CHAPTER 3

DESIGN OF OBSERVER AND OBSERVER BASED NMPC FORMULATION

3.1 INTRODUCTION

In this chapter the observer design and the observer based NMPC design have been presented.

3.2 UNCONSTRAINED OBSERVER DESIGN

Consider a nonlinear system represented by the following nonlinear differential equations:

$$\dot{\boldsymbol{x}} = \bar{f}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) \tag{3.1}$$

$$\boldsymbol{y} = \bar{\mathrm{g}}(\boldsymbol{x},\boldsymbol{u},\boldsymbol{d}) \tag{3.2}$$

Equation (3.1) describes a deterministic system evolution and can be obtained from the material and energy balances of the process under consideration. Equation (3.2) describes the relationship between the measurements and the state variables. In order to describe a discrete nonlinear system, Equations (3.1) and (3.2) can also be functionally represented in discrete form as:

$$\boldsymbol{x}(\mathrm{k}) = \mathrm{f}\left[\boldsymbol{x}(\mathrm{k}-1), \boldsymbol{u}(\mathrm{k}-1), \boldsymbol{d}(\mathrm{k}-1)\right] + \boldsymbol{w}(\mathrm{k}-1) \tag{3.3}$$

$$\boldsymbol{y}(\mathrm{k}) = \mathrm{g}\left[\boldsymbol{x}(\mathrm{k}-1), \boldsymbol{u}(\mathrm{k}-1)\right] + \boldsymbol{v}(\mathrm{k}) \tag{3.4}$$

where, $\mathbf{x}(\mathrm{k})$ is the system state vector ($\mathbf{x}(\mathrm{k}) \in \mathrm{R}^{\mathrm{n}}$) , $\mathbf{u}(\mathrm{k})$ is the known deterministic system input ($\mathbf{u}(\mathrm{k}) \in \mathrm{R}^{\mathrm{m}}$), $\mathbf{d}(\mathrm{k})$ is the unmeasured disturbance input ($\mathbf{d}(\mathrm{k}) \in \mathrm{R}^{\mathrm{q}}$) , $\mathbf{w}(\mathrm{k})$ is the state noise ($\mathbf{w}(\mathrm{k}) \in \mathrm{R}^{\mathrm{p}}$), $\mathbf{y}(\mathrm{k})$ is the measured variable ($\mathbf{y}(\mathrm{k}) \in \mathrm{R}^{\mathrm{r}}$) and $\mathbf{v}(\mathrm{k})$ is the measurement noise ($\mathbf{v}(\mathrm{k}) \in \mathrm{R}^{\mathrm{s}}$).The parameter k represents the sampling instant and the symbol f is a (possibly nonlinear) state transition function and g is a (possibly nonlinear) measurement function. We assume that measurements are made at discrete sampling instants with sampling period T. Note that the d(t) term described (3.1) is assumed to be piecewise constant for $\mathrm{kT} \le \mathrm{t} < (\mathrm{k}+1)\mathrm{T}$.

The objective is to find the conditional probability density function $\mathrm{p}\left[\mathbf{x}(\mathrm{k})|\mathbf{Y}^{\mathrm{k}}\right]$, where $\mathbf{Y}^{\mathrm{k}}$ denotes the set of all the available measurements up to time instant k [Rawlings and Bakshi, 2006]. The posterior density function of the state constitutes the complete solution to the sequential estimation problem. The posterior density is estimated in two stages: prediction and update. In the prediction step, the posterior density $\mathrm{p}\left[\mathbf{x}(\mathrm{k}) \mid \mathbf{Y}^{\mathrm{k}-1}\right]$ at previous time step is propagated into the next time step through the transition density as follows:

$$\begin{aligned} &p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k-1}\right] = \\ &\quad \int p\left[\boldsymbol{x}(k) \mid \boldsymbol{x}(k-1)\right] p\left[\boldsymbol{x}(k-1) \mid \boldsymbol{Y}^{k-1}\right] d\boldsymbol{x}(k-1) \end{aligned} \tag{3.5}$$

The update stage involves the application of Bayes' rule as follows:

$$p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k}\right]=\frac{p\left[\boldsymbol{y}(k) \mid \boldsymbol{x}(k)\right]}{p\left[\boldsymbol{y}(k) \mid \boldsymbol{Y}^{k-1}\right]} \times p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k-1}\right] \tag{3.6}$$

$$p\left[\boldsymbol{y}(k) \mid \boldsymbol{Y}^{k-1}\right]=\int p\left[\boldsymbol{y}(k) \mid \boldsymbol{x}(k)\right] p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k-1}\right] d\boldsymbol{x}(k) \tag{3.7}$$

