CHAPTER 3

DESIGN OF EXPERIMENTS AND RESPONSE SURFACE METHODOLOGY

3.1 Introduction

Experiments are performed by investigators in every field to discover something about a particular process or system. The word “experiment” denotes the type of study in which the investigator deliberately introduces certain changes in a process and makes observations or measurements in order to evaluate and compare effect of different changes. These changes are called treatments. Examples of treatments are different pulse current, pulse-on-time and gap voltage used to evaluate machinability of carbon silicon carbide composite. Machinability is the response in which the investigator is interested. In engineering, experimentation plays an important role in new product design, manufacturing process improvement and development.

In preliminary work, the objective may be simply to discover whether the applied treatment produces any measurable responses, while at a later stage in research the purpose maybe to verify or disprove certain hypothesis that have been put forward about the directions and sizes of the response to treatments. In applied work, measurement of the size of the response is often important to confirm the usefulness of a new treatment.
Modern concepts of experimental design are due primarily to R.A. Fisher, who developed them from 1919 to 1930 in planning of agriculture field experiments at Roth Amsted Experimental Station in England. He recognized that flaws in the way the experiment was performed and data generated often hampered the analysis of data from the system.

The main features of Fisher’s approach are as follows:
1. The requirement that an experiment itself furnishes a meaningful estimate of underlying variability to which the measurements of response to treatments are subject.
2. The use of randomization to provide these estimates of variability.
3. The use of blocking in order to balance the known extraneous sources of variation.
4. The principle that the statistical analysis of result is determined by the way in which the experiment was conducted.
5. The concept of factorial experimentation, which stresses advantages of investigating the effects of different factors or variables in single complex experiment, instead of devoting a separate experiment to each factor.

The second era of statistical design was catalyzed by the development of response surface methodology (RSM) by Box and Wilson. They recognized that in industrial experiments (1) the response variable can be observed immediately i.e. immediacy and (2) the experimenter can quickly learn crucial information i.e. sequentially [59]. Owing to inadequate training in basic statistical concepts, lack of computing resources and user friendly statistical software to support the application of statistically designed experiments, wide spread use of these methods in engineering did not take place.
During the third era of statistical design, the work of Genichi Taguchi had a significant impact on expanding the interest and use of designed experiments. Design of experiments became more widely used in automotive and aerospace manufacturing and many other industries. The successful integration of good experimental design practice into engineering and science is a key factor for future industrial competitiveness.

### 3.2 Role of Experimental Design

In general, experiments are conducted to study the performance of systems and processes. The model shown in Figure 3.1 represents the processes or system. Process can be visualized as a combination of machines, methods, people and other resources that transform some input into an output that has one or more observable responses. Some of the process variables $x_1, x_2, \ldots, x_p$ are controllable, whereas other variables $z_1, z_2, \ldots, z_p$ are uncontrollable. The objective of the experiment may include the following:

1. Determining the variables, which are most influential on the response ‘y’
2. Determining where to set the influential $x$’s so that $y$ is almost near the desired nominal value.
3. Determining where to set the influential $x$’s so that the variability in $y$’s is small.
4. Determining where to set the influential $x$’s so that the effect of the uncontrollable variables $z_1, z_2, \ldots, z_p$ are minimized.
In seeking an objective, the investigator selects the best model among the set of possible models and then estimates the parameters in the selected model. The investigator may not know what variables to measure, what ranges to be used and what series of experiments to run, till the experimental program is at least partially completed. In practice one plans one or more experiments, carries them out, analyses the result and modifies the experimental plan accordingly and outlines the strategy of experimentation as in Figure 3.2. One factor at a time in experimentation is the most widely used practice. This method consists of selecting a starting point, or baseline set of levels, for each factor then successively varying each factor over its range with the other factors held constant at the baseline level. This data is used to get the effect of individual variables on the responses and to select the optimal combination of parameters. The major disadvantage of one factor at a time strategy is that it fails to consider any possible interaction between the
factors. An interaction is the failure of one factor to produce the same effect on the response at different levels of another factor. One-factor at a time experiments are always less efficient than other methods based on a statistical approach to design.

