Chapter 5

Bishop’s and other properties

5.1 Introduction

Recall that for $T \in \mathcal{B}(H)$ satisfying SVEP and $x \in H$, consider the set
\[ \rho_T(x) = \{ \lambda_0 \in \mathbb{C} : \exists \text{ analytic function } f \text{ from some neighborhood } U_{\lambda_0} \text{ of } \lambda_0 \text{ to } H \text{ such that } (\lambda - T)f(\lambda) = x \text{ for all } \lambda \in U_{\lambda_0} \} \]
the analytic function $f$ is called a local resolvent function of $x$. The set $\rho_T(x)$ is called the local resolvent of $T$ at $x$. The set $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ is called the local spectrum of $T$ at $x$. For $F \subset \mathbb{C}$, $H_T(F) = \{ x \in H : \sigma_T(x) \subset F \}$ is called a spectral subspace of $T$. Also $SM(T)$ denotes the set of all maximal spectral spaces of $T$; i.e., $Y \in \text{Lat}(T)$ is in $SM(T)$, if $Y$ contains every $Z \in \text{Lat}(T)$ with $\sigma(T/Z) \subset \sigma(T/Y)$.

Bishop’s Property ( abbrev. $T$ has the property (β)) was introduced by E. Bishop [23] in connection with duality theory for spectral decomposition. An operator $T$ is said to satisfy the Bishop’s property ( abbrev. $T$ has property (β)) if for every open subset $U$ of complex plane $\mathbb{C}$ and for any sequence $\{f_n\}_{n=0}^{\infty}$ in $\hat{O}(U, H)$ such that whenever, $(\lambda - T)f_n(\lambda) \rightarrow 0$ uniformly on every compact subset of $U$, then $f_n(\lambda) \rightarrow 0$ uniformly on every compact subset.
of U. C. K. Fong [46] introduced the property $(\beta^*)$. An operator $T \in \mathbb{B}(H)$ is said to have the property $(\beta^*)$, if every open covering $\{G_1, ..., G_m\}$ of $\sigma(T)$, $H = H_T(G_1) + ... + H_T(G_m)$. In [44, 45] C. Foias introduced the notion of decomposability. An operator $T \in \mathbb{B}(H)$ is said to be decomposable if every open covering $\{G_1, ..., G_m\}$ of $\sigma(T)$, there exists a system $\{H_1, ..., H_m\}$ in $SM(T)$ such that $H = H_1 + ... + H_m$, $\sigma(T/H_i) \subset G_i$ and $(i = 1, ..., m)$ and proved that every decomposable operator has the property $(\beta)$. Probably the most interesting result concerning the property $(\beta)$ is in [66] that a continuous linear operator on a Hilbert space $H$ is decomposable in the sense of C. Foias if and only if $T$ and $T^*$ possess the property $(\beta)$. It has been observed in [8] that an operator $T \in B(H)$ is decomposable if and only if it has both the property $(\beta)$ and the property $(\beta^*)$. More significantly, Albrecht and Eschmeier [7] have shown that an operator $T$ has the property $(\beta)$ if and only if it is similar to the restriction of a decomposable operator to one of its invariant closed subspace, and that $T$ has the property $(\beta^*)$ if and only if it is similar to a quotient of a decomposable operator see [41, 42]. Furthermore, it has been shown in [7, 40] that properties $(\beta)$ and $(\beta^*)$ are dual to each other, in the sense that an operator $T \in \mathbb{B}(H)$ satisfies property $(\beta)$ if and only if $T^*$ satisfies property $(\beta^*)$.

The purpose of this chapter is to present a number of observations and remarks concerning these properties and their asymptotic behavior, along with some other related problems. And we shall introduce new concept of local spectral nature we call it as the bounded Bishop’s property.
5.2 Bishop’s property for operators and its dual.

Elementary properties of a spectral linear subspace are given in the following:

**Proposition 5.2.1.** [29, 43, 92] For an operator $T \in \mathcal{B}(H)$ and a closed subset $F$ in $\mathbb{C}$ we have

1. $H_T(\emptyset) = \{0\}, \; H_T(F) = H_T(\sigma(T) \cap F), \; H_T(\sigma(T)) = H.$

2. If $x \in H_T(F)$ and $f$ is its local resolvent function, then $f(\lambda) \in H_T(F)$ for $\lambda \in F^c$.

3. If $H_T(F) = H$, then $\lambda - T$ is surjective for all $\lambda \in F^c$. Moreover, if $T$ has SVEP, then $\sigma(T) \subset F$.

E. Albercht and J. Escheimer [7], have proved that the property $(\beta)$ and $(\beta^*)$ are dual of each other in the following sense.

**Theorem 5.2.2.** An operator $T \in \mathcal{B}(H)$ satisfies the property $(\beta)$ if and only if $T^*$ satisfies the property $(\beta^*)$.

Recall that an operator $T \in \mathcal{B}(H)$ is said to satisfies Dunford’s conditions $(C)$, if $H_T(F)$ is closed for every closed $F \subset \mathbb{C}$.

The following proposition is well known.

**Proposition 5.2.3.** [80, 38]

1. If $T \in \mathcal{B}(H)$ satisfies the property $(\beta)$, then $T$ satisfies the condition $(C)$.

2. If $T \in \mathcal{B}(H)$ satisfies the condition $(C)$, then $T$ has SVEP.
Recall that for \( T \in \mathcal{B}(H) \). An open set \( \Omega \subset \mathbb{C} \) is called a set of analytic uniqueness for \( T \) if, for every open set \( U \subset \Omega \) and \( f \) in \( \hat{O}(U,H) \), the identity \((\lambda - T)f(\lambda) = 0 \) on \( U \) implies \( f(\lambda) = 0 \) on \( U \). Denote by \( \Omega_T \) the maximal open set of analytic uniqueness for \( T \). \( S_T = \mathbb{C} \setminus \Omega_T \) is called the analytic residuum of \( T \). For \( x \in H \), denoted by \( \delta_T(x) \) the open subset of those points \( \lambda \in \mathbb{C} \) such that there exists an open neighborhood \( V \) of \( \lambda \) and a local resolvent for \( x \) on \( V \). Now we need the following lemma [43, 68].

**Lemma 5.2.4.** For every \( T \in \mathcal{B}(H) \), \( \sigma(T) = \bigcup_{x \in H} \sigma_T(x) = \sigma_s(T) \cup S_T \). Furthermore, if \( T \) has SVEP, then \( \sigma(T) = \sigma_s(T) \) and if \( T^* \) has SVEP, then \( \sigma(T) = \sigma_a(T) \). Finally \( \sigma_a(T^*) = \sigma_s(T) \) and \( \sigma_a(T) = \sigma_s(T^*) \).

Recall that An operator \( T \in \mathcal{B}(H) \) is said to have the spectral decomposition property (SDP) if every open covering \( \{G_1, ..., G_m\} \) of \( \sigma(T) \), there exists a system \( \{H_1, ..., H_m\} \) in \( \text{Lat}(T) \) such that \( H = H_1 + ... + H_m, \sigma(T/H_i) \subset G_i \), \( (i = 1, ..., m) \).

In the following theorem we list some characterization of decomposable operators.

**Theorem 5.2.5.** [6, 29, 64] Let \( T \in \mathcal{B}(H) \). Then the following are equivalent.

1. \( T \) is decomposable.

2. \( T \) has the SDP.

3. \( T \) has the property \((\beta)\) and the property \((\beta^*)\).

4. \( T \) has condition \((C)\) and property \((\beta^*)\).
5. For every open set \( G \) in \( C \), there is a subspace \( Y \in \text{Lat}(T) \) such that 
\[ \sigma(T/Y) \subseteq \overline{G} \text{ and } \sigma(T^Y) \subseteq G^c. \]

**Definition 5.2.1.** [40] A decomposable operator \( T \in \mathbb{B}(H) \) is said to be strongly decomposable if \( T/H \bigcap T(F) \) is decomposable for every closed subset \( F \subset C \).

**Definition 5.2.2.** Let \( T \in \mathbb{B}(H) \). We say that \( T \) is weakly decomposable if for every open covering \( \{G_1, ..., G_m\} \) of \( \sigma(T) \), there exists a system \( \{H_1, ..., H_m\} \) in \( SM(T) \) such that \( H = H_1 + ... + H_m \) and \( \sigma(T/H_i) \subset G_i \) \((i = 1, ..., m)\), we recall that \( SM(T) \) denotes the set of all maximal spectral spaces of \( T \).

**Definition 5.2.3.** Let \( T \in \mathbb{B}(H) \). We say that \( T \) is 2-weakly decomposable if for every open covering \( \{G_1, G_2\} \) of \( C \), there exist \( H_1, H_2 \in \text{Lat}(T) \) such that \( H = H_1 + H_2 \), \( \sigma(T/H_i) \subset G_i \) \((i = 1, 2)\).

**Definition 5.2.4.** An operator \( T \in \mathbb{B}(H) \) is said to be quasi-decomposable if it is weakly decomposable and satisfies condition \((C)\).