Equation (3.6) describes how the conditional posterior density function propagates from $p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k-1}\right]$ to $p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k}\right]$. It should be noted that the properties of the state transition equation (3.3) are accounted through the transition density function $p\left[\boldsymbol{x}(\mathrm{k}) \mid \boldsymbol{x}(\mathrm{k}-1)\right]$ while $p\left[\boldsymbol{y}(k) \mid \boldsymbol{x}(k)\right]$ accounts for the nonlinear measurement model (3.4). The prediction and update strategy provides an optimal solution to the state estimation problem, which, unfortunately, involves high-dimensional integration. The exact analytical solution to the recursive propagation of the posterior density is difficult obtain. However, when the process model is linear and noise sequences are zero mean Gaussian white noise sequences, Kalman filter describes the optimal recursive solution to the sequential state estimation problem.

While dealing with nonlinear systems, it becomes necessary to develop approximate and computationally tractable sub-optimal solutions to the above sequential Bayesian estimation problem. Gaussian approximation is the simplest method to approximate numerical integration problem due to its analytical tractability. The most popular approach in this class of nonlinear observers is extended Kalman filter (EKF), which assumes $p\left[\boldsymbol{x}(k) \mid \boldsymbol{Y}^{k}\right]$ to be Gaussian.

3.3 EXTENDED KALMAN FILTER (EKF)

The standard approach to estimate the system state of the discrete time nonlinear system (Refer Equations 3.3 and 3.4) using the well-known EKF (Muske and Edgar (1997)) is as follows:

The predicted mean is computed as

$$\hat{\mathbf{x}}(k \mid k-1) = \hat{\mathbf{x}}(k-1 \mid k-1) + \int_{k-1}^{k} f\left[\mathbf{x}(\tau), \mathbf{u}(k-1)\right] d\tau \tag{3.8}$$

The predicted covariance is computed as

$$\boldsymbol{P}(k \mid k-1) = \boldsymbol{\Phi}(k)\boldsymbol{P}(k-1 \mid k-1)\boldsymbol{\Phi}(k)^T + \boldsymbol{Q} \tag{3.9}$$

where

$$\boldsymbol{\Phi}(k) = \left[\frac{\delta f}{\delta \mathbf{x}}\right]_{\left[\hat{\mathbf{x}}(k-1/k-1), \mathbf{u}(k-1)\right]} ; \mathbf{C}(k) = \left[\frac{\delta g}{\delta \mathbf{x}}\right]_{\left[\hat{\mathbf{x}}(k/k-1), \mathbf{u}(k-1)\right]} \tag{3.10}$$

$\boldsymbol{\Phi}(k)$ in equation 3.10 is nothing but Jacobian matrices of partial derivatives of $\mathrm{f}[.]$ with respect to $\boldsymbol{x}$ and $\boldsymbol{C}(k)$ is the Jacobian matrix of partial derivatives of $\mathrm{g}[.]$ with respect to $\boldsymbol{x}$. Note that the extended Kalman filter (EKF) computes the covariance using the linear propagation (Refer Equation (3.9)).

The measurement prediction, computation of innovation and covariance matrix of innovation are as follows:

$$\hat{\boldsymbol{y}}(k \mid k-1) = g\left[\hat{\boldsymbol{x}}(k \mid k-1), \boldsymbol{u}(k-1)\right] \tag{3.11}$$

$$\boldsymbol{\gamma}(k \mid k-1) = \boldsymbol{y}(k) - \hat{\boldsymbol{y}}(k \mid k-1) \tag{3.12}$$

$$\boldsymbol{V}(k) = \boldsymbol{C}(k)\boldsymbol{P}(k \mid k-1)\boldsymbol{C}(k)^T + \boldsymbol{R} \tag{3.13}$$

It should be noted that the EKF make use of the actual measurement model for obtaining the predicted estimates of output.