It is most efficient to estimate the effects of several variables simultaneously. Each experimental design will contain a group of experimental runs. A new design is not necessarily employed for each iterative cycle. Sometimes a sequence of cycles will occur in which same data are controlled by same hypothesis. When it is not clear what modification should be made to an unsatisfactory hypothesis or alternatively, when further confirmation of an apparently satisfactory hypothesis is needed, additional data will be required. These are generated by further experimental runs arranged by new experimental design. In Figure 3.3 additional data from the jth experimental design D₁ is confronting deduced consequences of the ith hypothesis Hᵢ. The design is represented by movable window through which certain aspects of true state of nature, more or less distorted by noise, may be observed. Updating the hypothesis and comparing the deduced states of nature with actual data can lead to convergence on truth.
Design of experiment is the procedure of selecting a number of trials and conditions for running them and solving the problem with the desired precision [4]. Efficient methods of experimental design enable to obtain answers which are little affected by experimental error. Sensitive data analysis indicates the suitability of the current hypothesis and suggests new modifications to be effected. If the experimental design is poorly chosen, then resulting data does not contain much information and not much can be extracted, no matter how thorough or sophisticated the analysis may be.
3.3 Factorial Experimentation

Many experiments involve the study of the effects of two or more factors. In general, factorial designs are most efficient for this type of experiment. By factorial design, we mean that in each complete trial or replication of the experiment all possible combinations of experiments involving several factors where it is necessary to study the joint effect of the factors on the response. For example, if there are ‘a’ levels of factor A and ‘b’ levels of factor ‘B’, then each replicate contains all ‘ab’ treatment combinations. The investigator intends to examine different types of variables on response. In multifactor experiments, a single factor can be varied at a time.

Any discussion on methods of designing and analyzing experiments depends on the model set to represent the desire of experimenter. A model consists of some kind of representation of relationship between the imposed treatments and response. These relationships called parameters are to be estimated from the experimental results.

The advantage of the factorial approach is that it makes economical use of resources and provides convenient data for studying inter relationships of the effects of different variables [3,4,15,19,40,45,51,59,61,64].

For illustration, let us consider a response which is affected by these variables called factors A, B and C. Each factor can take any one of the two designed values, called levels. Denote the two levels of A by a_1 and a_2 and similarly the levels of B and C by b_1, b_2 and c_1 and c_2 respectively. The treatment consists of all possible combinations of the
levels of the factors. There are eight possible combinations i.e. it is a $2^3$ design as in Table 3.1. Suppose that one observation is taken on each of the eight combinations. The comparison $(y_2 - y_1)$ that is the difference between the observations for combinations $(2)$ and $(1)$ is clearly an estimate of difference in responses, when the factor $A$ changes from $a_2$ to $a_1$ while the factors $B$ and $C$ are held fixed at their lower levels. Similarly $(y_4 - y_3)$ gives the difference in response when factor $A$ changes from $a_2$ to $a_1$ while $B$ is held at a higher level and $C$ is at a lower level. The difference $(y_6 - y_5)$ and $(y_8 - y_7)$ supply two further differences in responses when the factor ‘$a$’ changes from $a_2$ to $a_1$. These are shown in Table 3.1. The average at these four differences provides change in responses when $A$ changes from $a_2$ to $a_1$. This average is based on two samples of size four and is called the main effect of $A$.

Table 3.1 Treatment Combination and Responses

<table>
<thead>
<tr>
<th>Treatment Combinations</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) $a_1$ $b_1$ $c_1$ .... $y_1$</td>
<td>(5) $a_1$ $b_1$ $c_2$ .... $y_5$</td>
</tr>
<tr>
<td>(2) $a_2$ $b_1$ $c_1$ .... $y_2$</td>
<td>(6) $a_2$ $b_1$ $c_2$ .... $y_6$</td>
</tr>
<tr>
<td>(3) $a_1$ $b_2$ $c_1$ .... $y_3$</td>
<td>(7) $a_1$ $b_2$ $c_2$ .... $y_7$</td>
</tr>
<tr>
<td>(4) $a_2$ $b_2$ $c_1$ .... $y_4$</td>
<td>(8) $a_2$ $b_2$ $c_2$ .... $y_8$</td>
</tr>
</tbody>
</table>