**Theorem 5.2.6.** Suppose \( T_1 \in \mathbb{B}(H) \) has the property \((\beta)\) and \( T_2 \in \mathbb{B}(H) \). If 
\[ C^m(T_1, T_2)(I) = 0, \text{ then } T_2 \text{ has the property } (\beta), \text{ where for } m \geq 1, C^m(T_1, T_2)(S) = \sum_{k=0}^{m}(-1)^k(m)\binom{m}{k} T_1^{m-k} S T_2^k. \]

**Proof.** Let \( \{f_n\}_{n=0}^\infty \) a sequence in \( \hat{O}(U, H) \) such that \( (\lambda - T_2)f_n(\lambda) \longrightarrow 0 \) uniformly on every compact subset of \( U \), we shall prove that \( f_n(\lambda) \longrightarrow 0 \)
uniformly on every compact subset of \( U \). Since

\[
\sum_{k=0}^{m} \binom{m}{k} (\lambda - T_1)^k (\lambda - T_2)^{m-k} \\
= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} T_1^i \lambda^{k-i} \right\} \left\{ \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \lambda^j T_2^{m-k-j} \right\}, \\
= \sum_{k=0}^{m} \binom{m}{k} \left\{ (-1)^k T_1^k + \sum_{i=0}^{k-1} (-1)^i \binom{k}{i} T_1^i \lambda^{k-i} \right\} \\
\{ T_2^{m-k} + \sum_{j=1}^{m-k} (-1)^j \binom{m-k}{j} \lambda^j T_2^{m-k-j} \}, \\
= \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} T_1^i T_2^{m-k} \lambda^{k-i} + \sum_{j=1}^{m-k} (-1)^j \binom{m-k}{j} \lambda^j T_1^j T_2^{m-k-j} \right\} \{ T_2^{m-k} \}.
\]

So we have

\[
\sum_{k=0}^{m} \binom{m}{k} (\lambda - T_1)^k (\lambda - T_2)^{m-k} = \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} T_1^i \lambda^{k-i} \right\} \left\{ \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \lambda^j T_2^{m-k-j} \right\}, \\
\text{where } h(p, q, \lambda) = \sum_{p=0}^{q} \sum_{k=0}^{m} \binom{m}{k} \left\{ \sum_{i=0}^{k} (-1)^i \binom{k}{i} T_1^i \lambda^{k-i} \right\} \left\{ \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \lambda^j T_2^{m-k-j} \right\}.
\]

Now since

\[
h(p, q, \lambda) = \lambda^{m-p-\sum_{k=p}^{m-q} \binom{m-q}{k} \binom{k}{p} \binom{m-k}{q}},
\]

by Proposition 2.1.3 (2), \( h(p, q, \lambda) = \lambda^{m-p-\sum_{k=p}^{m-q} \binom{m-q}{k} \binom{k}{p} \binom{m-k}{q} \binom{m-p}{q} \binom{m-k}{q} \). Hence

by Proposition 2.1.3(3), \( h(p, q, \lambda) = \lambda^{m-p-\sum_{k=p}^{m-q} \binom{m-q}{k} \binom{k}{p} \binom{m-p}{q} \binom{m-k}{q} \). Thus

\[
\sum_{k=p}^{m-q} \binom{m-q}{k} \binom{k}{p} \binom{m-p}{q} \binom{m-k}{q} = 0.
\]

Therefore

\[
h(p, q, \lambda) = \lambda^{m-p-q} \sum_{k=p}^{m-q} (-1)^k \binom{m}{p} \binom{m-p-q}{q} \binom{m-p}{q} \binom{m-k}{q},
\]

\[
= \lambda^{m-p-q} \binom{m}{p} \binom{m-p-q}{q} \sum_{k=p}^{m-q} (-1)^k \binom{m-p-q}{k-p},
\]

\[
= \lambda^{m-p-q} \binom{m}{p} \binom{m-p-q}{q} \{ 0 \} \sum_{k=0}^{m-q} (-1)^k \binom{m}{k} = 0
\]

\[= 0.\]
Hence
\[
\sum_{k=0}^{m} \binom{m}{k} (\lambda - T_1)^k (\lambda - T_2)^{m-k} = \sum_{k=0}^{m} \binom{m}{k} T_1^k T_2^{m-k} = C^m(T_1, T_2)(I) = 0.
\]

Now \[
\sum_{k=0}^{m} \binom{m}{k} (-1)^k T_1^k T_2^{m-k} f_n(\lambda) - (\lambda - T_1)^m f_n(\lambda) = -(\lambda - T_1)^m f_n(\lambda) = \sum_{k=0}^{m-1} \binom{m}{k} (\lambda - T_1)^k (\lambda - T_2)^{m-k} f_n(\lambda) = \sum_{k=0}^{m-1} \binom{m}{k} (\lambda - T_1)^k (\lambda - T_2)^{m-k-1} (\lambda - T_2) f_n(\lambda) \to 0.
\]

Since \( T_1 \) has the property \((\beta)\), \( f_n(\lambda) \to 0 \) uniformly on every compact subset of \( U \). Hence \( T_2 \) has the property \((\beta)\). \( \square \)

**Corollary 5.2.7.** Suppose \( T_1, T_2 \in \mathcal{B}(H) \) are commuting and \((T_1 - T_2)^m = 0\) for some \( m \). Then \( T_1 \) has the property \((\beta)\) if and only if \( T_2 \) has the property \((\beta)\).

**Theorem 5.2.8.** If \( T \in \mathcal{B}(H) \) is an \( n \)-normal operator, then \( T \) has the property \((\beta)\).

**Proof.** Since \( T \) is \( n \)-normal, by Proposition 3.2.3, \( T^* \) is \( n \)-normal. Hence \( TT^* = T^* T \). Since \( T^* \) is a normal operator, \( T^* \) has the property \((\beta)\). Since \( C(T, T^*)(I) = 0 \), \( T \) has the property \((\beta)\) by Theorem 5.2.6. \( \square \)

The converse of Theorem 5.2.8 need not be true. We note here that the unilateral shift on \( \ell^2 \) has the property \((\beta)\) but it is not \( n \)-normal.
**Proposition 5.2.9.** Let $N$ be nilpotent and $T = S + N$, where $NS = SN$. Then $S$ has the property $(\beta)$ if and only if $T$ has the property $(\beta)$.

**Proof.** Assume $T$ has the property $(\beta)$ and a sequence $\{f_n\}_{n=0}^{\infty}$ in $\hat{O}(U, H)$ such that $(\lambda - S)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since

$$(\lambda - (T - N))f_n(\lambda) \to 0$$

uniformly on every compact subset of $U$, $(\lambda - T)f_n(\lambda) + Nf_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Now

$$(\lambda - T)N^{k-1}f_n(\lambda) = N^{k-1}(\lambda - T)f_n(\lambda),$$

$$= N^{k-1}(\lambda - T)f_n(\lambda) + N^kf_n(\lambda)$$

$$= N^{k-1}(\lambda - T + N))f_n(\lambda) \to 0.$$  

Since $T$ has property $(\beta)$, $N^{k-1}f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Hence

$$(\lambda - T)N^{k-2}f_n(\lambda) = N^{k-2}(\lambda - T)f_n(\lambda),$$

$$= N^{k-2}(\lambda - T)f_n(\lambda) + N^{k-1}f_n(\lambda)$$

$$= N^{k-2}(\lambda - T + N))f_n(\lambda) \to 0.$$  

Since $T$ has the property $(\beta)$, $N^{k-2}f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Continuously we have $f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Therefore $S$ has the property $(\beta)$. By similar arguments one can prove the converse. 

**Corollary 5.2.10.** If $T_1, T_2 \in \mathbb{B}(H)$ are commuting and $C^m(T_1, T_2)(I) = 0$. Then $T_1$ has the property $(\beta)$ if and only if $T = T_1 + T_2$ has the property $(\beta)$.

**Proof.** Since $T_1T_2 = T_2T_1$, $C^m(T_1, T_2)(I) = (T_1 - T_2)^m = 0$. Thus
$N = T_1 - T_2$ is nilpotent. Since $T = T_1 + T_2 = 2T_1 - N$, by Proposition 5.2.9, $T_1$ has the property $(\beta)$ if and only if $T = T_1 + T_2$ has the property $(\beta)$. \qed

**Lemma 5.2.11.** Suppose $T_1, T_2 \in \mathbb{B}(H)$ have the property $(\beta)$. If $T_2T_1 = 0$, then $T = T_1 + T_2$ has the property $(\beta)$.

**Proof.** If $\{f_n\}_{n=0}^{\infty}$ is a sequence in $\hat{O}(U, H)$ such that $(\lambda - T)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$, $(\lambda - T_1 - T_2)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Hence $(\lambda T_2 - T_2T_1 - T_2^2)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since $T_2T_1 = 0$, $(\lambda - T_2)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since $T_2$ has the property $(\beta)$, $T_2f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Hence we have $(\lambda - T_1)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since $T_1$ has the property $(\beta)$, $f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Thus $T$ has the property $(\beta)$. \qed

**Proposition 5.2.12.** Let $T_1, T_2 \in \mathbb{B}(H)$ be invertible operators. Then $T_1T_2$ has the property $(\beta)$ if and only if $T_2T_1$ has the property $(\beta)$.