The Computation of Kalman Gain

$$\boldsymbol{K}(k) = \boldsymbol{P}(k \mid k-1)\boldsymbol{C}(k)^T \boldsymbol{V}^{-1}(k) \tag{3.14}$$

The updated state estimate is given by

$$\hat{\boldsymbol{x}}(k \mid k) = \hat{\boldsymbol{x}}(k \mid k-1) + \boldsymbol{K}(k)\boldsymbol{\gamma}(k \mid k-1) \tag{3.15}$$

The updated error covariance matrix is given by

$$\boldsymbol{P}(k \mid k) = \left[I - \boldsymbol{K}(k)\right]\boldsymbol{C}(k)\boldsymbol{P}(k \mid k-1) \tag{3.16}$$

It should be noted that the calculations of the predicted error covariance matrix and updated error covariance matrix and the gain of the EKF are the same as those of the linear Kalman filter.

We have assumed that the initial state and the sequence $\{\boldsymbol{w}(k)\}$ and $\{\boldsymbol{v}(k)\}$ are white, Gaussian and independent of each other.

$$E\left[\boldsymbol{w}(k)\right] = 0 \tag{3.17}$$

$$\mathrm{E}\left[\boldsymbol{v}(\mathrm{k})\right] = 0 \tag{3.18}$$

$$Cov\left[\boldsymbol{w}(k)\right] = \mathbf{Q} \tag{3.19}$$

$$Cov\left[\boldsymbol{v}(k)\right] = \mathbf{R} \tag{3.20}$$

$$Cov\left[\boldsymbol{w}(k), \boldsymbol{w}(j)\right] = 0;\ j \neq k \tag{3.21}$$

$$Cov\left[\boldsymbol{v}(k), \boldsymbol{v}(j)\right] = 0; \quad j \neq k \tag{3.22}$$

$$Cov\left[\boldsymbol{w}(k), \boldsymbol{v}(j)\right] = 0 \tag{3.23}$$

Equations (3.17) and (3.18) imply that the random variables **w**(k) and **v**(k) have zero mean. The respective covariance matrices (**Q** and **R**) are given by Equations (3.19) and (3.20). Equations (3.21) and (3.22) imply that the disturbances at different times are not correlated; similarly the measurement errors at different time instants are not correlated. Equation (3.23) stipulates that the disturbances and measurement errors are not cross correlated.

In the EKF described above, it is assumed that all the non-random inputs and parameters of the process are precisely known. However, the inputs to the process such as feed temperature, feed concentration and feed flow rate are likely to change from their nominal values. Hence, in order to generate accurate state estimates, the state equations have to be augmented with an additional differential equation of the form

$$\frac{d\boldsymbol{\beta}}{dt} = 0 \tag{3.24}$$

In the above equation β is the vector of unknown inputs/parameters that have to be estimated. Since there is no *a priori* knowledge about the above variables, it is assumed that these variables don't change with time, which is the implication of the above equation.

There are variety of forms of Extended Kalman filter formulations, such as iterated EKF and second order EKF, depending on how predicted state covariance updates are performed. Muske and Edgar (1997) have given a detailed account of various forms of EKF.

3.4 DESIGN OF FUZZY OBSERVER

3.4.1 Fuzzy Dynamic Model

A T-S fuzzy model has been proposed to represent a nonlinear system using locally linearized models Takagi and Sugeno (1985). Two different methods for developing a T-S fuzzy model have been suggested in the literature, namely (i) the black box identification via fuzzy clustering technique Babuska and Verbruggen (1997) and (ii) Linearization of an existing nonlinear system around the centers of the fuzzy region partitioning the state space. The T-S fuzzy model is nothing but a piecewise interpolation of local linear models through membership functions. The T-S fuzzy model is described by IF-THEN rule which represent local linear relations of the nonlinear system. The rule to describe the nonlinear system around an operating point is as follows:

Rule i (i =1: N)

If z_1(k) is $M_{i,1}$ and .... and z_g(k) is $M_{i,g}$ then

$$\boldsymbol{x}_i(k) = \boldsymbol{\Phi}_i(\mathbf{x}(k-1) - \overline{\mathbf{x}}_i) + \boldsymbol{\Gamma}_\mathrm{i}(\boldsymbol{u}(k-1) - \overline{\boldsymbol{u}}_i) + \boldsymbol{\Psi}_i\mathbf{w}(k-1) \tag{3.25}$$