Similarly effect of $B$ and $C$ may be found by taking average of differences as shown in Table 3.2. Thus testing of eight treatment combinations in a factorial experiment gives the estimate effect of each of the factors $A$, $B$ and $C$ based on a sample size of four. If separate experiments were devoted to each factor as in the ‘one variable at a time’
approach, 24 combinations would have to be tested in order to furnish estimates based on
the samples of size four. The economy in the factorial approach is achieved as every
observation contributes information on all factors. If factors interact one at a time,
experimental approach may lead to wrong conclusions. When interactions exist, their
nature being unknown, factorial design is necessary to avoid misleading conclusions. The
factorial design is an efficient method. An efficient method is, one which obtains the
required information with required degree of precision and with minimum expenditure of
effort. With large number of factors to be analyzed fractional factorial approach is
advantageous.

Table 3.2 Main Effects Table

(a) Main effect of A

<table>
<thead>
<tr>
<th>S.No.</th>
<th>Difference</th>
<th>Levels</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>B</td>
</tr>
<tr>
<td>1.</td>
<td>$y_2 - y_1$</td>
<td>-</td>
</tr>
<tr>
<td>2.</td>
<td>$y_4 - y_3$</td>
<td>+</td>
</tr>
<tr>
<td>3.</td>
<td>$y_6 - y_5$</td>
<td>-</td>
</tr>
<tr>
<td>4.</td>
<td>$y_8 - y_7$</td>
<td>+</td>
</tr>
</tbody>
</table>

Main Effect of $A = \frac{1}{4}(1+2+3+4)$
Main effect of B = \( \frac{1}{4}(1+2+3+4) \)

Main effect of C = \( \frac{1}{4}(1+2+3+4) \)

In general if there are k factors with 2 levels for each factor, then the number of observations required to depict such a design is \( 2 \times 2 \times 2 \times \cdots \times 2 = 2^k \) and is called a \( 2^k \) factorial design. These two values may be quantitative, such as two values of temperature, pressure or time or may be qualitative such as two machines, two operations, the ‘high’
and ‘low’ levels of a factor, etc. The $2^k$ designs are particularly useful in the early stages of the experimental work, when there are likely to be many factors to be investigated. It provides the smallest number of runs with which $k$ factors can be studied in a complete factorial design. As there are only two levels for each factor, we assume that the response is approximately linear over the range of the factor levels chosen. In systems subjected to experiments, this is often a reasonable assumption.

### 3.4 A $2^3$ Factorial design

Though the general methods for analysis of factorial designs are discussed, there are several special cases of the general factorial design that are important because they are widely used in research work and also because they form the base for other designs of considerable practical relevance. For convenience, a factor is denoted by a capital letter, when all the factors are at their lower level, the treatment combination is represented by symbol (l) as in Table 3.3 if any one of the factors is at its higher level, its small number represents the treatment combination. The symbols in the first column of Table 3.3 give the conventional representation of eight treatment combinations. Presence of small letters means, the factor it represents stands at its higher level, while absence of the letter means that the factor is at its lower level. The treatment ‘bc’ denotes the higher levels of B and C and lower levels of A. The corresponding responses $y_1$ to $y_8$ for the various treatment combinations are also given. Figure 3.4 shows the geometrical representation of treatment combinations. The corners of the cube represent the eight treatment combinations.
### Table 3.3 Symbols for Treatment Combinations

<table>
<thead>
<tr>
<th>Sr.No.</th>
<th>Treatment Combination</th>
<th>Levels of Factors</th>
<th>Response</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>A</td>
<td>B</td>
</tr>
<tr>
<td>1</td>
<td>(1)</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>2</td>
<td>A</td>
<td>+</td>
<td>–</td>
</tr>
<tr>
<td>3</td>
<td>B</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>Ab</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>5</td>
<td>C</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>6</td>
<td>Ac</td>
<td>+</td>
<td>–</td>
</tr>
<tr>
<td>7</td>
<td>Bc</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>8</td>
<td>Abc</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

+ Higher Level  - Lower level

### 3.4.1 Calculation of main effects

The meaning of the ‘effect’ of a factor is the change in the response as the factor moves from lower level to higher level. To find the main effect of the factor, say A the response corresponding to all treatments containing ‘a’ and all those not containing A are averaged.