**Proof.** Let $U$ be an open subset of $\mathbb{C}$ containing $\sigma(T_2T_1)$ such that $0 \notin U$. Suppose that $T_1T_2$ has the property $(\beta)$. Let $(\lambda - T_2T_1)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$, for a sequence $\{f_n\}_{n=0}^{\infty}$ in $\hat{O}(U, H)$. Hence $(\lambda T_1 - T_1T_2T_1)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Thus $(\lambda - T_1T_2)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since $T_1T_2$ has the property $(\beta)$, $T_1f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since $T_1$ is invertible, $f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Hence $T_2T_1$ has the property $(\beta)$. \qed

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Definition 5.2.5. [9] For an operator $T \in \mathcal{B}(H)$, the Aluthge transformation $\tilde{T}$ is defined by

$$
\tilde{T} = |T|^\frac{1}{2} U |T|^\frac{1}{2},
$$

where $T = U |T|$ is the polar decomposition of $T$.

Corollary 5.2.13. Let $T \in \mathcal{B}(H)$ be an invertible operator. Then $T$ has the property $(\beta)$ if and only if the Aluthge transform $\tilde{T}$ of $T$ has the property $(\beta)$.

Proof. Since $T = U |T|$, $T = (U |T|^\frac{1}{2}) |T|^\frac{1}{2}$. If $T$ has the property $(\beta)$, then by Proposition 5.2.12 $|T|^\frac{1}{2} (U |T|^\frac{1}{2})$ has the property $(\beta)$ and vice versa. □

Theorem 5.2.14. [7] An operator $T \in \mathcal{B}(H)$ has the property $(\beta^*)$ if and only if there exist a decomposable operator $R$ and a continuous linear surjection $Q$ such that $TQ = QR$.

Proposition 5.2.15. Let $T \in B(H)$ and $V \in \mathcal{B}(H)$ be isometry. Then $T$ has the property $(\beta)$ if and only if $VTV^*$ has the property $(\beta)$.

Proof. Assume that $T$ has the property $(\beta)$. Then $T^*$ has the property $(\beta^*)$. Therefore there exist a decomposable operator $R$ and a continuous linear surjection $Q$ such that $T^*Q = QR$. Thus $(VQ)R = VT^*V^*VQ$. Hence $VT^*V^*$ has property $(\beta^*)$. So $VTV^*$ has the property $(\beta)$. Conversely, assume that $VTV^*$ has the property $(\beta)$. Then $VT^*V^*$ has the property $(\beta^*)$. Thus there exist a decomposable operator $R$ and a continuous linear surjection $Q$ such that $(VT^*V^*)Q = QR$. So $(V^*Q)R = T^*(V^*Q)$. Hence $T^*$ has the property $(\beta^*)$. So $T$ has the property $(\beta)$. □
Theorem 5.2.16. Let $T \in B(H)$ be an $n$-normal operator. Then $f(T)$ and $f(T^*)$ have the property $(\beta)$ for every $f \in H(\sigma(T))$, and hence $f(T)$ is decomposable.

Proof. Since $T$ is $n$-normal, by Theorem 5.2.2, and Theorem 5.2.8, $T$ has the property $(\beta^*)$. Let $\{G_i\}_{i=1}^m$ be an open covering for $\sigma(f(T))$. Therefore $\sigma(T) \subseteq \bigcup_{i=1}^m f^{-1}(G_i)$. Hence $H = H_T(f^{-1}(G_1)) + \ldots + H_T(f^{-1}(G_m))$. Since $f^{-1}(G_i) \subseteq f^{-1}(G_i)$, $H = H_T(f^{-1}(G_1)) + \ldots + H_T(f^{-1}(G_m))$. Therefore by [80], $H = H_{f(T)}(G_1) + \ldots + H_{f(T)}(G_m)$. Thus $f(T)$ has the property $(\beta^*)$. Now since $T$ is $n$-normal, $T^*$ satisfies the property $(\beta^*)$. Hence $\bar{f}(T^*) = (f(T))^*$ has the property $(\beta^*)$. Therefore $(\bar{f}(T^*))^* = f(T)$ has the property $(\beta)$. By the same arguments one can prove that $f(T^*)$ has the property $(\beta)$. And by Theorem 5.2.5, we get $f(T)$ is Decomposable. \qed

In fact J. Eschmeier and M. Putinar [41], have proved that, if $T$ has the property $(\beta)$, then $f(T)$ has the property $(\beta)$ for every $f \in H(\sigma(T))$. Theorem 5.2.16 is a corollary to this result and Theorem 5.2.8. Our intense is to give direct and simple proof.

Recall that an invariant subspace $Y$ of $T$ is said to be T-strongly analytic subspace if $T^Y$ has the property $\beta$, where $T^Y$ is the induced operator on the quotient space $H/Y$.

Corollary 5.2.17. Let $T \in B(H)$ be an $n$-normal operator and $f \in H(\sigma(T))$. If $Y \in \text{Lat}(T)$ is $T$-strongly analytic subspace, then $Y$ is $f(T)$-strongly analytic subspace.

Proof. Since $T$ is an $n$-normal operator, by Theorem 5.2.8, $T$ has the property $(\beta)$. Hence by [47], $Y$ is $T$-strongly analytic. By [39, Theorem 2.3 and 2.5],
\( f(T^Y) = f(T)^Y \). Therefore by Theorem 5.2.16, \( f(T)^Y \) has the property \((\beta)\). 
Thus \( Y \) is \( f(T) \)-strongly analytic. 

It is well known that, if \( T \in \mathbb{B}(H) \) satisfies the condition \((C)\), then \( H_T(F) \in SM(T) \) and \( \sigma(T/H_T(F)) \subset F \cap \sigma(T) \) for every closed subset \( F \) of \( \mathbb{C} \) [29, Proposition 1.3.8]. Now we shall state the converse.

**Lemma 5.2.18.** For an operator \( T \in \mathbb{B}(H) \) and every closed subset \( F \) of \( \mathbb{C} \). If \( \sigma(T/H_T(F)) \subset F \), then \( T \) satisfies the condition \((C)\), and hence it has SVEP.

**Proof.** Let \( F \) a closed subset of \( \mathbb{C} \) and \( \sigma(T/H_T(F)) \subset F \). Let \( x \in \overline{H_T(F)} \).
Then \( f(\lambda) = (\lambda - T/H_T(F))^{-1}x \) is analytic and \( (\lambda - T)f(\lambda) = x \) on \( \rho(T/H_T(F)) \supseteq \mathbb{C} \setminus F \) i.e., \( x \in H_T(F) \). Thus \( H_T(F) \) is closed. Hence \( T \) satisfies condition \((C)\), and hence \( T \) has SVEP. 

**Remark:** In general \( \sigma(T/H_T(F)) \subset F \cap \sigma(T) \) need not be true for \( T \in \mathbb{B}(H) \) and a closed subset \( F \) of \( \mathbb{C} \) [90].

**Definition 5.2.6.** An operator \( T \in \mathbb{B}(H) \) has 2-weak \((\beta^*)\) property if 
\( H_T(G_1) + H_T(G_2) \) is dense in \( H \) for every pair of open subsets \( G_1 \) and \( G_2 \) covering \( \sigma(T) \).

**Proposition 5.2.19.** Let \( T \in \mathbb{B}(H) \). If \( T \) has 2-weak \((\beta^*)\) property, then \( T^* \) has SVEP.

**Proof.** Let \( V \) be an open subset of \( \mathbb{C} \), and let \( f : V \to H \) be an analytic function satisfying \( (\lambda - T^*)f(\lambda) = 0 \) on \( V \). Without loss of generality, we may assume that \( V \) is connected. Let \( \{G_1, G_2\} \) be an open cover of \( \sigma(T) \)
such that \( G_1 \subset V \) and \( V \setminus G_2 \neq \emptyset \). Since \( T \) has the 2-weak \((\beta^*)\) property, \( H = \overline{H_T(G_1)} + \overline{H_T(G_2)} \). For \( x_1 \in H_T(G_1) \) and \( \lambda \in V \setminus G_2 \), there exists an analytic function \( f_1 : U \rightarrow H \) on some neighborhood \( U \) of \( \lambda \) such that \((\lambda - T)f_1(\lambda) = x_1 \) on \( U \). Therefore
\[
\langle x_1, f(\lambda) \rangle = \langle (\lambda - T)f_1(\lambda), f(\lambda) \rangle \\
= \langle f_1(\lambda), (\lambda - T^*)f(\lambda) \rangle \\
= 0
\]
on \( U \). Thus \( \langle x_1, f(\lambda) \rangle = 0 \) on \( V \). Similarly, one has \( \langle x_2, f(\lambda) \rangle = 0 \) for all \( x_2 \in H_T(G_2) \). Then \( \langle x, f(\lambda) \rangle = 0 \) for all \( x \in \overline{H_T(G_1)} + \overline{H_T(G_2)} \) and \( \lambda \in V \). Since \( \lambda \in V \) and \( \overline{H_T(G_1)} + \overline{H_T(G_2)} \) is dense, \( f(\lambda) = 0 \) on \( V \). Hence \( T^* \) has SVEP.

Now by Proposition 5.2.19 and [6, Lemma 5], we have the following.

**Corollary 5.2.20.** If an operator \( T \in \mathbb{B}(H) \) has the 2-weak SDP, then \( T \) and \( T^* \) have the SVEP.

**Corollary 5.2.21.** If an operator \( T \in \mathbb{B}(H) \) has the 2-weak \((\beta^*)\) property, then \( \sigma(T) = \sigma_a(T) \).