$$\boldsymbol{y}_i(k) = \boldsymbol{C}\boldsymbol{x}_i(k) \tag{3.26}$$

where, $\mathbf{z}_\mathbf{j}(k)$ are the premise variables and $M_{ij}(k)$ are the fuzzy sets. $\boldsymbol{\Phi}_i, \boldsymbol{\Gamma}_i, \boldsymbol{C}$ and $\boldsymbol{\Psi}_i$ are known time invariant matrices of appropriate dimensions. In this work it is assumed that such a model of the process can be developed from the first principles by linearizing them around different operating steady state values. The global system behaviour is described by a fuzzy fusion of all linear model outputs. For a given input vector **u**(k), the global state and output of fuzzy model are inferred as follows:

$$\boldsymbol{x}(k) \ = \ \sum_{i=1}^{N} h_i(\mathbf{z}(k))[\boldsymbol{\Phi}_i(\boldsymbol{x}(k-1)-\overline{\boldsymbol{x}}_i)+\boldsymbol{\Gamma}_i(\boldsymbol{u}(k-1)-\overline{\boldsymbol{u}}_i)+\overline{\boldsymbol{x}}_i] \tag{3.27}$$

$$\boldsymbol{y}(k) = \boldsymbol{C}\boldsymbol{x}(k) \tag{3.28}$$

where, the membership grades $h_i(\mathbf{z}(k))$ are defined as

$$h_i(\mathbf{z}(k)) \ = \ \frac{\mu_i(\mathbf{z}(k))}{\mu(k)} \tag{3.29}$$

$$\mu_i(\mathbf{z}(k)) = \prod_{j=1}^{g} M_{ij} \tag{3.30}$$

$$\mu(k) \ = \ \sum_{i=1}^{N} \mu_i(\mathbf{z}(k)) \tag{3.31}$$

It should be noted that the grade of membership should be $h_i(\mathbf{z}(k)) \in [0,1]$ and $\sum_{i=1}^{N} h_i(\mathbf{z}(k)) = 1$.

3.4.2 Fuzzy Kalman Filter (FKF)

For a nonlinear dynamic system described by the T-S fuzzy model, FKF can be designed to estimate the system state vector. For the FKF design, it is assumed that the linearised models are locally observable, (i.e., all ($\mathbf{C}_i$, $\mathbf{\Phi}_i$) (i=1 to N) pairs are observable). The local linear models are un-correlated and each of the N local model parameters is time invariant.

A local linear observer can be designed for each local linear dynamic model using Kalman filter theory. At an operating point, the local observer is associated with each fuzzy rule as given below:

Rule i (i =1: N)

If z_1(k) is $M_{i,1}$and .... and z_q(k) is $M_{i,g}$then

$$\hat{\boldsymbol{x}}_i(k|k-1) = \boldsymbol{\Phi}_i(\hat{\mathbf{x}}(k-1|k-1) - \bar{\mathbf{x}}_i) + \boldsymbol{\Gamma}_i(\mathbf{u}(k-1) - \bar{\mathbf{u}}_i) \tag{3.32}$$

$$\hat{\boldsymbol{y}}_\mathrm{i}(\mathrm{k}\,|\,\mathrm{k}-1) = \boldsymbol{C}\hat{\boldsymbol{x}}_\mathrm{i}(\mathrm{k}\,|\,\mathrm{k}-1) \tag{3.33}$$

$$\gamma_i(k\,|\,k-1) = (\boldsymbol{y}(k) - \bar{\boldsymbol{y}}_i) - \hat{\boldsymbol{y}}_i(k\,|\,k-1) \tag{3.34}$$

$$\hat{\boldsymbol{x}}_i(k\,|\,k) = \hat{\boldsymbol{x}}_i(k\,|\,k-1) + \boldsymbol{K}_i(k)\;\gamma_i(k\,|\,k-1) \tag{3.35}$$

The Kalman gain matrix ($\mathbf{K}_\mathbf{i}(k)$) in Equation (3.48) can be calculated from the following set of equations.

$$\boldsymbol{P}_\mathrm{i}(k|k-1) = \boldsymbol{\Phi}_i\mathbf{P}_i(k-1|k-1)\boldsymbol{\Phi}_i^T + \boldsymbol{\Psi}_i\mathbf{Q}(k-1)\boldsymbol{\Psi}_i^T \tag{3.36}$$

$$\boldsymbol{V}_i(k) = \boldsymbol{C}\boldsymbol{P}_i(k\,|\,k-1)\boldsymbol{C}^T + \boldsymbol{R}_i(k) \tag{3.37}$$

$$\boldsymbol{K}_i(k) = \boldsymbol{P}_i(k \mid k-1)\boldsymbol{C}^T\boldsymbol{V}_i^{-1}(k) \tag{3.38}$$

$$\boldsymbol{P}_i(k \mid k) = \left[I - \boldsymbol{K}_i(k)\boldsymbol{C}\ \right]\boldsymbol{P}_i(k \mid k-1) \tag{3.39}$$