Average response at higher level of A = \( \frac{y_2 + y_4 + y_6 + y_8}{4} \)

Average response at lower level of A = \( \frac{y_1 + y_3 + y_5 + y_7}{4} \)

Main effects of A = \( \frac{((y_2 + y_4 + y_6 + y_8)/4) - ((y_1 + y_3 + y_5 + y_7)/4)}{4} \)

\[
= \frac{(y_2 + y_4 + y_6 + y_8) - (y_1 + y_3 + y_5 + y_7)}{4} \quad (3.1)
\]
Main Effects of \( B = \frac{(-y_2 + y_4 - y_6 + y_8 - y_1 + y_3 - y_5 + y_7)}{4} \) \hspace{1cm} (3.2)

Main Effects of \( C = \frac{(-y_2 + y_4 + y_6 + y_8 - y_1 - y_3 + y_5 + y_7)}{4} \) \hspace{1cm} (3.3)

There is a convenient order to write the treatment combinations and effects. For one factor \( a \), for two factors add \( b \), \( ab \) derived by multiplying the first two by additional letter \( b \). For three factors add \( c \), \( ac \), \( bc \), \( abc \), derived by multiplying the first four additional factor \( c \) and so on.

![Geometrical representation of treatment combinations](image)

Figure 3.4 Geometrical representation of treatment combinations

In a similar way the expression for other main effects and interactions are given in Table 3.4. In this table it may be noted that corresponding to effect \( A \), treatment combination containing ‘a’ have the plus sign and all not containing ‘a’ have the minus sign. All other
main effects can be written out in the same manner. The signs for any interaction are
equal to those obtained by multiplying together the signs for the main effects
corresponding to the letters in the interaction. Table 3.4 gives the level of the factors in
each treatment combination and appropriate signs to apply to the response for the purpose
of estimating the main and interacting effects.

Table 3.4 Table of Signs for Calculating Effects in $2^3$ Experiments

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Total</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_1x_2$</th>
<th>$x_2x_3$</th>
<th>$x_1x_2x_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>a</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>b</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>ab</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>c</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>ac</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>bc</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>abc</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

**3.4.2 Interaction**

Variables A and B may not have additively and are therefore said to be interacting.

$$AB\text{ interaction} = \frac{(y_1 + y_4 + y_5 + y_8)}{4} - \frac{(y_2 + y_3 + y_6 + y_7)}{4} \quad (3.4)$$

Similarly the interactions AC and BC can be obtained. If the value of a two factor
interaction is different at different level of third factor, there will be a three-factor
interaction ABC. The procedure of $2^3$ factorial can be readily extended to $2^k$ factorial design dealing with factors.

### 3.5 Interactive Design and Analysis of Factorial Experiments

Pre-world war II era, application of design of experiments was mainly to the bio-agriculture field. Experimental design was constrained by computing considerations. Much care was taken to employ an orthogonal fraction even at the expense of a large number of observations. If orthogonality could not be achieved, designs were employed that entailed computationally simple analysis.

At present the major application area of designed experiment has shifted from bio-agriculture to industry. The high speed electronic computer provided enough computing power to handle the analysis of any design that might realistically be employed in research. Two factors have increased the usefulness of the computer in design, construction and subsequent data analysis.

