lambda) \) is discontinuous on \( \Pi_{00}(T) \).

*Proof.* Since \( T \) is \( n \)-normal operator, by Proposition 3.3.4, \( T \) satisfies Weyl’s theorem. Hence by [51], \( \gamma_T(\lambda) \) is discontinuous for each \( \lambda \in \Pi_{00}(T) \).

Next we show that the Weyl’s spectral mapping theorem holds for \( n \)-normal operators. Furthermore, if \( T \) is an \( n \)-normal operator, then Weyl’s theorem holds for \( f(T) \), where \( f \) is analytic in a neighborhood of \( \sigma(T) \).
Proposition 3.3.10. Let $T \in \mathcal{B}(H)$ be an $n$-normal operator. Then

$$\sigma_\omega(f(T)) = f(\sigma_\omega(T)) \text{ for every } f \in H(\sigma(T)).$$

Proof. By [49, 73], $\sigma_\omega(f(T)) \subseteq f(\sigma_\omega(T))$ for every $f \in H(\sigma(T))$. Next suppose $\lambda \notin \sigma_\omega(f(T))$. Then $f(T) - \lambda$ is Weyl. We may write

$$f(T) - \lambda = cg(T) \prod_{j=1}^{m} (T - \alpha_j)$$

where $c, \alpha_1, ..., \alpha_m \in \mathbb{C}$ and $g(T)$ is invertible. Since the factors of the right-hand side of the last equation commute, every $(T - \alpha_j)$ is Fredholm for all $j = 1, ..., m$. Since $T$ is $n$-normal, by Lemma 3.2.27, $T$ has SVEP. Hence by [3], $i(T - \alpha_j) \leq 0$. Since

$$0 = i(f(T) - \lambda),$$
$$= i(g(T)) + \sum_{j=1}^{m} i(T - \alpha_j),$$
$$= \sum_{j=1}^{m} i(T - \alpha_j)$$
$$\leq 0,$$

$i(T - \alpha_j) = 0$ for all $j = 1, ..., m$. Thus $T - \alpha_j$ is Weyl for all $j = 1, ..., m$. Therefore $\lambda \notin f(\sigma_\omega(T))$. Hence $\sigma_\omega(f(T)) = f(\sigma_\omega(T))$. \qed

Recall that an operator $T \in \mathcal{B}(H)$ is isloid if $iso(\sigma(T)) \subset \Pi_{00}(T)$.

Lemma 3.3.11. Let $T \in \mathcal{B}(H)$ be an $n$-normal operator $n > 1$. Then $T$ is isoloid.

Proof. Assume $\lambda \in \sigma(T)$ is an isolated point of $\sigma(T)$ with associated Riesz idempotent $E_\lambda$. Then
\[ T = T/E_\lambda H \oplus T/(I - E_\lambda)H \]

and

\[ \sigma(T/E_\lambda H) = \{\lambda\}, \sigma(T/(I - E_\lambda)H) = \sigma(T) \setminus \{\lambda\}. \]

Since \( T^n = T^n/E_\lambda H \oplus T^n/(I - E_\lambda)H \), by Proposition 3.2.3, an operator \( T^n/E_\lambda H = (T/E_\lambda H)^n \) is normal. Hence \( T/E_\lambda H \) is \( n \)-normal. Thus by Lemma 3.3.3, there exists a positive integer \( m > 1 \) such that \( (T/E_\lambda H - \lambda)^m = 0 \) but \( (T/E_\lambda H - \lambda)^{m-1} \neq 0 \). Let \( 0 \neq x_0 \in E_\lambda H \) such that \( y_0 = (T/E_\lambda H - \lambda)^{m-1}x_0 \neq 0 \). Hence \( (T/E_\lambda H - \lambda)y_0 = 0 \). Thus \( \lambda \) is an eigenvalue of \( T/E_\lambda H \), and hence eigenvalue of \( T \). Hence \( T \) is isoloid. \( \square \)

**Theorem 3.3.12.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then Weyl’s theorem holds for \( f(T) \) for every \( f \in H(\sigma(T)) \).

Proof. Since \( T \) is \( n \)-normal, by Lemma 3.3.11, \( T \) is isoloid. Hence by [55],

\[ f(\sigma(T) \setminus \Pi_{00}(T)) = \sigma(f(T)) \setminus \Pi_{00}(f(T)) \text{ for every } f \in H(\sigma(T)). \]

So \( \sigma(f(T)) \setminus \Pi_{00}(f(T)) = f(\sigma(T) \setminus \Pi_{00}(T)) = f(\sigma_\omega(T)) = \sigma_\omega(f(T)) \). Hence Weyl’s theorem holds for \( f(T) \). \( \square \)

**Theorem 3.3.13.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then

\[ \sigma_{\omega}(T) = \sigma_a(T) \setminus \Pi_{a0}(T). \]

Proof. Since \( T \) is an \( n \)-normal operator, by Proposition 3.3.4, and Lemma 3.2.27, \( \sigma(T) \setminus \sigma_\omega(T) = \Pi_{00}(T) = \Pi_0(T) \) and \( T \) has SVEP. Hence by [25, Corollary 2.45], \( \sigma(T^*) = \sigma_a(T^*) \). Since \( T \) has SVEP, by [14, Theorem 3.6],

\[ \Pi_{00}(T) = \overline{\Pi_{00}(T^*)} = \overline{\Pi_{a0}(T^*)}, \sigma(T) = \overline{\sigma(T^*)} = \overline{\sigma_a(T^*)}. \]

47
Now, we shall prove that \( \sigma_{aw}(T^*) = \sigma_{\omega}(T^*) \). Evidently \( \sigma_{aw}(T^*) \subseteq \sigma_{\omega}(T^*) \).