In equations (3.36) and (3.39), $\mathbf{P_i}(\mathrm{k} \mid \mathrm{k}-1)$ and $\mathbf{P_i(k \mid k)}$ are the covariance matrices of errors in predicted and updated state estimates of ith local observer respectively. The overall state estimation is a nonlinear combination of individual local observer outputs. The overall observer dynamics will then be a weighted sum of individual linear observers given by:

$$\hat{\boldsymbol{x}}(k/k) = \sum_{i=1}^{N} h_i(\mathbf{z(k)})\{\hat{\boldsymbol{x}}_i(k/k-1) + \left[\boldsymbol{K}_i(k)[(\boldsymbol{y}(k) - \bar{\boldsymbol{y}}_i) - \hat{\boldsymbol{y}}_i(k/k-1)]\right] + \bar{\boldsymbol{x}}_i\} \tag{3.40}$$

3.5 NONLINEAR MODEL PREDICTIVE CONTROLLER (NMPC)

3.5.1 The Principle of NMPC

A simple block representation of NMPC is shown in Figure 3.1. Model Predictive Control (MPC) is one of the most widely used control techniques in the chemical industry. MPC casts the control problem in the form of an optimization, which makes it convenient to handle constraints and nonlinear models explicitly (Rao et al. 2006). The two components of the NMPC are the controller and state estimator. The estimator part of the NMPC provides optimal estimate of the current state of the process. The controller part of the NMPC on the other hand computes the unknown optimal future manipulated inputs over a future horizon. The computation of the unknown future manipulated variables over the future horizon is performed in an optimization framework by minimization of a suitable objective function with constraints. The objective function is usually chosen as the sum of the squares of the differences between the predicted values and the setpoint values over the prediction horizon of N_p time steps. The objective function also includes a second term, which is the squared sum of manipulated variable changes over the control horizon (N_c).