**Proof.** Since \( T \) has the 2-weak \((\beta^*)\) property, by Proposition 5.2.19, \( T^* \) has SVEP. Hence by Lemma 5.2.4 \( \sigma(T) = \sigma_a(T) \). \( \square \)

**Lemma 5.2.22.** Let \( T \in \mathbb{B}(H) \). Suppose for every open subset \( U \) of \( \mathbb{C} \) such that for every sequence \( \{f_n\} \) in \( \hat{O}(U,H) \) with \( (\lambda - T)f_n(\lambda) \rightarrow 0 \) uniformly on all compact subsets of \( U \), \( \{f_n\} \) is bounded on each compact subsets of \( U \). Then \( T \) has the property \((\beta)\).
Proof. Let \( \{f_n\} \) a sequence in \( \hat{O}(U, H) \) such that \((\lambda - T)f_n(\lambda) \to 0 \) uniformly on a compact set \( K \) of \( U \), and
\[
\epsilon_n = \sup_{\lambda \in K} \| (\lambda - T)f_n \|. \quad \text{Since } \epsilon_n \to 0, \quad \sup_{\lambda \in K} \| (\lambda - T)\frac{f_n}{\sqrt{\epsilon_n}} \| = \sqrt{\epsilon_n} \to 0.
\]
For \( g \in \hat{O}(U, H) \) and a compact set \( K \subset U \), define \( \|g\|_K = \sup\{\|g(\lambda)\|: \lambda \in K\} \). Therefore \( \|f_n\|_K \leq M \) for some \( M > 0 \). Thus
\[
\|f_n\| - K \leq M\sqrt{\epsilon_n} \to 0.
\]
Thus \( f_n(\lambda) \to 0 \) uniformly on all compact subset of \( U \). So \( T \) has the property \((\beta)\).

In [8, Corollary 1.2 (a)], it is been proved that an operator \( T \in \mathcal{B}(H) \) has the property \((\beta)\), whenever \( L_T \) has the property \((\beta)\), where \( L_T(A) = TA \), \( A \in \mathcal{B}(H) \). Now we shall prove the converse.

**Proposition 5.2.23.** Let \( T \in \mathcal{B}(H) \) has the property \((\beta)\). Then \( L_T \) has the property \((\beta)\).

Proof. Suppose \( T \) has property \((\beta)\) and \( \{f_n\} \) a sequence of \( \hat{O}(U, \mathcal{B}(H)) \) such that \((\lambda - L_T)f_n(\lambda) \to 0 \) as \( n \to 0 \) uniformly on every compact subset of \( U \). Then for a compact subset \( K \) of \( U \), \((\lambda - T)f_n(\lambda)x \to 0 \) for all \( x \in H \) uniformly on \( K \). Thus \( f_n(\lambda)x \to 0 \) uniformly on \( K \) for all \( x \in H \). Hence by uniform boundedness theorem \( f_n(\lambda) \) bounded on \( K \). Therefore
\[
\sup_{n \in \mathbb{N}, \lambda \in K} \|f_n(\lambda)\| < \infty.
\]
Hence by Lemma 5.2.22, \( L_T \) has property \((\beta)\). \( \square \)
Lemma 5.2.24. Let \( S, T \in \mathcal{B}(H) \) be \( n \)-normal operators. If \( AT = SA \), where \( A \in \mathcal{B}(H) \), then \( AH_T(F) \subset H_S(F) \) for any closed subset \( F \) of \( \mathbb{C} \).

Proof. Since \( S \) and \( T \) are \( n \)-normal operators, by Lemma 3.2.27, \( S \) and \( T \) have SVEP. If \( x \in H_T(F) \), then \( \sigma_T(x) \subset F \). Hence \( F^c \subset \rho_T(x) \). So there exists an analytic \( H \)-valued function \( f \) defined on \( F^c \) such that

\[
(\lambda - T)f(\lambda) = x, \ \lambda \in F^c.
\]

Since \( AT = SA \), \( (\lambda - S)Af(\lambda) = A(\lambda - T)f(\lambda) = Ax, \ \lambda \in F^c \). Since \( Af : F^c \to H \) is analytic, \( F^c \subset \rho_S(Ax) \), i.e., \( \sigma_S(Ax) \subset F \). Hence

\[
Ax \in H_S(F), \ \text{i.e.,} \ AH_T(F) \subset H_S(F).
\]

\( \square \)

The following proposition is [29, Theorem 3.3 (ii) \( \implies \) (i)].

Proposition 5.2.25. Assume \( \lim_{n \to \infty} \| C^n(S, T)A \|^\frac{1}{n} = 0 \), where \( S, T, A \in \mathcal{B}(H) \). Then \( AH_T(F) \subset H_S(F) \) for any closed subset \( F \) of \( \mathbb{C} \).

Corollary 5.2.26. Let \( A, R, T \in \mathcal{B}(H) \). Assume that \( A \) has finite dimensional null space such that \( \lim_{n \to \infty} \| C^n(R, T)A \|^\frac{1}{n} = 0 \) and \( V \) surjective such that \( \lim_{n \to \infty} \| C^n(R, T)V \|^\frac{1}{n} = 0 \). Then \( \sigma(R) \subset \sigma(T) \).

Proof. Let \( D \) be an open disk in \( \Omega_R \) and \( f \in \hat{O}(D, H) \) such that

\[
(\lambda - T)f(\lambda) = 0.
\]

Hence by Proposition 5.2.1, \( f(\lambda) \in H_T(\mathbb{C} \setminus D) \). Since \( f(\lambda) \) is an eigenvector of \( T \) corresponding to eigenvalue \( \lambda, f(\lambda) \in H_T(\{\lambda\}), (\lambda \in D) \). Since \( D \subset \Omega_R \), by Proposition 5.2.25,

\[
Af(\lambda) \in H_R(\{\lambda\}) \cap H_R(\mathbb{C} \setminus D) = \{0\}.
\]
Hence $Af(\lambda) = 0$. Let $H_1$ be a subspace generated by $f(\lambda)$ for all $\lambda \in D$. Since $AH_1 = \{0\}$ and $\text{dim}(\text{ker}A) < \infty$, $H_1$ is a finite dimensional invariant subspace of $T$. If $f(\lambda) \neq 0$, $T/H_1$ would have infinitely many eigenvalues, which is a contradiction. Thus $f(\lambda) = 0$ on $D$. Thus $\Omega_R \subset \Omega_T$. Therefore by Definition 2.1.54, $S_T \subset S_R$. By Lemma 5.2.4, it is enough to show that $\sigma_s(R) \subset \sigma_s(T)$. Now by [70, Lemma 2], and Proposition 5.2.25, we get

$$\sigma_s(R) = \bigcup_{y \in H} \gamma_R(y) = \bigcup_{x \in H} \gamma_R(Vx) \subset \bigcup_{x \in H} \gamma_T(x) = \sigma_s(T).$$

Hence $\sigma(R) \subset \sigma(T)$.  

**Definition 5.2.7.** [69] For $S, T \in \mathbb{B}(H)$, an operator $A \in \mathbb{B}(H)$ is said to be asymptotically intertwines $S$ and $T$ if, $\|C^m(S,T)A\|^\frac{1}{m} \rightarrow 0$ as $m \rightarrow \infty$.

**Definition 5.2.8.** [69] Operators $T, S \in \mathbb{B}(H)$ are said to be asymptotically similar if there exists an invertible $A \in \mathbb{B}(H)$ such that $A$ asymptotically intertwines $S$ and $T$ and $A^{-1}$ asymptotically intertwines $T$ and $S$.

As immediate consequences of Corollary 5.2.26, we have the following corollary.

**Corollary 5.2.27.** If $R, T \in \mathbb{B}(H)$ are asymptotically similar, then

$$\sigma(T) = \sigma(R).$$

**Definition 5.2.9.** [69] Let $S, T \in \mathbb{B}(H)$, we say that $T$ is asymptotically quasi-affine transform of $S$ if $\|C^m(S,T)A\|^\frac{1}{m} \rightarrow 0$ for some quasi-affinity $A \in \mathbb{B}(H)$.
Proposition 5.2.28. Suppose that $T$ is asymptotically quasi-affine transform of $S$. If $T$ has the property $(\beta^*)$ and $S$ has the property $(\beta)$, then $S$ is quasi-decomposable.

Proof. Assume $\|C^n(S,T)A\|^{\frac{1}{n}} \to 0$ for some quasi-affinity $A$. Then by Proposition 5.2.25, $AH_T(F) \subset H_S(F)$ for each closed subset $F$ of $\mathbb{C}$. Since $T$ has the property $(\beta^*)$, $H = H_T(G_1) + \ldots + H_T(G_n)$. Since $AH_T(F) \subset H_S(F)$ and $AH$ is dense in $H$, $H = H_T(G_1) + \ldots + H_T(G_n)$. Since $S$ has the property $(\beta)$, by Proposition 5.2.3, $H_S(G_i)$ is closed and $\sigma(S/H_S(G_i)) \subset G_i$ for $i = 1, \ldots, n$. □ 

Recall that the operators $T, S \in \mathbb{B}(H)$ are said to be quasi-similar if they are quasi-affine transform of each others.