Next, let \( \lambda \notin \sigma_{aw}(T^*) \). Then \( T^* - \lambda \in SF_+ \), i.e., \( T^* - \lambda \) is upper semi-Fredholm and \( i(T^* - \lambda) \leq 0 \). Since \( T \) has SVEP, by [1, Corollary 3.19], \( i(T^* - \lambda) \geq 0 \). Thus \( i(T^* - \lambda) = 0 \) and \( T^* - \lambda \) is Fredholm. Hence \( \lambda \notin \sigma_{\omega}(T^*) \).

So \( \sigma_{\omega}(T^*) \subseteq \sigma_{aw}(T^*) \). Therefore \( \sigma_{\omega}(T^*) = \sigma_{aw}(T^*) \). Thus \( \sigma_{a}(T^*) \setminus \sigma_{aw}(T^*) = \Pi_{a0}(T^*) \).

\[ \text{Proposition 3.3.14.} \text{ Let } T \in B(H) \text{ be an } n \text{-normal operator. Then} \]
\[ \sigma_{aw}(f(T)) = f(\sigma_{aw}(T)) \text{ for every } f \in H(\sigma(T)). \]

\[ \text{Proof.} \text{ It is well known [55], that } \sigma_{aw}(f(T)) \subseteq f(\sigma_{aw}(T)). \]

Next suppose that \( \lambda \notin \sigma_{aw}(f(T)) \). Then
\[ f(T) - \lambda \in SF_+(H) \text{ and } f(T) - \lambda = cg(T) \prod_{j=1}^{m}(T - \alpha_j), \]
where \( c, \alpha_1, ..., \alpha_m \in \mathbb{C} \) and \( g(T) \) invertible. Since \( T \) is \( n \)-normal, by Lemma 3.2.27, \( T \) has SVEP. Hence by [3], \( i(T - \alpha_j) \leq 0 \) for each \( j = 1, ..., m \). Hence \( i(f(T) - \lambda) \leq 0 \). Therefore \( \lambda \notin f(\sigma_{aw}(T)) \), and hence \( \sigma_{aw}(f(T)) = f(\sigma_{aw}(T)). \)

\[ \text{The following lemma is analogous to [36, Theorem 2.1], here we replace } \]
\[ \text{“T satisfies Weyl’s theorem” by “T satisfies Browder’s theorem”}. \]

\[ \text{Theorem 3.3.15.} \text{ Let } T \in B(H) \text{ such that the Browder’s theorem holds at } T. \text{ Then } \sigma_b \text{ is continuous at } T \text{ if and only if } \sigma \text{ is continuous at } T. \]

\[ \text{Proof.} \text{ Suppose } \sigma_b \text{ is continuous at } T. \text{ Let } \{T_n\} \text{ be a sequence in } B(H) \text{ such that } T_n \longrightarrow T. \text{ Since } \sigma \text{ is upper semi-continuous at } T \in B(H), \text{ we have to prove } \sigma \text{ is lower semi-continuous at } T, \text{ i.e., } \sigma(T) \subseteq \lim_{n \to \infty} \inf \sigma(T_n). \]

48
Let $\lambda \in \sigma(T)$. Then $\lambda \in \sigma(T) \setminus \sigma_b(T)$ or $\lambda \in \sigma_b(T)$. If $\lambda \in \sigma(T) \setminus \sigma_b(T)$, then by Browder’s theorem, $\lambda \in \Pi_0(T)$. Hence $\lambda$ is an isolated point of $\sigma(T)$. Thus by [59, Theorem IV.3.16], $\lambda \in \lim_{n \to \infty} \inf \sigma(T_n)$. If $\lambda \in \sigma_b(T) \subset \sigma(T)$, then

$$
\lambda \in \sigma_b(T) \subset \lim_{n \to \infty} \inf \sigma_b(T_n) \subset \lim_{n \to \infty} \inf \sigma(T_n).
$$

Therefore $\sigma$ is continuous at $T$.

Now, let $\sigma$ be continuous at $T$. Since

$$
\pi_0(T) \cap \sigma_e(T) \subset \pi_0(T) \cap \sigma_b(T) = \pi_0(T) \cap (\sigma(T) \setminus \Pi_0(T)) \subset \Gamma_0, \text{ by } [14, \text{ Theorem 14.17}], \sigma_b \text{ is continuous at } T. \quad \square
$$

**Lemma 3.3.16.** Let $T \in \mathcal{B}(H)$ be such that Browder’s theorem holds at $T$. If $\sigma_a$ is continuous at $T$, then $\sigma_b$ is continuous at $T$.