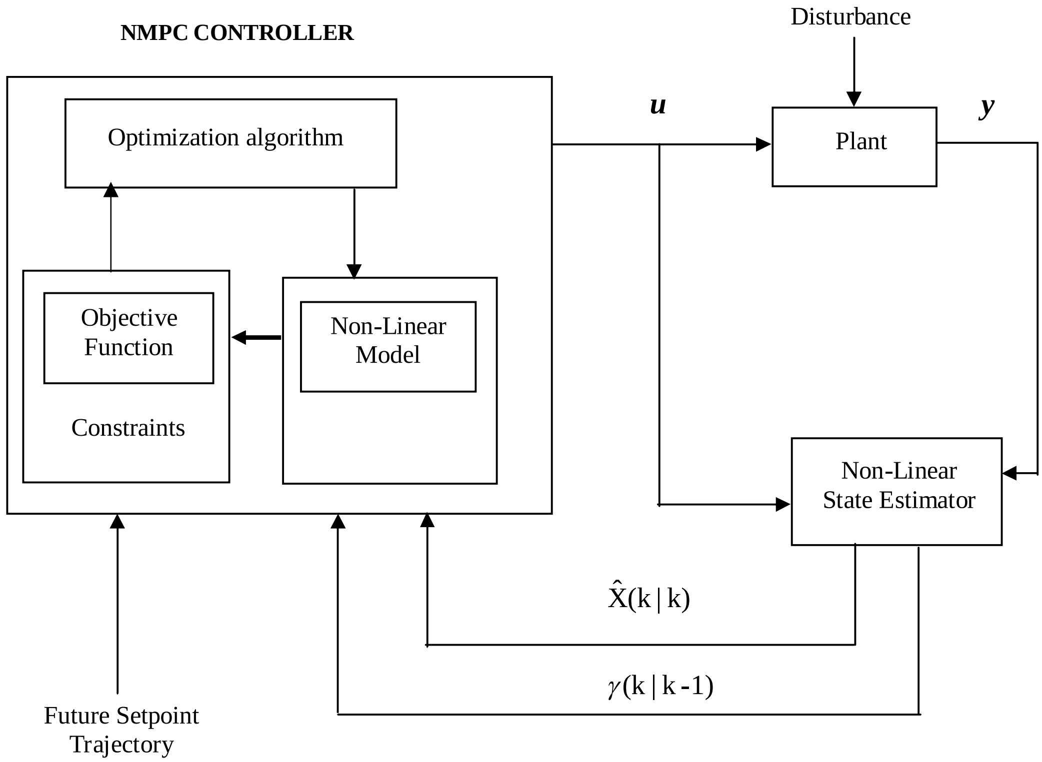

Figure 3.1 Block diagram of NMPC Control Scheme

The NMPC formulation uses nonlinear model for the prediction of the process behavior and a quadratic performance index for determining the unknown manipulated variables. In order to predict the future values of the state and output variables, the nonlinear model must be integrated using initial conditions (current values) for the state variables. In order to estimate the initial conditions, extended Kalman filter has been widely used. In the NMPC approach absolute, velocity constraints for manipulated variables as well as state and output variables constraints are explicitly included. It should be noted that it is possible to use different nonlinear models for carrying out future predictions in the controller. Significant progress has been made in the field of dynamic process optimization with rapid online optimization algorithm, which exploits the specific structure of the optimization problems arising in NMPC having been developed. However, the global solution for optimization can't be guaranteed and the development of rapid and stable optimization techniques remains a major objective in the NMPC research. Thus the key characteristics of NMPC are that it allows the use of nonlinear model for prediction, explicit considerations state, input and output constraints. Moreover, in the NMPC a specified performance criterion is minimized on-line. An explicit solution can be obtained only if the objective function is quadratic, the model is linear and there are no constraints, otherwise an iterative numerical optimization method has to be used to determine the future manipulated input variables. Further, the NMPC scheme is implemented in a moving horizon framework i.e. only the first move $\mathbf{u}(k/k)$ is implemented on the plant and the optimization problem is reformulated at the next sampling instant based on the updated information from the plant (refer Figure 3.2).