Consider $d_{SP}(T,S) = \lim_{m \to \infty} \sup \|C^n(T,S)(I)\|^{\frac{1}{n}}$ and

$P(T,S) = \max\{d_{SP}(T,S), d_{SP}(S,T)\}.$

Note that if $TS = ST$, then $d_{SP}(T,S) = d_{SP}(S,T) = P(T,S)$. We say that $S, T \in \mathbb{B}(H)$ are spectral equivalent (or quasi-nilpotent equivalent) if $P(T,S) = 0$ and we denote this relation by $T \sim_{SP} S$. We say that a sequence $\{T_k\}$, in $\mathbb{B}(H)$ converges spectrally to $T \in B(H)$ (and denote it by $T_k \to_{SP} T$) if $P(T_k, T) \to 0$.

Theorem 5.2.29. Let $T \in \mathbb{B}(H)$ be an $n$-normal operator. If $R \in \mathbb{B}(H)$ such that $T \sim_{SP} R$, then $R$ has the property $(\beta^*)$.

Proof. Since $T$ is an $n$-normal operator, by Theorem 5.2.8 and Theorem 5.2.2, $T$ has property $(\beta^*)$. Since $T \sim_{SP} R$, by applying Proposition 5.2.25, twice $H_T(F) = H_T(F)$ for each closed subset $F$ of $\mathbb{C}$. Hence $R$ has property $(\beta^*)$. □
Remark: We can apply Theorem 5.2.29 for the property \((\beta)\) and the proof follows by duality between the property \((\beta)\) and the property \((\beta^*)\) and the fact that \(T \sim^{SP} R\) if and only if \(T^* \sim^{SP} R^*\).

Recall that \(F_{P(T,S)}\) consists of those points (in the complex plane) whose distance to \(F\) is less than or equal to \(P(T,S)\).

Lemma 5.2.30. Let \(T, R \in \mathcal{B}(H)\) and \(F\) be a compact subset in \(\mathbb{C}\). Then

\[
H_T(F) \subset H_R(F_{P(T,R)}).
\]

Proof. Let \(x \in H_T(F)\) and \(f \in \hat{O}(U,H)\) such that \((\lambda - T)f(\lambda) = x\). Then by [13, Lemma 2.1], the function

\[
g(\lambda) = \sum_{n=0}^{\infty} (-1)^n \frac{C_n^m(S,T)(U)}{n!} f^{(n)}(\lambda),
\]

is analytic in \(\mathbb{C}\setminus F_{P(T,S)}\). Hence by using the calculus of [28, Theorem 3], \((\lambda - S)g(\lambda) = x\). Therefore \(x \in H_R(F_{P(R,T)})\). So \(H_T(F) \subset H_S(F_{P(T,R)})\). \(\square\)

Theorem 5.2.31. Let \(\{T_n\}\) be a sequence in \(\mathcal{B}(H)\) and \(T \in \mathcal{B}(H)\) be such that \(T_n \longrightarrow^{SP} T\). Then

1. If \(T_n\) has the property \((\beta^*)\), for each \(n\), then \(T\) also has the property \((\beta^*)\).

2. If \(T_n\) has the property \((\beta)\), for each \(n\), then \(T\) also has the property \((\beta)\).

Proof. (1) Let \(\{G_i\}_{i=1}^m\) be an open covering of \(\sigma(T)\) and \(F_i\) be a compact subset of \(\mathbb{C}\) such that \(F_i \subset G_i\) and \(\sigma(T) \subset \bigcup_{i=1}^m F_i^c\). Then by [4, Proposition 2.2], we can find an integer \(N\) such that \(\bigcup_{i=1}^m F_i^c \supset \sigma(T_n)\) for all \(n \geq N\). So for \(n \geq N\),

\[
F_{P(T,T_n)}^c = \{\lambda \in \mathbb{C} : dist(\lambda,F_i^c) \leq P(T,T_n)\} \subset G_i.
\]

Now by Lemma 5.2.30,
for $n \geq N$, $H_{T_n} F_i \subset H_T (F^i_{P(T,T_n)}) \subset H_T (G_i)$. Since $T_n$ has the property ($\beta^*$), $H = \sum_{i=1}^{m} H_T (G_i)$. Hence $T$ has the property ($\beta^*$). We can prove part (2) by using the part (1) and the fact that $P(T_n, T) = P(T^*_n, T^*)$. □

The following is a Corollary of [13, Theorem 2.6], we have given a simple proof.

**Corollary 5.2.32.** If $\{T_n\}$ is a sequence in $\mathcal{B}(H)$ of decomposable operators and $T_n \longrightarrow^{SP} T$, then $T$ is decomposable.

**Proof.** Assume that $\{T_n\}$ be a sequence of decomposable operators in $\mathcal{B}(H)$ and $T \in \mathcal{B}(H)$ such that $T_n \longrightarrow^{SP} T$. Then by Theorems 5.2.31, 5.2.5, $T$ has both the property ($\beta$) and the property ($\beta^*$). Hence $T$ is decomposable. □

Now we shall study the property ($\beta$) for some operator matrices.

**Theorem 5.2.33.** Let $T \in \mathcal{B}(H \oplus H)$ be the following operator matrix

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $C$ is nilpotent operator. Then $A$ has property ($\beta$) if and only if $T$ has property ($\beta$).

**Proof.** Suppose that $C^k = 0$. Assume $A$ has property ($\beta$). Put $T_1 = A \oplus 0$ and $T_2 = \begin{pmatrix} 0 & B \\ 0 & C \end{pmatrix}$. Then $T = T_1 + T_2$ and $T_2 T_1 = 0$. Since $A$ has property ($\beta$), $T_1$ has the property ($\beta$). If $\{f_n\}_{n=0}^\infty$ is a sequence in $\hat{O}(U, H \oplus H)$ (i.e., $f_n = (g_n, h_n) \in \hat{O}(U, H \oplus H)$) such that $(\lambda - T_2) f_n(\lambda) \longrightarrow 0$ uniformly on every compact subset of $U$, then we have

$$\begin{pmatrix} \lambda g_n(\lambda) - B h_n(\lambda) \\ (\lambda - C) h_n(\lambda) \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since $C^k = 0$, $\lambda C^{k-1} h_n(\lambda) \longrightarrow C^k h_n(\lambda) = 0$ uniformly on every compact
subset of $U$. If $\lambda \neq 0$ and $C^{k-1}h_n(\lambda)$ is analytic, then $C^{k-1}h_n(\lambda) \to 0$ uniformly on every compact subset of $U$. And $\lambda C^{k-2}h_n(\lambda) \to C^{k-1}h_n(\lambda) = 0$ uniformly on every compact subset of $U$. Hence $C^{k-2}h_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Hence by continuing this way, $h_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Thus $\lambda g_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Therefore $T_2$ has the property $(\beta)$. Clearly $T_1$ has the property $(\beta)$. By lemma 5.2.11 we get $T$ has property $(\beta)$.

Next assume that $T$ has the property $(\beta)$. If $\{f_n\}_{n=0}^\infty$ be a sequence in $\hat{O}(U, H)$ such that $(\lambda - A)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Then $(\lambda - T)(f_n(\lambda), 0) = (\lambda - A)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. Since $T$ has the property $(\beta)$, $f_n(\lambda) \to 0$ uniformly on every compact subset of $U$. □

**Proposition 5.2.34.** Let $T \in \mathbb{B}(\bigoplus_{k=1}^m H)$ be the following operator matrix

$$
T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1m} \\
0 & T_{22} & \cdots & T_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_{mm}
\end{pmatrix}.
$$

Assume that $T$ has the property $(\beta)$. If $T_{11}T_{ij} = T_{ij}T_{jj}$ for $i = 1, \ldots, j$ and $j = 1, \ldots, m$, then $T_{jj}$ has the property $(\beta)$ for $j = 1, \ldots, m$.

**Proof.** Assume that $\{f_n\}_{n=0}^\infty$ be a sequence in $\hat{O}(U, \bigoplus_{k=1}^m H)$ such that

$(\lambda - T_{jj})f_{j,n}(\lambda) \to 0$ for $j = 1, \ldots, n$, uniformly on every compact subset of
$U$. Then for $j = 1, \ldots, n$,

$$(\lambda I - T)(T_{ij}f_{j,n}(\lambda), 0, \ldots, 0) = ((\lambda - T_{11})T_{ij}f_{j,n}(\lambda), 0, \ldots, 0)$$

$$= ((\lambda T_{ij} - T_{11})T_{ij}f_{j,n}(\lambda), 0, \ldots, 0)$$

$$= (T_{ij}(\lambda - T_{jj})f_{j,n}(\lambda), 0, \ldots, 0) \to 0$$

uniformly on every compact subset of $U$, where $i = 1, \ldots, j$. Since $T$ has the property $(\beta)$ for $j = 1, \ldots, m$, we get $T_{ij}f_{j,n}(\lambda) \to 0$ uniformly on every compact subset of $U$ for $i = 1, \ldots, j$. Hence for $j = 1$, $(\lambda - T)(f_{1,n}, 0, \ldots, 0) = (\lambda - T_{11}f_{1,n}(\lambda), 0, \ldots, 0) \to 0$ uniformly on every compact subset of $U$ and for $j = 2, \ldots, m$, $(\lambda - T)(0, 0, \ldots, f_{j,n}(\lambda), 0, \ldots, 0) = (-T_{1j}f_{j,n}(\lambda), \ldots, (-T_{j-1,j}f_{j,n}(\lambda))$, $((\lambda - T_{jj})f_{j,n}(\lambda), 0, \ldots, 0) \to 0$ uniformly on every compact subset of $U$. Since $T$ has the property $(\beta)$, we conclude that $f_{j,n}(\lambda) \to 0$ uniformly on every compact subset of $U$ for $j = 1, \ldots, m$. Hence $T_{jj}$ has the property $(\beta)$.  