**Proof.** If $\sigma_a$ is continuous at $T$, then by [33], $\sigma$ is continuous at $T$. Since by Theorem 3.3.15, Browder’s theorem holds at $T$, $\sigma_b$ is continuous. \quad \square

**Lemma 3.3.17.** If $T \in \mathcal{B}(H)$ is an $n$-normal operator, then $T$ satisfies Browder’s theorem.

**Proof.** Since $T$ is $n$-normal, by Proposition 3.3.4, $T$ satisfies Weyl’s theorem. Hence $T$ satisfies Browder’s theorem. \quad \square

**Proposition 3.3.18.** Let $T \in \mathcal{B}(H)$ be an $n$-normal operator. Then $\sigma_a$ is continuous at $T$.

**Proof.** Since $\sigma_a$ is upper semi-continuous, its remain to prove $\sigma_a$ is lower semi-continuous at $T$. Let $\lambda \in \sigma_a(T)$. Then either $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$ or $\lambda \in \sigma_{aw}(T)$. Since $\sigma_{aw}(T)$ is upper semi-continuous [34], its enough to prove $\sigma_{aw}(T)$ is lower

49
semi-continuous. Assume \( \sigma_{a\omega} \) is not lower semi-continuous. Then there exist a sequence \( \{T_k\} \) in \( \mathcal{B}(H) \), and \( \epsilon > 0 \), \( \lambda_0 \in \sigma_{a\omega}(T) \) and \( \epsilon \)-neighborhood \( (\lambda_0)_\epsilon \) of \( \lambda_0 \) such that \( \sigma_{a\omega}(T_k) \cap (\lambda_0)_\epsilon = \emptyset \) for infinity many values of \( k \).

Without lost generalization we may assume that \( \sigma_{a\omega}(T_k) \cap (\lambda_0)_\epsilon = \emptyset \) for all \( k \). So \( T_k - \lambda \in SF^+_{\omega}(H) \) for every \( \lambda \in (\lambda_0)_\epsilon \) such that \( \lambda \neq \lambda_0 \) and for all \( k \). Hence \( i(T_k - \lambda) \leq 0 \), \( \alpha(T_k - \lambda) < \infty \) and \( (T_k - \lambda)H \) is closed for all \( k \). Thus \( i(T_k^* - \overline{\lambda}) \geq 0 \) and \( \beta(T_k^* - \overline{\lambda}) < \infty \). Since \( T_k^* \) has SVEP for all \( k \), \( i(T_k^* - \overline{\lambda}) \leq 0 \) for all \( k \). Hence \( i(T_k^* - \overline{\lambda}) = 0 \) and \( \alpha(T_k^* - \overline{\lambda}) = \beta(T_k^* - \overline{\lambda}) < \infty \) for all \( k \). By continuity of index [52, Problem 103], we get

\[
i(T^* - \overline{\lambda}) = \lim_{n \to \infty} i(T_n^* - \overline{\lambda}) = 0.
\]

So \( T^* - \overline{\lambda} \) is Fredholm with \( i(T^* - \overline{\lambda}) = 0 \). Therefore \( T - \lambda \) is Fredholm with \( i(T - \lambda) = 0 \). So \( T - \lambda \in SF^+_{\omega}(H) \) which is a contradiction. Hence \( \sigma_{a\omega} \) is lower semi-continuous.

Now let \( \lambda \in \sigma_a(T) \setminus \sigma_{a\omega}(T) \). Then by Theorem 3.3.13, \( \lambda \in \sigma_a(T) \). Hence by [59, Theorem IV.3.16], \( \lambda \in \operatorname{lim} \inf_{n \to \infty} \sigma_a(T_n) \). So \( \sigma_a \) is continuous at \( T \). \( \square \)

**Corollary 3.3.19.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then \( \sigma_b \) is continuous at \( T \), and hence \( \sigma \) is continuous at \( T \).

**Proof.** Since \( T \) is an \( n \)-normal operator, by Lemma 3.3.17, \( T \) satisfies Browder’s theorem. Hence by Proposition 3.3.18, \( \sigma_a \) is continuous at \( T \). Thus by Lemma 3.3.16, \( \sigma_b \) is continuous at \( T \). So by Theorem 3.3.15, \( \sigma \) is continuous at \( T \). Since \( T \) satisfies Browder’s condition, \( \sigma_\omega(T) = \sigma_b(T) \). So \( \sigma_\omega \) is continuous at \( T \). \( \square \)
Proposition 3.3.20. Let $T \in \mathbb{B}(H)$ be an $n$-normal operator. Then $\sigma_{es}$ is continuous at $T$.

Proof. Since $T$ is $n$-normal, $T$ has SVEP. Hence by [1, theorem 3.65 (ii)], $\sigma_{es}(T) = \sigma_{aw}(T^*)$. Since $T^*$ is $n$-normal, $\sigma_{aw}$ is continuous at $T^*$. Thus $\sigma_{es}$ is continuous at $T$. $\square$

Lemma 3.3.21. Let $T \in \mathbb{B}(H)$ be $n$-normal. Then $\sigma_{ab}$ is continuous at $T$.

Proof. Suppose $\lambda \in \sigma_a(T) \setminus \sigma_{aw}(T)$. Then $T - \lambda \in SF_+(H)$. Since $T$ has SVEP, $i(T - \lambda) = 0$. So $\lambda \notin \sigma_{aw}(T)$. Thus $\lambda \in \Pi_{00}(T)$. Hence $\lambda \in \Pi_{ab}(T)$, which is a contradiction with [82, 83, 84],

$$\sigma_{ab}(T) = \sigma_{aw}(T) \cup \{\lambda \in \sigma_a(T) : \lambda \text{ is not an isolated point of } \sigma_a(T)\}.$$ 

Hence $\lambda \notin \sigma_{ab}$. Therefore $\sigma_{ab}(T) = \sigma_{aw}(T)$. Thus by Proposition 3.3.18, $\sigma_{ab}$ is continuous at $T$. $\square$

Definition 3.3.1. An operator $T \in \mathbb{B}(H)$ is called an essentially $n$-normal operator if $\pi(T)$ is $n$-normal, where $\pi : \mathbb{B}(H) \to \mathbb{B}(H)/K(H)$ is the canonical map.