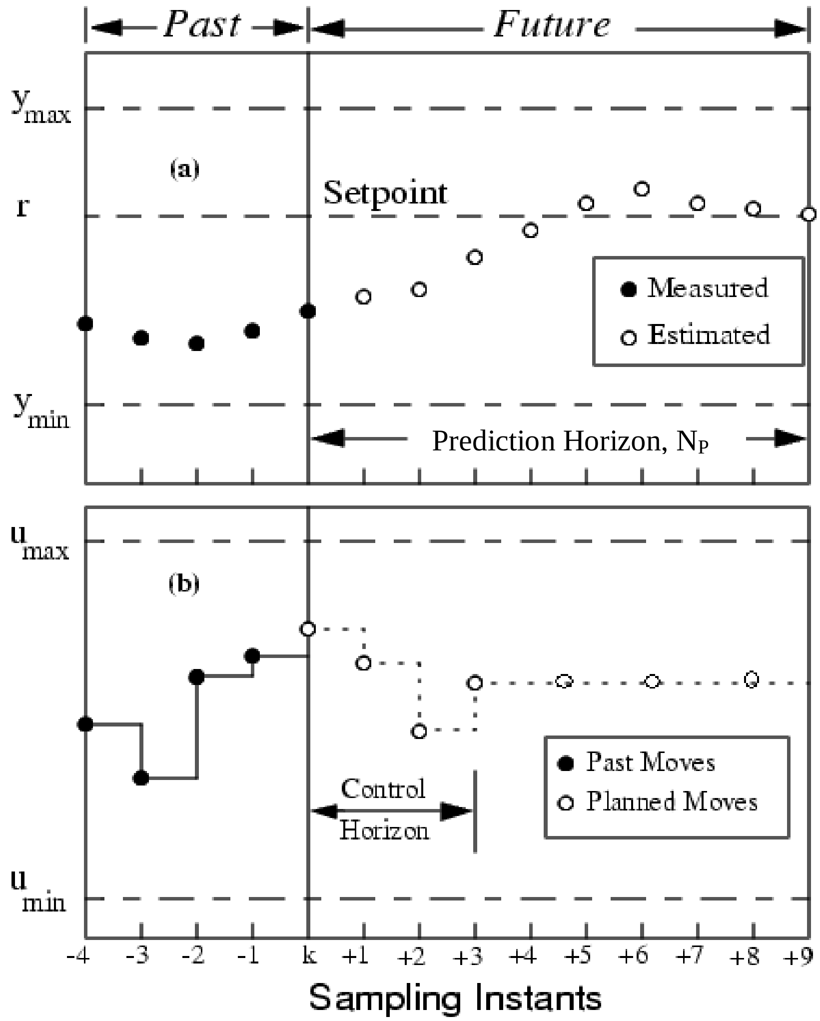

Figure 3.2 Basic concept of NMPC

3.5.2 NMPC formulation using T-S fuzzy dynamic model

In the proposed MPC formulation, at every sampling instant, the fuzzy dynamic model (Refer Equations 3.27 and 3.28) is used for predicting the future behaviour of the plant over a finite number of future time steps say N_P which is called prediction horizon. A set of N_c future manipulated input moves $\{ \boldsymbol{u}(k/k), \boldsymbol{u}(k+1/k) \ldots \boldsymbol{u}(k+N_c-1/k) \}$ (where N_c is called the control horizon) are determined by constrained optimization with the objective of minimizing the predicted deviation of the process output from the target over the prediction horizon as well as minimizing the expenditure of control effort in driving the process output to target, subject to pre-specified operating constraints.

The fuzzy dynamic model (refer Equations 3.27 and 3.28) can be used recursively to obtain multi-step prediction. Given a sequence of future control moves $\{\mathbf{u}(k/k)...\mathbf{u}(k+1/k)...\mathbf{u}(k+N_c-1/k)\}$, a N_P step ahead output prediction can be written as follows:

$$\boldsymbol{x}(k+j+1|k)=\sum_{i=1}^{N}h_i(\mathbf{z}(k)\Big[\boldsymbol{\Phi}_i\Big[\big(\boldsymbol{x}(k+j|k)-\overline{\boldsymbol{x}}_i\Big]+\boldsymbol{\Gamma}_{u,i}\big(\boldsymbol{u}(k+j|k)-\overline{\boldsymbol{u}}_i\big)+\overline{\boldsymbol{x}}_i\Big]$$
$$for\ j=0.........N_c-1 \tag{3.41}$$

$$\boldsymbol{x}(k+j+1|k)=\sum_{i=1}^{N}h_i(\mathbf{z}(k)\Big[\boldsymbol{\Phi}_i\Big[\big[\boldsymbol{x}(k+j|k)\big]-\overline{\boldsymbol{x}}_i\Big]+\Gamma_{u,i}(\boldsymbol{u}(k+N_c-1|k)-\overline{\boldsymbol{u}}_i)+\overline{\boldsymbol{x}}_i\Big]$$
$$for\ \ j=N_c.....N_P-1 \tag{3.42}$$

$$\boldsymbol{y}(k+j|k)=\boldsymbol{C}\,\boldsymbol{x}(k+j|k);\qquad for\ \ j=1...N_P \tag{3.43}$$

To account for plant model mismatch and unmeasured disturbances, a simple unmeasured disturbance estimator similar to the dynamic matrix control scheme (Bequette, 1993) is incorporated as follows:

$$\boldsymbol{y}_c(k+j|k)=\boldsymbol{y}(k+j|k)+\boldsymbol{d}(k+j|k) \tag{3.44}$$

where

$$\boldsymbol{d}(k+j|k)=\boldsymbol{d}(k|k)=\big[\boldsymbol{y}_m(k)-\boldsymbol{y}(k)\big]\ for\ j=1...N_p \tag{3.45}$$

In the above equation (3.45), $\boldsymbol{y}_m(k)$ represents the measured output at the k[th] instant and $\mathbf{y}(k)$ represents the model output at the k[th] instant. Given a future setpoint trajectory $\mathbf{y}_r(k+j/k),(j=1......N_p)$, the NMPC controller design problem can be formulated as:

$$\min_{\boldsymbol{u}(k/k)........\boldsymbol{u}(k+N_p-1/k)} J \tag{3.46}$$

where

$$J=\left\{\sum_{j=1}^{N_p}\left[\boldsymbol{E}(k+j/k\right]^T \boldsymbol{W}_E\left[E(k+j|k)\right]\right\}+\sum_{j=0}^{N_c-1}\Delta\boldsymbol{u}\left(k+j|k\right)^T \boldsymbol{W}_U\left[\Delta u(k+j|k)\right] \tag{3.47}$$

$$\boldsymbol{E}(k+j|k)=\boldsymbol{y}_r(k+j|k)-\boldsymbol{y}_c(k+j|k) \tag{3.48}$$

$$\Delta\boldsymbol{u}(k+j|k)=\boldsymbol{u}(k+j|k)-\boldsymbol{u}(k+j-1|k) \tag{3.49}$$