The first factor is advent of interactive computing. One advantage of interactive computing over batch processing is the greater continuity of thought it offers to the user. The second factor facilitating computer aided design work is development of programming languages specifically to handle vectors, matrices and other arrays. Standard softwares are also available to solve design of experiment problems. Interactive design of factorial experiments will have applications in the industrial experimentation.
3.6 Two - Level Factorial Design

As the number of factors in a $2^k$ factorial design increases, the number of runs required for a complete replicate of the design rapidly outgrows the resources of most experimenters. For example, a complete replicate of the $2^6$ design requires 64 runs. In this design only 6 of the 63 degrees of freedom correspond to main effects and only 15 degrees of freedom corresponds to two-factor interactions.

If the experimenter can reasonably assume that certain high-order interactions are negligible, information on the main effects and low-order interactions may be obtained by running only a fraction of the complete factorial experiments. Such a design is called a fractional factorial design. These fractional factorial designs are among the most widely used types of design for product and process design and for process improvement. A major use of fractional factorial design is in screening experiments to identify the factors that have large effects. The factors that are identified as important are then investigated more thoroughly on subsequent experiments.

If the experimenter cannot afford to run all $2^3 = 8$ treatment combinations of a three factor two level situation and can afford four runs, then he can use a one-half fraction of the $2^3$ design, i.e. $2^{3-1} = 4$ treatment combinations. A one-half fraction of the $2^3$ design is often called $2^{3-1}$ design.
3.7 Response Surface Methodology

Response surface Methodology (RSM) is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable or response and the goal is to optimize this response. We denote the independent variables by $x_1, x_2, x_3, \ldots, x_k$. It is assumed that these variables are continuous and controllable by the experimenter with negligible error. The response ‘$y$’ is said to be a random variable. RSM is used for the design and analysis of experiments, it seeks to relate an average response to the value of quantitative variables that effect the response. RSM answers different kinds of questions, such as the following [59].

1. How is a particular response affected by a given set of input variables over some specified region of interest?
2. To what level are the inputs to be controlled, to give a product simultaneously satisfying desired specifications?
3. What values of inputs will yield a maximum for a specific response and what is the nature of response surface close to the maximum?

The relationship between the dependent variable and the independent variable can be represented as:

$$y = f(x_1, x_2, x_3, \ldots, x_k) + \varepsilon$$  \hspace{1cm} (3.5)

where, $\varepsilon$ represents the noise or error observed in the response ‘$y$’.

If we denote the expected response by

$$E(y) = f(x_1, x_2, x_3, \ldots, x_k) = \eta$$  \hspace{1cm} (3.6)

then, the surface is represented by
is called the response surface. This surface is drawn between some response such as material removal rate whose levels are denoted by ‘m’ and number of quantitative variables or factors whose levels are denoted by \( x_1, x_2, x_3, \ldots, x_k \). The feature of the surface of greatest interest is often the value of variables \( (x_1, x_2, x_3, \ldots, x_k) \) for which ‘m’ is the maximum or minimum. In most RSM problems, the form of the relationship between the response and the independent variables is unknown. Thus, the first step in RSM is to find suitable approximation for the true functional relationship between \( y \) and the set of independent variables. Usually, a low-order polynomial in some region of the independent variable is employed.

If the response is well modeled by a linear function of the independent variable, then the approximating function is the first-order model \([3, 4, 15, 19, 40, 45, 51, 59, 61, 64]\).

\[
y = \beta_0 x_0 + \beta_1 x_1 + \cdots + \beta_k x_k + \varepsilon \tag{3.8}
\]

If there is a curvature in the system, then a polynomial of higher degree must be used, such as second order model.

\[
y = \beta_0 x_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_i x_i^2 + \sum \sum_{i<j} \beta_{ij} x_i x_j + \varepsilon \tag{3.9}
\]

Almost all RSM problems use one or both of these models. Of course, it is unlikely that a polynomial model will be reasonable approximation of the true functional relationship over the entire space of the independent variables, but for a relatively small region they usually work.
The method of least squares is used to estimate the parameters in the approximating polynomials. The response surface analysis is then performed using the fitted surface. If the fitted surface is an adequate approximation of the true response function, then analysis of the fitted surface will be approximately equivalent to analysis of the actual system. The model parameters can be estimated most effectively if proper designs are used to collect the data. Designs for fitting response surface are called response surface designs.