Lemma 3.3.22. Let $T \in \mathbb{B}(H)$ be an essentially $n$-normal operator. Then $\sigma_e$ is continuous at $T$.

Proof. Since $T$ is essentially $n$-normal, $\pi(T)$ is $n$-normal. Hence $\sigma$ is continuous at $\pi(T)$. Since $\sigma_e(T) = \sigma(\pi(T)), \sigma_e$ is continuous at $T$. $\square$
3.4 Spectral properties

In this section we discuss some spectral properties of \( n \)-normal operators.

**Lemma 3.4.1.** Let \( T \in \mathbb{B}(H) \) be an \( n \)-normal compact operator. Then \( T \) is decomposed into the direct sum \( T = T_1 \oplus T_2 \), where \( T_1 \) is nilpotent operator of nilpotence index \( \leq n \) and \( T_2^n = \bigoplus_{m=1}^{\infty} \lambda_m P_m \), where \( \lambda_m \) (\( m = 1, \ldots \)) are the distinct nonzero eigenvalues of \( T^n \) and \( P_m \) (\( m = 1, \ldots \)) is the orthogonal projection of \( H \) onto \( N(T^n - \lambda I) \).

**Proof.** Since \( T \) is \( n \)-normal and compact, \( T^n \) is normal compact. Thus by [52, p.92], \( T^n = 0 \oplus S \) with respect to the decomposition \( N(T^n) \oplus R(T^n) \), where \( S \) is diagonalizable. Assume that

\[
T = \begin{pmatrix} T_1 & T_3 \\ T_4 & T_2 \end{pmatrix}
\]

with respect to the decomposition \( N(T^n) \oplus R(T^n) \). Since \( T \) commutes with \( T^n \), \( T_3 S = ST_4 = 0 \) and \( ST_2 = T_2 S \). Since \( S \) is injective and has dense range \( T_3 = T_4 = 0 \). So \( T = T_1 \oplus T_2 \) with \( ST_2 = T_2 S \) and \( T_1^n = 0 \). Since \( S = \sum_{m=1}^{\infty} \lambda_m P_m \), where \( \lambda_m \) (\( m = 1, 2, \ldots \)) are the distinct nonzero eigenvalues of \( T^n \) and \( P_m \) (\( m = 1, 2, \ldots \)) is the orthogonal projection of \( H \) onto \( N(T^n - \lambda I) \).

Since \( S = T_2^n \) is compact, \( P_m \) is finite rank and \( \lambda_m \longrightarrow 0 \) as \( m \longrightarrow 0 \). Hence \( T_2^n = \bigoplus_{m=1}^{\infty} \lambda_m P_m \), where \( \lambda_m \) (\( m = 1, \ldots \)) are the distinct nonzero eigenvalues of \( T^n \) and \( P_m \) (\( m = 1, \ldots \)) is the orthogonal projection of \( H \) onto \( N(T^n - \lambda I) \).

**Lemma 3.4.2.** If \( T \in \mathbb{B}(H) \) is an \( n \)-normal operator, then \( T \) has finite ascent.

**Proof.** Since \( T \) is \( n \)-normal, \( T^n \) is normal. Hence \( N(T^n) = N(T^{2n}) \). Let \( n < m < 2n \). Then it is clear \( N(T^n) \subset N(T^m) \). Now let \( 0 \neq x \in N(T^m) \).
Then \( x \in N(T^{2n}) = N(T^n) \). Thus \( N(T^m) \subset N(T^n) \). Hence \( T \) has finite ascent.

Recall that An operator \( T \in \mathcal{B}(H) \) is called \( a \)-isoloid if every isolated point of \( \sigma_a(T) \) is eigenvalue.

**Proposition 3.4.3.** If \( T \in \mathcal{B}(H) \) is an \( n \)-normal operator, then \( T \) is an \( a \)-isoloid operator.

**Proof.** Since \( T \) is an \( n \)-normal operator, by Proposition 3.3.4, \( T \) satisfies Weyl’s theorem. Since \( T \) and \( T^* \) both are \( n \)-normal, \( T \) and \( T^* \) are both have SVEP. Hence by [1, Corollary 2.45 (iii)], \( \sigma(T) = \sigma_a(T) \).

Now assume that \( \lambda \in iso(\sigma_a(T)) = iso(\sigma(T)) \). Since \( T \) is \( n \)-normal operator, by Lemma 3.3.11, \( T \) isoloid. Hence \( \lambda \) is eigenvalue of \( T \). Therefore \( T \) is \( a \)-isoloid.

It is well known that Browder’s theorem holds for \( T \) if and only if

\[
\sigma_b(T) = \sigma_\omega(T).
\]

Next we shall prove similar equivalence for generalized Browder’s theorem.

**Proposition 3.4.4.** Let \( T \in \mathcal{B}(H) \). Then generalized Browder’s theorem holds for \( T \) if and only if \( \sigma_{B_\omega}(T) = \sigma_{Bb}(T) \).