**Remark 5.2.1.** If we set $B = T_{11} \oplus \ldots \oplus T_{nn}$ and $C = T - B$ in Proposition 5.2.34, then $T = B + C$, and $C^n = 0$. But we note that $BC \neq CB$.

**Proposition 5.2.35.** Let $T \in \mathcal{B}(\oplus_{k=1}^{m} H)$ with the following operator matrix

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1m} \\ 0 & T_{22} & \cdots & T_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & T_{mm} \end{pmatrix}.$$  

If $T_{jj}$ has the property $(\beta)$ for $j = 1, \ldots, m$, then $T$ has the property $(\beta)$.

**Proof.** Let $\{f_n\}_{n=0}^{\infty}$ be a sequence in $\hat{O}(U, \oplus_{k=1}^{m} H)$. If $(\lambda - T)f_n(\lambda) \to 0$ uniformly on every compact subset of $U$, then
\[
\begin{pmatrix}
(\lambda - T_{11})f_{1,n}(\lambda) - T_{12}f_{2,n}(\lambda) - T_{13}f_{3,n}(\lambda) - \ldots - T_{1n}f_{m,n}(\lambda) \\
(\lambda - T_{22})f_{2,n}(\lambda) - T_{23}f_{3,n}(\lambda) - T_{24}f_{4,n}(\lambda) - \ldots - T_{2n}f_{m,n}(\lambda) \\
\vdots \\
(\lambda - T_{m-1,n-1})f_{m-1,n}(\lambda) - T_{m-1,n}f_{m,n}(\lambda) \\
(\lambda - T_{mn})f_{m,n}(\lambda)
\end{pmatrix} \rightarrow 0
\]

uniformly on every compact subset of \(U\). Hence \((\lambda - T_{mn})f_{m,n}(\lambda) \rightarrow 0\) uniformly on every compact subset of \(U\). Since \(T_{m,m}\) has the property \((\beta)\), \(f_{m,n}(\lambda) \rightarrow 0\) uniformly on every compact subset of \(U\). Thus \((\lambda - T_{m-1,m-1})f_{m-1,n}(\lambda) \rightarrow 0\) uniformly on every compact subset of \(U\). Since \(T_{m-1,m-1}\) has property \((\beta)\), \(f_{m-1,n}(\lambda) \rightarrow 0\) uniformly on every compact subset of \(U\). By induction \(f_{j,n}(\lambda) \rightarrow 0\) uniformly on every compact subset of \(U\) for \(j = 1, \ldots, m\). Thus \(T\) has the property \((\beta)\). \(\square\)

From Propositions 5.2.34 and 5.2.35 we get the following corollary.

**Corollary 5.2.36.** Let \(T \in \mathbb{B}(\bigoplus_{k=1}^{m} H)\) with the following operator matrix

\[
T = \begin{pmatrix}
T_{11} & T_{12} & \cdots & T_{1m} \\
0 & T_{22} & \cdots & T_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & T_{mm}
\end{pmatrix}
\]

where \(T_{1i}T_{ij} = T_{ij}T_{jj}\) for \(i = 1, \ldots, j\) and \(j = 1, \ldots, m\). Then \(T_{kk}\) has the property \((\beta)\) for all \(k = 1, 2, \ldots, m\) if and only if \(T\) has the property \((\beta)\).

The following corollary is a special case of Corollary 5.2.36.
Corollary 5.2.37. Let $T \in \mathbb{B}(H \oplus H)$ be the following operator matrix,

$$T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix},$$

where $T_1S = ST_2$. Then $T_1$ and $T_2$ have the property ($\beta$) if and only if $T$ has the property ($\beta$).

5.3 Bounded Bishop’s property

Now we shall introduce a new concept related to Bishop’s property ($\beta$) that will called it bounded Bishop’s property ($b\beta$). One of the prime motivation for the study of property ($b\beta$) is fits strictly between the two fundamental concepts in local spectral theory, namely, the single valued extension property and Bishop’s property ($\beta$).

Definition 5.3.1. We say that $T \in \mathbb{B}(H)$ has the property ($b\beta$) if, for any uniformly bounded sequence of bounded analytic functions $f_n : U \rightarrow H$, ($U$ open subset of $\mathbb{C}$) such that $(T - \lambda)f_n(\lambda) \rightarrow 0$ as $n \rightarrow \infty$ uniformly on all compact subsets of $U$, we have $f_n(\lambda) \rightarrow 0$ uniformly on all compact subsets of $U$.

Remark: It is clear that:

1. If an operator $T \in \mathbb{B}(H)$ has the property ($\beta$), then $T$ has the property ($b\beta$).

2. If an operator $T \in \mathbb{B}(H)$ has the property ($b\beta$), then $T$ has SVEP.

We will show by examples that both the property ($\beta$) implies to the property ($b\beta$) and SVEP implies to the property ($b\beta$) not need be true.
The following theorem in [58, Theorem 3.14.1], will need to prove Proposition 5.3.2.

**Theorem 5.3.1.** [Vitali’s theorem] Let \( \{f_n\} \) be a uniformly bounded sequence of bounded functions in \( \hat{O}(U, H) \), where \( U \) is open subset of \( \mathbb{C} \). Assume that in \( U \) there is a subset \( F \) on which \( \{f_n\} \) is pointwisely convergent and that \( F \) has an accumulation point in \( U \). Then \( \{f_n\} \) converges uniformly on all compact subsets of \( U \).

**Proposition 5.3.2.** Let \( T \in \mathcal{B}(H) \) such that \( \sigma(T) \) has no interior point. Then \( T \) has the property \((b\beta)\).

**Proof.** Let \( f_n : U \rightarrow H \) be an uniformly bounded sequence of bounded functions in \( \hat{O}(U, H) \) such that \( (T - \lambda)f_n(\lambda) \rightarrow 0 \) as \( n \rightarrow \infty \) uniformly on each compact subset of \( U \). We notice that \( f_n(\lambda) \rightarrow 0 \) as \( n \rightarrow \infty \) uniformly on each compact subset of \( U \cap \rho(T) \). For \( \lambda \in U \cap \sigma(T) \), there exists a sequence \( \{\lambda_k\} \) in \( U \cap \rho(T), (\rho(T) = \mathbb{C}) \) which converges to \( \lambda \). Hence

\[
\lim_{n \to \infty} \|f_n(\lambda)\| = \lim_{k,n \to \infty} \|f_n(\lambda_k)\| = 0.
\]

Therefore for every \( \lambda \in U \), \( f_n(\lambda) \rightarrow 0 \). Thus by Theorem 5.3.1, \( f_n(\lambda) \) converges uniformly to 0 on each compact subset of \( U \). So \( T \) has the property \((b\beta)\).

**Example 5.3.1.** Yoo in [94], has quoted an example of an operator having \( \sigma(T) = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \) which does not have property \((\beta)\). Since \( T \) has spectrum without interior point, by Proposition 5.3.2, \( T \) has property \((b\beta)\).
Lemma 5.3.3. Let $T \in B(H)$ with $\|T\| = 1$. Suppose for every $r$ positive and for every positive integer $n$, there exists a non zero $x \in H$ such that $\|T^n x\| = \|x\|$, and $\|T^{n+1} x\| < r^{n+1} \|x\|$. Then $T$ does not have property $(b\beta)$.

Proof. Suppose $0 < r < 1$ and $n$ is a natural number. Then there is $x \in H$ with $\|T^n x\| = \|x\| = 1$ and $\|T^{n+1} x\| < r^{n+1}$. For $\lambda \in \mathbb{C}$, define $P_n(\lambda) = T^n x + \lambda T^{n-1} x + \ldots + \lambda^n x$. Then $(\lambda - T)P_n(\lambda) = -T^{n+1} x + \lambda^{n+1} x$. Hence for $\lambda \in \mathbb{C}$ with $|\lambda| \leq r$, $\|(\lambda - T)P_n(\lambda)\| \leq \|T^{n+1} x\| + |\lambda|^{n+1} \leq 2r^{n+1}$. Since $\|T\| = \|x\| = 1$, $\|P_n(\lambda)\| \leq 1 + r + r^2 + \ldots = \frac{1}{1-r}$. But $\|P_n(0)\| = \|T^n x\| = 1$. This show that $P_n$ is uniformly bounded on the disk $D$ of radius $r$ centered at 0, $(\lambda - T)P_n(\lambda)$ converges to zero uniformly on $D$, but $P_n(0)$ does not converges to zero. Hence $T$ does not have property $(b\beta)$. \qed

Definition 5.3.2. [31, p.24] If $\{H_n\}$ be a sequence of Hilbert spaces, let $H = \left\{ \{h_n\}_{n=1}^{\infty} : h_n \in H_n \text{ for all } n \text{ and } \sum_{n=1}^{\infty} \|h_n\|^2 < \infty \right\}$. For $h = \{h_n\}$ and $g = \{g_n\}$ in $H$ define

$$\langle h, g \rangle = \sum_{n=1}^{\infty} \langle h_n, g_n \rangle.$$ 

Then $\langle \ldots \rangle$ is an inner product on $H$ and the norm relative to this inner product is $\|h\| = \left( \sum_{n=1}^{\infty} \|h_n\|^2 \right)^{\frac{1}{2}}$. With this inner product $H$ is a Hilbert space, called the direct sum of $\{H_n\}$ and denoted by $\bigoplus_{n=1}^{\infty} H_n$.