3.8 First Order Design

These designs do not provide any estimate of the experimental error variance. This can be obtained by:

1. replication of the whole experiment
2. by the use of an estimate from previous experimentation, if there is convincing evidence that error variance remains stable through time
3. adding to the $2^k$ factorial a number of tests made at the point at which all ‘x’ have the value ‘0’ in the coded scale.

The linear equation in $k(x)$ variables contains $(k+1)$ regression coefficients that must be estimated. The smallest experiment to which a linear equation can be fitted is one that has $(k+1)$ observations.

If there is no lack of fit and sufficient precision is obtained, on the basis of this, direction of steepest ascent is determined and exploration is continued. Otherwise try with transformations of one or more variables and response. Careful blocking and expanding
the size of design can increase precision. If satisfactory fit and precision is not obtained then second order design are to be restored.

3.9 Second-Order Design

The general form of the second-degree polynomial can be represented as equation 4.0.

\[
y = (b_0 + b_1x_1 + \cdots + b_kx_k) + (b_{12}x_1 + b_{13}x_1 + \cdots + b_{(k-1)k}x_{(k-1)}x_k) + (b_{11}x_1^2 + b_{22}x_2^2 + \cdots + b_{kk}x_k^2) \tag{4.0}
\]

In order to estimate the regression coefficients in this model, each variable must take at least three different levels. Use of factorial designs of \(3^k\) will be necessary in this case. Main disadvantage of a \(3^k\) factorial design is that with more than three variables, experiments become large. Further, Box and Wilson pointed out that co-efficient of \(b_{11}, b_{22}, \ldots, b_{kk}\) of the squared terms are estimated with relatively low precision.

Box and Wilson developed a new design for fitting the second order response surface. The composite designs are constructed by adding further treatment combinations to the first order design. Central composite design consist of additional \((2k+1)\) treatments, \((0,0,0------0); (-\alpha,0,0------0); (\alpha,0,0------0); ( 0,-\alpha,,0,------0);\ldots;----------(0,0,0------\alpha). Total number of treatment combinations is \((2^k+2k+1)\). The value of ‘\(\alpha\)’ can be chosen to make the regression co-efficient orthogonal to one another. Central composite design can be fitted into a sequential program of experimentation.
3.9.1 Non-Central Composite Design

This design has \( k \) extra points, one for each factor. Non-central Composite Design is used when \( 2^k \) factorial experiments have suggested that the point of maximum response is near to one of the factor combinations than to the centre. For three-factor systems, central composite and non-central composite designs are illustrated in Figure 3.5.

3.9.2 Rotatable Second Order Design

The design used for fitting the second order response surface should be easy to compute. Box and Hunter proposed the criterion of rotatability. In a rotatable design, standard error is the same for all points.

Box and Hunter showed that a rotatable design is obtained by making test at \( n_s \) points equally spaced around circumference of a circle in the \( x_1, x_2 \) plane with centre (0,0) plus one or more tests at the centre.

The points on the circumference lie at the vertices of a regular polygon inscribed in a circle. Since there are six regression coefficients to be determined when \( k = 2 \), the smallest design consists of a pentagon plus one point at the centre. The replicated points at the centre have two purposes. They provide \( (n_0-1) \) degrees of freedom for estimating the experimental error and they determine the precision of \( y \) at the centre. If there are many replications of the centre point, the standard error of \( y \) is low at the centre and with a few replications at the centre standard error of \( y \) may be greater. As a compromise, Box and Hunter suggested that the number of centre points be chosen so that standard error of \( y \) is
approximately the same at the centre as at all points on circle with radius ‘1’ in coded unit. For rotatability, the axis arms of measure polytope should be $\alpha = 2^{k/4}$. The total number of points required for rotatable central composite design is $2^k + 2^k + n_0$, where $n_0$ equals the number of points at the origin.