**Proof.** For any \( T \in \mathcal{B}(H) \), \( \sigma_{B_\omega}(T) \subseteq \sigma_{Bb}(T) \) [19]. Now assume that \( T \) satisfies generalized Browder’s theorem and let \( \lambda \notin \sigma_{B_\omega}(T) \). Then \( \lambda \in \Pi(T) \). So \( T - \lambda \) is Drazin invertible. Let \( F \in \mathbb{F}_0(H) \) with \( TF = FT \). Then by [20], \( T + F - \lambda \) is Drazin invertible. Therefore \( \lambda \notin \sigma_D(T + F) \). Hence \( \lambda \notin \sigma_{Bb}(T) \). Hence \( \sigma_{Bb}(T) \subseteq \sigma_{B_\omega}(T) \). Thus \( \sigma_{Bb}(T) = \sigma_{B_\omega}(T) \). Conversely assume that
σ_{Bb}(T) = σ_{Bω}(T). We have to prove that generalized Browder’s theorem holds for \( T \). Suppose that \( \lambda \in σ(T) \setminus σ_{Bω}(T) \). Then \( \lambda \in σ(T) \setminus σ_{Bb}(T) \). Hence \( T − \lambda \) is not invertible and there is \( F \in 𝔻_0(H) \) such that \( FT = TF \) and \( T + F − \lambda \) is Drazin invertible. Since \( TF = FT \), by [20], \( T − \lambda \) is Drazin invertible. Therefore \( T − \lambda \) has finite ascent and descent. Since \( \lambda \in σ(T) \), \( \lambda \in Π(T) \). Thus \( σ(T) \setminus σ_{Bb}(T) \subseteq Π(T) \). Now let \( \lambda \in Π(T) \). Then \( T − \lambda \) is Drazin invertible but not invertible. Since \( \lambda \) is an isolated point of \( σ(T) \), by [19], \( T − \lambda \) is B-Weyl. Hence \( \lambda \in σ(T) \setminus σ_{Bω}(T) \). Thus \( Π(T) \subseteq σ(T) \setminus σ_{Bω}(T) \).

Therefore generalized Browder’s theorem holds for \( T \).

Lemma 3.4.5. Let \( T \in 𝒮(H) \). \( T \) satisfies generalized Browder’s theorem if and only if \( σ(T) \setminus σ_{Bω}(T) \subseteq Π_0(T) \).

Proof. Assume \( T \) satisfies generalized Browders’s theorem.

Let \( \lambda \in σ(T) \setminus σ_{Bω}(T) \). Then \( \lambda \in σ(T) \setminus σ_{Bb}(T) \). Hence there exists \( F \in 𝔻_0(H) \) such that \( TF = FT \) and \( T + F − \lambda \) is Drazin invertible but \( T − \lambda \) is not invertible. Therefore \( T − \lambda \) is Drazin invertible but not invertible. Hence \( \lambda \in σ(T) \setminus σ_D(T) \), and so \( \lambda \in Π_0(T) \). Thus \( σ(T) \setminus σ_{Bω}(T) \subseteq Π_0(T) \).

Next assume that \( σ(T) \setminus σ_{Bω}(T) \subseteq Π_0(T) \). Since \( σ_{Bω}(T) \cup Π_0(T) \subseteq σ(T) \), \( σ(T) = σ_{Bω}(T) \cup Π_0(T) \). Now it is enough to prove that \( σ_{Bb}(T) \subseteq σ_{Bω}(T) \).

Next suppose \( \lambda \notin σ_{Bω}(T) \). Then \( T − \lambda \) is B-Weyl and not invertible. Hence \( \lambda \in Π_0(T) \). By [53], \( T − \lambda \) is Drazin invertible. So \( \lambda \notin σ_D(T) \). Thus by [20], there exists \( F \in 𝔻_0(H) \), \( FT = TF \) such that \( σ_D(T) = σ_D(T + F) \). Hence \( \lambda \notin σ_{Bb}(T) \) and so \( σ_{Bb}(T) = σ_{Bω}(T) \). Thus by Proposition 3.4.4, generalized Browder’s theorem holds for \( T \).
Theorem 3.4.6. Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then \( T \) satisfies generalized Browder’s theorem.

Proof. Assume \( \lambda \in \sigma(T) \setminus \sigma_{B\omega}(T) \). Then \( T - \lambda \) is \( B \)-Weyl but not invertible. Since \( T - \lambda \) is \( B \)-Weyl, by [19], \( T - \lambda = T_1 \oplus T_2 \), where \( T_1 \) is Weyl and \( T_2 \) is nilpotent. Since by Lemma 3.2.27, \( T \) has SVEP, \( T_1 \) and \( T_2 \) also have SVEP. Hence Browder’s theorem holds for \( T_1 \) and \( T_2 \). So Browder’s theorem holds for \( T_1 \oplus T_2 \). Thus \( T - \lambda \) is Browder. Hence \( \lambda \) is an isolated point of \( \sigma(T) \).

Thus by Lemma 3.4.5, \( T \) satisfies generalized Browder’s theorem. \( \square \)

For a normal operator, generalized Weyl’s theorem holds [21]. Now consider generalized Weyl’s Theorem for \( n \)-normal operators through the local spectral theory.

Proposition 3.4.7. Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then \( T \) satisfies generalized Weyl’s theorem.