If $T_n \in B(H_n)$ for any $n \in \mathbb{N}$. Define an operator $T : H \rightarrow H$ by $\{Tx\}_n = T_n x_n$, for all $n \in \mathbb{N}$. Then if $M = \sup_{n \in \mathbb{N}} \|T_n\|_{B(H_n)} < \infty$, then $\|Tx\|^2 = \sum_{n=1}^{\infty} \|T_n x_n\|^2 \leq \sum_{n=1}^{\infty} \|T_n\|_{B(H_n)}^2 \|x_n\|^2 \leq M^2 \|x\|^2$. Thus $T \in B(H)$.
We say that $T$ is the direct sum of $\{T_n\}_{n=1}^{\infty}$.

Next, we shall give an example of operator having SVEP but without the property $(b\beta)$.

**Example 5.3.2.** Let $T_m, (m \geq 2)$ be the $m \times m$ nilpotent Jordan blocks

$$T_m = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}.$$ 

Consider as an operator defined on the Hilbert space $\mathbb{C}^m$. Let $T$ be the direct sum $\bigoplus_{m=2}^{\infty} T_m$ defined on the direct sum $H = \bigoplus_{m=2}^{\infty} \mathbb{C}^m$ of Hilbert spaces. We have $\|T_m\| = \|T_{m-1}^m\| = 1$ and $(T_m)^m = 0$. Hence

$$\|T\| = \sup\{\|T_m\| : m \geq 2\} = 1.$$ 

Let $U$ be an open set in $\mathbb{C}$ and $f = \bigoplus_{m=2}^{\infty} f_m : U \rightarrow H$ be an analytic function such that $(\lambda - T)f_m(\lambda) = 0$, i.e., $(\lambda - T_m)f(\lambda) = 0$ for every $m \geq 2$. Therefore $0 = (T_m)^m f_m(\lambda) = \lambda^m f_m(\lambda)$. So $f_m(\lambda) = 0$ ($m \geq 2$) for all $0 \neq \lambda \in U$. Since $f$ is continuous, $f(\lambda) = 0$ for all $\lambda \in U$. Thus $T$ has SVEP. Now give $n$, take $x_n = (0, 0, \ldots, 0, 1) \in \mathbb{C}^{n+1}$, $x = (0, 0, \ldots, 0, x_n, 0, \ldots)$. Then $\|x\| = 1, T^j x = T_{n+1}^j x_n$. So $\|T^j x\| = \|x\| = 1$ for $j = 1, 2, \ldots, n$ and for $0 < r < 1$, $\|T^{n+1} x\| < r^{n+1}$. Thus by Lemma 5.3.3, $T$ does not have the property $(b\beta)$.

Give $T \in \mathbb{B}(H)$, a subspace $Y \in \text{Lat}(T)$ gives two related linear operators; the restriction $T/Y$ acting on $Y$ and the induced operator $T^Y$ acting on the
quotient space $H/Y$. Obviously the property $(b\beta)$ is inherited by the restriction to invariant subspace but not by $T^Y$.

The following example is in [37, Example 5.29].

**Example 5.3.3.** Let $T$ be the bilateral shift on $\ell^2(\mathbb{Z})$ and $Y = \overline{SP}\{e_{-1}, e_{-2}, \ldots\}$, where $\{e_n\}_{n \in \mathbb{Z}}$ is the usual orthonormal basis for $\ell^2(\mathbb{Z})$. Since $T$ is unitary, $\sigma(T)$ is a subset of unit circle (i.e., $\sigma(T)$ has no interior point). So by Proposition 5.3.2, $T$ has property $(b\beta)$, but $T^Y$ is unitarily equivalent to the left shift on $\ell^2(\mathbb{N})$ and hence it does not have SVEP. Hence $T^Y$ dose not have property $(b\beta)$.

**Proposition 5.3.4.** Let $T \in B(H)$. If for some $Y \in \text{Lat}(T)$ $T/Y$ and $T^Y$ both have the property $(b\beta)$, then $T$ has the property $(b\beta)$.

**Proof.** Let $\{f_n\}$ be an uniformly bounded sequence of bounded functions in $\hat{O}(U, H)$ satisfying $(\lambda - T)f_n(\lambda) \longrightarrow 0$ uniformly on all compact subset of $U$. Let $\lambda_0 \in U$ and $r > 0$ such that $D \subset U$, where $D = \{\lambda : |\lambda - \lambda_0| < r\}$. Now, extend $f_n$, $(n = 1, 2, \ldots)$ as a power series in $D$ as

$$f_n(\lambda) = \sum_{j=0}^{\infty} a_n(\lambda - \lambda_0)^j.$$  

(5.3.1)

Since $(\lambda - T)f_n(\lambda) \longrightarrow 0$ and hypothesis on $T^Y$, $\phi f_n \longrightarrow 0$ uniformly in the quotient norm of $H/Y$ on $\overline{D}$, where $\phi : H \longrightarrow H/Y$ is the quotient map. Hence by 5.3.1 we get $\phi f_n = \sum_{j=0}^{\infty} \phi a_n(\lambda - \lambda_0)^j$. Let $\epsilon_n = \sup\{||\phi f_n(\lambda)|| : |\lambda - \lambda_0| = r\}$. Then $\epsilon_n \longrightarrow 0$. By Cauchy’s inequality $||\phi a_n|| \leq \frac{\epsilon_n}{r^j}$. Next, for each $n$ and $j$, choose $b_{n_j} \in Y$ such that

$$||a_{n_j} - b_{n_j}|| \leq \frac{3\epsilon_n}{r^j(j+1)^2},$$  

(5.3.2)
from which it follows that

$$\|b_n\| \leq \|a_{nj}\| + \frac{3\epsilon_n}{r^j(j+1)^2}. \quad (5.3.3)$$

Define $h_n(\lambda) = \sum_{j=0}^{\infty} b_{nj}(\lambda - \lambda_0)^j$. By 5.3.3, $\{h_n\}$ is a $Y$-valued sequence of uniformly bounded analytic functions on $D$ and

$$\|h_n(\lambda) - f_n(\lambda)\| = \|\sum_{j=0}^{\infty} (a_{nj} - b_{nj})(\lambda - \lambda_0)^j\|
\leq \sum_{j=0}^{\infty} \|a_{nj} - b_{nj}\| r^j
\leq 3\sum_{j=0}^{\infty} \frac{\epsilon_n}{(j+1)^2}
= 3\epsilon_n \frac{\pi}{6} \longrightarrow 0.$$

Moreover, $(\lambda - T)f_n(\lambda) \to 0$ and 5.3.2 imply that

$$(\lambda - T)h_n(\lambda) = (\lambda - T)(h_n(\lambda) - f_n(\lambda)) + (\lambda - T)f_n(\lambda) \to 0$$
uniformly on all compact subsets of $D$. Since $T/Y$ has the property $(b\beta)$, $h_n(\lambda) \to 0$ uniformly on all compact subsets of $D$. By 5.3.2, $f_n(\lambda) \to 0$ uniformly on all compact subset of $D$. \hfill \Box

**Lemma 5.3.5.** If $S,T \in \mathbb{B}(H)$ are unitarily equivalent such that $T$ has the property $(b\beta)$, then $S$ has the property $(b\beta)$.

**Proof.** Let $\{f_n\}$ be a uniformly bounded sequence of bounded functions in $\hat{O}(U,H)$ such that $(\lambda - S)f_n(\lambda) \to 0$. Since $S,T$ are are unitarily equivalent, there is unitary operator $V$ such $V^*(\lambda - T)Vf_n(\lambda) \to 0$. Since $V^*$ unitary, $(\lambda - T)Vf_n(\lambda) \to 0$. Since $T$ has the property $(b\beta)$, $Vf_n(\lambda) \to 0$. Since $V$ unitary, $f_n(\lambda) \to 0$. Thus $S$ has the property $(b\beta)$. \hfill \Box
We have mentioned earlier that the property \((b\beta)\) is inherited on restriction to invariant subspace. The following example shows that the converse need not be true.

**Example 5.3.4.** Let \(H = \ell^2(\mathbb{N}) \oplus \ell^2(\mathbb{N})\) and \(T_1, T_2 : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N})\) be the right, left shift operators, respectively. Let \(T = T_1 \oplus T_2\). Then \(T\) does not have SVEP \([89]\). So \(T\) does not have property \((b\beta)\). Now if \(Y = \{0\} \oplus \ell^2(\mathbb{N})\), then \(Y \in \text{Lat}(T)\), and \(T/Y\) is unitarily equivalent to \(T_1\). Since \(T_1\) is a restriction of bilateral shift on \(\ell^2(\mathbb{Z})\), \(T_1\) has the property \((b\beta)\). Hence \(T/Y\) has property \((b\beta)\).

**Lemma 5.3.6.** Let \(T \in \mathbb{B}(H)\) be such that \(T\) is onto but it is not one to one. Then \(T\) does not have property \((b\beta)\).