![Central Composite Design](image)

Figure 3.5 Central and non-central composite design

### 3.9.3 Determination of Factor Levels for optimum Condition

At the outset, the experimenter must decide which factors are to be included in the experiments. Sometimes there are initially as many as dozen or more factors that might influence the response. Some preliminary weeding out of factors that seem likely to be of minor importance is necessary. The range within which the level of each factor is to be varied must also be selected [3,4,15,19,40,45,51,59,61,64].
3.10 The Method of Steepest Ascent

RSM is a sequential procedure. Often when we are at a point on the response surface that is remote from the optimum, such as the current operating conditions in Figure 4.3, there is little curvature in the system and the first order model will be appropriate. The objective is to lead the experimenter rapidly and efficiently along a path of improvement toward the vicinity of the optimum. Once the region of the optimum has been found, a more elaborate model, such as the second-order model, may be employed, and an analysis may be performed to locate the optimum. The analysis of a response surface can be thought of as ‘climbing a hill’, where the top of the hill represents the point of maximum response. If the true optimum is a point of minimum response, then we may think of ‘descending into a valley’.

The method of steepest ascent is a sequential procedure for moving sequentially along the path of steepest ascent, that is, in the direction of the maximum increase in the response. Of course, if minimization is desired, then we call this technique the method of steepest descent [59]. The eventual objective of the RSM is to determine the optimum operating conditions for the system or to determine a region of the factor space in which operating requirements are satisfied. When we are remote from the optimum, we usually assume that a first order model is an adequate approximation to the true surface in a small region of the x’s.

Box and Wilson proposed the method of steepest ascent/descent. The maximum is located by means of a series of experiments, each planned from the result of the proceeding ones.
At the end of each experiment a polynomial approximation to response surface is fitted to the results and is used to determine the nature of the next experiment. The first experiment has two purposes:

1. To fit linear equation
2. To test whether the linear approximation fits within the limits of experimental errors.

The $2^k$ factorial or functional designs are useful for this purpose.

When the first experiment is complete the region of experimentation is shifted to another set of level of $x$’s. This set is to be chosen so that maximum expected increase in response occurs. If the centre of the first experiment is taken as the origin, the problem is to move from the origin, with $x$ co-ordinates $(0,0,0----0)$ to the point say $P$, with co-ordinates $(x'_1,x'_2--------x'_k)$, so that the response $\Phi(x'_1,x'_2--------x'_k)$ is maximized.

For the fitted first order model, given by equation 4.8, the contours of $\eta$ are a series of parallel lines. The direction of steepest ascent is the direction in which $\eta$ increases more rapidly. This direction is parallel to the normal to the fitted response surface. We usually take as the path of steepest ascent the line through the centre of the region of interest and normal to the fitted surface.

Thus the steps along the path are proportional to the regression co-efficients $(\beta'_i)$. The experimenter based on process knowledge or other practical considerations determines the actual step size. Experiments are conducted along the path of steepest ascent until no further increase in response is observed.
\[ \eta = \beta_o + \sum_{i=1}^{k} \beta_i x_i \]  

(4.1)

The change in the response depends upon the size of the jump that is made from 0 to point P. By geometrical analogy, the distance r from 0 to P is defined as:

\[ r = \sqrt{\left( x_{1}^2 + x_{2}^2 + x_{3}^2 \ldots + x_{k}^2 \right)} \]  

(4.2)

Path of this steepest ascent is determined. Exploration is continued with new experiments. In course of time a situation is reached in which \(2^k\) factorial designs give one of the following:

1. The linear equation appears to fit, but all coefficients \(b_i\) are small. His is the indication of approach of a plateau.

2. The lack of fit terms shows that the linear approximation is inadequate. This indicates that the experiment is carried out in the region in which curvature of the surface exists.

For further exploration second order designs are used. Second order designs are reconstructed by adding additional points to the last \(2^k\) factorial experiments.
3.11 Conclusion

Design of experiments is the basis for the methods used in this investigation to analyze and model the effect of various machining variables on the machinability parameters of carbon silicon carbide composite, which are of greater importance to process planners. Response Surface methodology is used to obtain the empirical models of machinability parameters in terms of the variables of machining after identifying their relative contribution. The method of steepest ascent/descent can be effectively used to identify the optimum values of the responses.