Proof. Suppose \( \lambda \in \sigma(T) \setminus \sigma_{B\omega}(T) \). Then \( T - \lambda \) is \( B \)-Weyl but not invertible. Since \( T \) is an \( n \)-normal operator, by Lemma 3.2.27, and Theorem 3.4.6, \( T \) has SVEP and \( T \) satisfies generalized Browder’s theorem respectively. Thus \( \sigma_{B\omega}(T) = \sigma_{Bb}(T) \). Now we shall prove that \( \lambda \) is an isolated point. Suppose that \( \lambda \notin \sigma_{Bb}(T) \). Then there exists \( F \in \mathcal{F}_0(H) \) such that \( FT = TF \) and \( \lambda \notin \sigma_D(T + F) \). Since \( T + F - \lambda \) is Drazin invertible and \( TF = FT \), by [20], \( T - \lambda \) is Drazin invertible. Therefore \( T - \lambda \) has finite ascent and descent. So \( \lambda \) is an isolated point of \( \sigma(T) \). Since \( T \) \( n \)-normal, by Lemma 3.3.11, \( T \) is isoloid. Therefore \( \lambda \in E(T) \). Now let \( \lambda \in E(T) \) and \( E_\lambda \) the Riesz idempotent for \( \lambda \).
Then

\[ T = T/E_\lambda H \oplus T/(I - E_\lambda)H, \]

where \( \sigma(T/E_\lambda H) = \{\lambda\} \) and \( \sigma(T/(I - E_\lambda)H) = \sigma(T)\setminus\{\lambda\} \). Now if \( \lambda = 0 \), then \( T/E_\lambda H \) is \( n \)-normal and quasinilpotent. Thus by Lemma 3.3.3, \( T/E_\lambda H \) is nilpotent. Hence \( T \) is the direct sum of an invertible operator and nilpotent. Hence by [19], \( T \) is \( B \)-Weyl. Thus \( 0 \in \sigma(T)\setminus\sigma_{B\omega}(T) \).

If \( \lambda \neq 0 \), then since \( \sigma(T/E_\lambda H) = \{\lambda\} \), \( T/E_\lambda H - \lambda \) is nilpotent. Thus \( T - \lambda \) is \( B \)-Weyl. Therefore \( \lambda \in \sigma(T)\setminus\sigma_{B\omega}(T) \). Hence \( T \) satisfies generalized Weyl’s theorem. \( \square \)

The following well known lemma [19] is required in the proof of Lemma 3.4.9.

**Lemma 3.4.8.** Let \( S, T \in \mathbb{B}(H) \) be two commuting \( B \)-Fredholm operators. Then \( ST \) is a \( B \)-Fredholm operator and \( i(ST) = i(S) + i(T) \).

**Lemma 3.4.9.** Suppose \( S, T \in \mathbb{B}(H) \) are commuting \( n \)-normal operators. Then \( ST \) is \( B \)-Weyl if and only if \( S \) and \( T \) are both \( B \)-Weyl.

**Proof.** Assume \( ST \) is \( B \)-Weyl. Then \( ST \) is \( B \)-Fredholm of index zero. Hence \( (ST)^n \) is \( B \)-Fredholm and \( i((ST)^n) = 0 \). Since \( (ST)^n = S^nT^n \), by Lemma 3.4.8 \( S^n \) and \( T^n \) are both \( B \)-Fredholm and normal. So \( i(T^n) = i(S^n) = 0 \). Hence by Lemma 3.4.8 \( S \) and \( T \) are \( B \)-Fredholm and \( 0 = i(T^n) = i(T) + \cdots + i(T) \). \( n \)-times Thus \( i(T) = 0 \). Similarly, \( i(S) = 0 \). So \( S \) and \( T \) are \( B \)-Weyl.

Conversely, let \( S \) and \( T \) are both \( B \)-Weyl. Then \( S \) and \( T \) are both \( B \)-Fredholm and \( i(S) = i(T) = 0 \). Hence by Lemma 3.4.8, \( ST \) is \( B \)-Fredholm and \( i(ST) = i(S) + i(T) = 0 \). Therefore \( ST \) is \( B \)-Weyl. \( \square \)
Next we show that the generalized Weyl’s spectral mapping theorem holds for \( n \)-normal operators. Furthermore, generalized Weyl’s theorem holds for \( f(T) \), where \( T \) is an \( n \)-normal operator and \( f \) is analytic in a neighborhood of \( \sigma(T) \).

**Theorem 3.4.10.** Let \( T \in \mathcal{B}(H) \) be an \( n \)-normal operator. Then

\[
f(\sigma_{B\omega}(T)) = \sigma_{B\omega}(f(T)), \text{ for every } f \in H(\sigma(T)).
\]

**Proof.** Suppose \( \lambda \notin f(\sigma_{B\omega}(T)) \). Let \( f(T) - \lambda = g(T)\Pi_{j=1}^{m}(T - \lambda_{j}) \), where \( \lambda_{1}, ..., \lambda_{m} \in \mathbb{C} \) and \( g(T) \) is invertible. Since \( \lambda \notin f(\sigma_{B\omega}(T)) \),

\[
g(\mu)\Pi_{j=1}^{m}(\mu - \lambda_{j}) \neq 0 \text{ for every } \mu \in \sigma_{B\omega}(T).
\]

Hence \( \mu \neq \lambda_{j} \) for every \( \mu \in \sigma_{B\omega}(T) \) and \( j = 1, ..., m \). Thus \( T - \lambda_{j} \) is \( B \)-Weyl, \((j=1,...,m)\). Since \( g(T) \) is invertible,

\[
i(f(T) - \lambda) = \Sigma_{j=1}^{m}i(T - \lambda_{j}) + i(g(T));
\]

\[
= 0.
\]