*Proof.* By Theorem 2 of \([43]\), \(T\) does not have SVEP. So \(T\) does not have property \((b\beta)\). \(\square\)

As in the case of SVEP and the property \((\beta)\), the property \((b\beta)\) is neither preserved under the adjoint operation nor the sum operation. For the first one, we just take the right shift operator on \(\ell^2(\mathbb{N})\) whose its adjoint is the left shift. For the sum operation, take any operator without property \((b\beta)\) or take \(T\) be a left shift operator. then \(\frac{1}{2}(T + T^*)\) and \(\frac{1}{2}(T - T^*)\) are normal and so they have property \((b\beta)\). But \(T = \frac{1}{2}(T + T^*) + \frac{1}{2}(T - T^*)\) dose not have property \((b\beta)\) by [43, 69].

**Lemma 5.3.7.** Let \(T \in \mathbb{B}(H)\). Then the following hold.

1. If \(T\) has bounded left inverse but it is not onto, then \(T^*\) does not have the property \((b\beta)\).
2. If \( T \) has right inverse but has not left inverse, then \( T \) does not have the property \((b\beta)\).

\textbf{Proof.} (1). Since \( T \) has bounded left inverse, \( R(T) \) is closed. Hence \( R(T^*) \) is also closed. Since \( T \) is one to one, \( T^* \) onto. Since \( T \) is not onto, \( T^* \) is not one to one. Hence by Lemma 5.3.6, \( T^* \) does not have property \((b\beta)\).

(2) the hypothesis here just an other way of Lemma 5.3.6. \( \Box \)

\textbf{Proposition 5.3.8.} Let \( T \in B(H) \). If \( T \) has the property \((b\beta)\) and \( T^{-1} \) exists, then \( T^{-1} \) also has the property \((b\beta)\).

\textbf{Proof.} Let \( \{f_n\} \) be an uniformly bounded sequence of bounded functions in \( \hat{O}(U, H) \), \( T \) has property \((b\beta)\) and suppose \( (\lambda - T^{-1})f_n(\lambda) \longrightarrow 0 \) uniformly on every compact subset of \( U \). Let \( K \subset U \) be a compact subset. Now \( 0 \notin \sigma(T^{-1}) \) and so there exists an \( r > 0 \) such that \( \{\lambda \in \mathbb{C} : |\lambda| \leq r\} \) is disjoint from \( \sigma(T^{-1}) \).

Define the compact subsets \( K_1 \) and \( K_2 \) by
\[
K_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq r\} \cap K,
K_2 = \{\lambda \in \mathbb{C} : |\lambda| \geq r\} \cap K
\]
and we note that \( K = K_1 \cup K_2 \). Since \( K_1 \subset \rho(T^{-1}) \), \( f_n(\lambda) \longrightarrow 0 \) uniformly for \( \lambda \in K_1 \). For \( \lambda \in W = \{\lambda \in \mathbb{C} : |\lambda| \geq \frac{r}{2}\} \cap U \), we have
\[
(\lambda I - T^{-1})f_n(\lambda) = -(\lambda^{-1}I - T)(\frac{T}{\lambda^2})^{-1}f_n(\lambda).
\]
Therefore \( (\lambda^{-1}I - T)(\frac{T}{\lambda^2})^{-1}f_n(\lambda) \longrightarrow 0 \) uniformly on \( W \).

Define \( g_n(\lambda) = (\lambda T)^{-1}f_n(\lambda) \ (n = 1, 2, ...) \). Then \( \{g_n(\frac{1}{\lambda})\} \) is uniformly bounded on \( W \). Thus for \( \lambda \) in a compact subset of \( W \), \((\lambda^{-1}I - T)g_n(\frac{1}{\lambda}) \longrightarrow 0 \) uniformly and since \( T \) has property \((b\beta)\), \( g_n(\frac{1}{\lambda}) \longrightarrow 0 \) uniformly for \( \lambda \) in \( K_2 \subset W \). Now
for \( \lambda \in K_2 \), the operator \( \frac{T}{\lambda} \) is bounded so we have \( f_n(\lambda) = \frac{T}{\lambda}g_n(\frac{1}{\lambda}) \longrightarrow 0 \) uniformly. thus \( f_n(\lambda) \longrightarrow 0 \) uniformly in \( K = K_1 \cup K_2 \). Hence \( T^{-1} \) has the property \((b\beta)\).

\[ \]

**Theorem 5.3.9.** Let \( T \in \mathcal{B}(H) \) has the property \((b\beta)\). Then the following hold.

1. The operator \( \alpha T \) has the property \((b\beta)\) for any scalar \( \alpha \in \mathbb{C} \).
2. The operator \( (\alpha I - T) \) has the property \((b\beta)\) for any scalar \( \alpha \in \mathbb{C} \).
3. The resolvent operator \( (\lambda I - T)^{-1} \) has the property \((b\beta)\) for any \( \lambda \in \rho(T) \).

**Proof.** The proofs of (1) and (2) are similar, we shall prove only (2).

Let \( \{f_n\} \) be an uniformly bounded sequence of bounded functions in \( \hat{O}(U,H) \) and \( \alpha \in \mathbb{C} \). Suppose that \( (\lambda I - (\alpha I - T))f_n(\lambda) \longrightarrow 0 \) uniformly on compact subset of \( U \). For \( \lambda \in U - \alpha \), for \( n = 1, 2, ..., \) define

\[ g_n(\lambda) = f_n(\lambda + \alpha). \]

Then \( g_n \) are uniformly bounded sequence of bounded analytic functions on \( U - \alpha \) and \( ((\lambda - \alpha) - T)g_n(\lambda - \alpha) \longrightarrow 0 \) uniformly on all compact subsets of \( U \). Since \( T \) has property \((b\beta)\), for any compact subset \( K \subset U \), \( g_n(\lambda - \alpha) \longrightarrow 0 \) uniformly on \( K \). So \( f_n(\lambda) = g_n(\lambda - \alpha) \longrightarrow 0 \) uniformly on \( K \).

We can get part (3) by part (2) and Theorem 5.3.8. \( \square \)

We have seen that if \( T \in \mathcal{B}(H) \) has the property \((b\beta)\), then \( \lambda \in \rho(T) \) if and only if \( R(\lambda I - T) = H \).
Theorem 5.3.10. A bounded linear operator $T_1 \oplus T_2$ on a Hilbert space $H_1 \oplus H_2$ has the property $(b\beta)$ if and only if $T_i$ has the property $(b\beta)$ on $H_i$ ($i = 1, 2$).

Proof. Suppose that $T_1 \oplus T_2$ has the property $(b\beta)$ on a Hilbert space $H_1 \oplus H_2$. Let $\{f_n\}$ be an uniformly bounded sequence of bounded functions in $\hat{O}(U, H_1)$ and $(\lambda I - T_1)f_n(\lambda) \longrightarrow 0$ uniformly on every compact subset of $U$. Let $I_i$ be the identity operator on $H_i$ ($i = 1, 2$) and define $g_n : U \longrightarrow H_2$ for $n = 1, 2, \ldots$, by $g_n(\lambda) = 0$. Thus we have $(\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2))(f_n(\lambda) \oplus g_n(\lambda)) \longrightarrow 0$ uniformly on every compact subset of $U$. Since $T_1 \oplus T_2$ has property $(b\beta)$, $(f_n(\lambda) \oplus g_n(\lambda)) \longrightarrow 0$ uniformly in every compact subset of $U$. By definition of convergence in $H_1 \oplus H_2$, $f_n(\lambda) \longrightarrow 0$ uniformly in every compact subset of $U$, and hence $T_1$ has the property $(b\beta)$. By the same argument we can show that $T_2$ has the property $(b\beta)$.

Conversely, Assume $T_i$ on $H_i$ ($i = 1, 2$), have the property $(b\beta)$. Let $\{f_n^1 \oplus f_n^2\}$ be an uniformly bounded sequence of bounded functions in $\hat{O}(U, H_1 \oplus H_2)$ with $\{f_i^i\}$ be an uniformly bounded sequence of bounded functions in $\hat{O}(U, H_i)$ and $(\lambda(I_1 \oplus I_2) - (T_1 \oplus T_2))(f_n^1(\lambda) \oplus f_n^2(\lambda)) \longrightarrow 0$ uniformly on every compact subset of $U$. Since $T_i$ has property $(b\beta)$, $f_n^i(\lambda) \longrightarrow 0$ uniformly on every compact subset of $U$ for ($i = 1, 2$). Therefore $f_n^1(\lambda) \oplus f_n^2(\lambda) \longrightarrow 0$ uniformly on every compact subset of $U$. \qed
5.4 Concluding Remarks

Bishop’s property ($\beta$) is a central concept in the theory of generalized spectral decomposition. In this chapter we have studied some operators with the property ($\beta$). We have proved that, if $T$ is $n$-normal operator and $f \in H(\sigma(T))$, then $f(T)$ is decomposable. Also we have studied the property ($\beta$) for $T_1, T_2$ in $\mathcal{B}(H)$ with $C^m(T_1, T_2)(I) = 0$. We have studied the property ($\beta$) for an $n \times n$ triangular operator matrix $T$. We also have introduced a new concept of local spectral theory which it is called bounded Bishop’s property (abbrev. $T$ has the property ($b\beta$)). We have seen that this property fits strictly between the single valued extension property and the property ($\beta$). We also have shown that the property ($b\beta$) is stable under the inverse operation and direct sum of operators.