Therefore \( f(T) - \lambda \) is \( B \)-Weyl. Hence \( \lambda \notin \sigma_{B\omega}(f(T)) \). Thus

\[
\sigma_{B\omega}(f(T)) \subseteq f(\sigma_{B\omega}(T)).
\]

Now let \( \lambda \notin \sigma_{B\omega}(f(T)) \). Then \( f(T) - \lambda \) is \( B \)-Weyl and

\[
f(T) - \lambda = g(T)\Pi_{j=1}^{m}(T - \lambda_{j}),
\]

where \( \lambda_{1}, ..., \lambda_{m} \in \mathbb{C} \) and \( g(T) \) is invertible. Since the factors of the right-hand side of the last equation commute, by Lemma 3.4.8, every \( (T - \lambda_{j}) \) is \( B \)-Fredholm for all \( j = 1, ..., m \) [18]. Since \( T \) \( n \)-normal, \( T \) has SVEP. Thus by [62], \( i(T - \lambda_{j}) \leq 0 \). Since
Proposition 3.4.11. Let $f \in \mathcal{B}(H)$ be an $n$-normal operator. Then $f(T)$ satisfies generalized Weyl’s theorem for every $f \in H(\sigma(T))$.

Proof. First prove that $\sigma(f(T)) \setminus E(f(T)) = f(\sigma(T) \setminus E(T))$

Assume $\lambda \in \sigma(f(T)) \setminus E(f(T))$. Then by the spectral mapping theorem,

$$\lambda \in f(\sigma(T)).$$

Hence $\lambda \in f(\sigma(T)) \setminus E(f(T))$. Suppose that $\lambda$ is not an isolated point of $f(\sigma(T))$. Then there exists a sequence $\{\lambda_m\} \subseteq f(\sigma(T))$ such that $\lambda_m \to \lambda$.

Since $\lambda_m \in f(\sigma(T))$, $\lambda_m = f(\mu_m)$ for some $\mu_m \in \sigma(T)$. Since $\sigma(T)$ is compact, there is a convergent subsequence $\{\mu_{m_k}\}$ of $\{\mu_m\}$ say $\mu_{m_k} \to \mu \in \sigma(T)$.

Hence $f(\mu_{m_k}) \to \lambda$, and $\lambda = f(\mu)$. Since $\mu \in \sigma(T) \setminus E(T)$, $\lambda = f(\mu) \in f(\sigma(T) \setminus E(T))$. Now, suppose that $\lambda \in \text{iso } f(\sigma(T))$. Since $\lambda \notin E(f(T))$ (by assumption), $\lambda$ can not be an eigenvalue of $f(T)$. Suppose that $\mu_0 \in E(T)$ such that $\lambda = f(\mu_0)$. Then $h(z) = f(z) - \lambda$, $h(\mu_0) = 0$. Hence by [86, Theorem 10.18], $h(z) = (z - \mu_0)^m h_1(z)$, $h_1(\mu_0) \neq 0$. Thus $h(T) = h_1(T)(T - \mu_0)^m$. Since $\mu_0$ is eigenvalue of $T$, there exists $x \in H$, $\|x\| = 1$ such that $(T - \mu_0)x = 0$. So $h(T)x = h_1(T)(T - \mu_0)^m x = 0$. Therefore $(f(T) - \lambda)x = 0$ which is a contradiction. Therefore $\sigma(f(T)) \setminus E(f(T)) \subseteq f(\sigma(T) \setminus E(T))$.

Next suppose that $\lambda \in f(\sigma(T) \setminus E(T))$. Then by the spectral mapping theorem, $\lambda \in f(\sigma(T))$. Let $\lambda \in E(f(T))$. Then $\lambda$ is an isolated point of $\sigma(f(T))$. 

\[0 = i(f(T) - \lambda) = i(g(T)) + \sum_{j=1}^{m} i(T - \lambda_j) = \sum_{j=1}^{m} i(T - \lambda_j) \leq 0,\]

$i(T - \lambda_j) = 0$ for all $j = 1, \ldots, m$. Thus $T - \lambda_j$ is $B$-Weyl for all $j = 1, \ldots, m$.

Therefore $\lambda \notin f(\sigma_{B}\omega(T))$. Hence $\sigma_{B}\omega(f(T)) = f(\sigma_{B}\omega(T))$. \qed
Let $f(T) - \lambda = g(T)\Pi_{j=1}^{m}(T - \lambda_{j})$, where $\lambda_{1}, ..., \lambda_{m} \in \mathbb{C}$ and $g(T)$ is invertible. Since $f(T) - \lambda$ is not invertible, there exists $\lambda_{j} \in \{\lambda_{1}, ..., \lambda_{m}\}$ such that $\lambda_{j} \in \sigma(T)$. Since $\lambda$ is isolated point of $\sigma(f(T))$, $\lambda_{j}$ is isolated in $\sigma(T)$. Since $T$ is $n$-normal, by Lemma 3.3.11, $T$ is isoloid. Hence $\lambda_{j}$ is an eigenvalue of $T$. Since $\lambda \in E(f(T))$ such that $\lambda_{j} \in E(T)$. Thus $\lambda = f(\lambda_{j})$ for some $\lambda_{j} \in E(T)$. Hence $\lambda \in f(E(T))$ which is a contradiction. Therefore $\lambda \notin E(f(T))$. So $\lambda \in \sigma(f(T)) \setminus E(f(T))$. Thus $\sigma(f(T)) \setminus E(f(T)) = f(\sigma(T) \setminus E(T))$. Since $T$ is $n$-normal, $T$ satisfies generalized Weyl’s theorem. Thus

$$\sigma(f(T)) \setminus E(f(T)) = f(\sigma(T) \setminus E(T)) = f(\sigma_{B\omega}(T)) = \sigma_{B\omega}(f(T)).$$

Therefore $f(T)$ satisfies generalized Weyl’s theorem. \hfill \qed

The following lemma is similar to [78, Theorem 4].

**Lemma 3.4.12.** Let $T \in \mathbb{B}(H)$ satisfies generalized Weyl’s theorem and $F \in \mathbb{F}_{0}(H)$ such that $FT = TF$. If $E(T + F) = \Pi(T + F)$, then $T + F$ satisfies generalized Weyl’s theorem.

**Proof.** Since $T$ satisfies generalized Weyl’s theorem, by [45], $\sigma_{B\omega}(T) = \sigma_{D}(T)$. By [53], $\sigma_{B\omega}(T) = \sigma_{B\omega}(T + F)$. Since $TF = FT$, by [20], $\sigma_{D}(T + F) = \sigma_{D}(T)$. So $\sigma_{B\omega}(T + F) = \sigma_{D}(T + F)$. Therefore since $\Pi(T + F) = E(T + F)$, by [20], $T + F$ satisfies generalized Weyl’s theorem. \hfill \qed

**Theorem 3.4.13.** Let $T \in \mathbb{B}(H)$ be an $n$-normal operator and $F \in \mathbb{F}_{0}(H)$ such that $FT = TF$. Then $f(T) + F$ satisfies generalized Weyl’s theorem for every $f \in H(\sigma(T))$.

**Proof.** First we shall prove that $T + F$ satisfies generalized Weyl’s theorem, for that in view of Lemma 3.4.12, it is enough to show that $\Pi(T + F) = E(T + F)$. 

59
Since $\Pi(T+F) \subset E(T+F)$ is true, we have to prove that $E(T+F) \subset \Pi(T+F)$.

Let $\lambda \in E(T+F)$. Then by [74], $\lambda$ is an isolated point of $\sigma(T)$. Since $T$ is $n$-normal, by Proposition 3.4.7, $T$ satisfies generalized Weyl’s theorem. Hence $\lambda \in E(T) = \Pi(T)$. Since $\Pi(T) = \pi(T + F)$, $\lambda \in \Pi(T + F)$. Now, since $T$ is $n$-normal, by Proposition 3.4.11, $f(T)$ satisfies generalized Weyl’s theorem for every $f \in H(\sigma(T))$. Hence by Lemma 3.4.12, $f(T) + F$ satisfies generalized Weyl’s theorem. □

Recall that an operator $T \in \mathbb{B}(H)$ satisfies the property $(w)$ if $\sigma_a(T) \setminus \sigma_{aw}(T) = \Pi_{00}(T)$, the property $(b)$ if $\sigma_a(T) \setminus \sigma_{aw}(T) = \Pi_0(T)$, and the property $(gw)$ if $\Delta^g_a(T) = \Pi_{00}(T)$, where $\Delta_g(T) = \sigma(T) \setminus \sigma_{B\omega}(T)$ and $\Delta^g_a(T) = \sigma_a(T) \setminus \sigma_{aB\omega}(T)$.

**Proposition 3.4.14.** Let $T \in \mathbb{B}(H)$ be an $n$-normal operator. Then the property $(b)$ holds at $f(T)$ for every $f \in H(\sigma(T))$.

**Proof.** Since $T$ is an $n$-normal operator, by Proposition 3.2.3, $T^*$ is $n$-normal. Hence by Lemma 3.2.27, $T^*$ has SVEP and it is polaroid. Thus by [11], $T$ satisfies the property $(gw)$. Since $T$ is $n$-normal, by Proposition 3.4.11, generalized Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$. Since $T$ has SVEP and SVEP is stable under the functional calculus, $f(T)$ has SVEP. Therefore by [11], $f(T)$ satisfies the property $(gw)$ for every $f \in H(\sigma(T))$. So by [11], $f(T)$ satisfies the property $(w)$. Hence by [22], $f(T)$ satisfies the property $(b)$. □

Recall that an operator $T \in \mathbb{B}(H)$ is said to $a$-polaroid if $\text{iso}(\sigma_a(T)) \subset \Pi(T)$. 

60
Lemma 3.4.15. If $T \in \mathbb{B}(H)$ is an $n$-normal operator, then $T$ is $a$-polaroid.

Proof. Since $T$ is an $n$-normal operator, $T^*$ has SVEP. Hence by [1, Corollary 2.45], $\sigma(T) = \sigma_a(T)$. Hence $T$ is an $a$-polaroid operator. \qed

3.5 Concluding Remarks

This chapter has been intended to introduce $n$-normal operators on a Hilbert space $H$. We have given some basic properties of these operators. And we have given some condition implies normality of an $n$-normal operator. We also have proved Weyl’s theorem, Browder’s theorem, generalized Weyl’s theorem, generalized Browder’s theorem for $n$-normal operators. Finally, the continuity of the spectrum, the approximate point spectrum, the Browder spectrum, the Weyl spectrum and the Weyl approximate point spectrum of $n$-normal operators on a Hilbert space are established.