Subject to the following constraints:

$$\boldsymbol{u}^L\le\boldsymbol{u}(k+j|k)\le\boldsymbol{u}^H \;\; for\; j=0...N_c-1 \tag{3.50}$$

$$\boldsymbol{y}^L\le\boldsymbol{y}_c(k+j|k)\le\boldsymbol{y}^H \;\; for\;\; j=1...N_p \tag{3.51}$$

$$\Delta\boldsymbol{u}(k+N_c|k)=\Delta\boldsymbol{u}(k+N_c+1|k)=.....\;\Delta\boldsymbol{u}(k+N_p-1|k)=\overline{0} \tag{3.52}$$

The resulting constrained optimization problem can be solved using any standard optimization technique. It should be noted that the desired closed loop performance of the proposed NMPC scheme using T-S fuzzy dynamic model can be achieved by appropriately selecting the prediction horizon $\mathrm{N_p}$, control horizon $\mathrm{N_c}$, the error weighting matrix (W_{E}) and the input move weight matrix (W_{U}). Also, the NMPC scheme is implemented in a moving horizon framework i.e. only the first move $\mathbf{u}(k/k)$ is implemented on the plant and the optimization problem is reformulated at the next sampling instant based on the updated information from the plant.

3.5.3 NMPC formulation using Fuzzy Kalman Filter (FKF)

In the proposed observer based MPC formulation, at each sampling instant, the FKF is used for predicting the future behavior of the plant over a finite time horizon of length N_p from the current time instant k.

Let us assume that at any instant we are forced to choose only N_c future manipulated variable moves $\mathbf{u}(k\,|\,k).......\mathbf{u}(k+1\,|\,k).......\mathbf{u}(k+N_C-1\,|\,k)$ with the following constraints on the remaining future input moves $\mathbf{u}(k+N_C\,|\,k) = \mathbf{u}(k+N_C+1\,|\,k) = \mathbf{u}(k+N_P-1\,|\,k) = \mathbf{u}(k+N_C-1\,|\,k)$ where N_c is defined as the control horizon. As the expected value of the future innovation is zero, optimal predicted estimates of those variables can be recursively obtained using the FKF (Refer section 3.4.2). At any sampling instant k, the FKF based MPC problem is defined as an optimization problem where the future manipulated inputs are determined by minimizing the following objective function

$$\min_{\boldsymbol{u}(k\,|\,k).......\boldsymbol{u}(k+N_p-1\,|\,k)} \sum_{j=1}^{N_P} \boldsymbol{E}(k+j\,|\,k)^{T} \boldsymbol{W}_E\, \boldsymbol{E}(k+j\,|\,k) + \sum_{j=0}^{N_C-1} \Delta\boldsymbol{u}(k+j\,|\,k) \boldsymbol{W}_U\, \Delta\boldsymbol{u}(k+j\,|\,k) \tag{3.53}$$

Subject to the following:

$$\hat{\mathbf{x}}(k+j+1\,|\,k) = \sum_{i=1}^{N} h_i(\mathbf{z}(k))[\boldsymbol{\Phi}_i(\hat{\mathbf{x}}(k+j\,|\,k)-\bar{\mathbf{x}}_i) + \Gamma_{u,i}(\boldsymbol{u}(k+j\,|\,k)-\bar{\boldsymbol{u}}_i) + \boldsymbol{K}_i(k)\gamma_i(k\,|\,k-1) + \bar{\mathbf{x}}_i] \quad j=(1:N_p-1) \tag{3.54}$$

$$\hat{\boldsymbol{y}}(k+j+1|k) = \boldsymbol{C}\hat{\mathbf{x}}(k+j+1|k) + \varepsilon(k) \quad (j = 0....N_p - 1) \tag{3.55}$$

$$\boldsymbol{u}(k+N_c|k) = \boldsymbol{u}(k+N_c+1|k) = \boldsymbol{u}(\mathrm{k}+\mathrm{N_p}\text{ -1|k}) = \boldsymbol{u}(\mathrm{k}+\mathrm{N_c}\text{ -1|k}) \tag{3.56}$$

$$\mathbf{x}^{\mathrm{L}} \le \hat{\mathbf{x}}(k+j|k) \le \mathbf{x}^{\mathrm{U}} \qquad (\mathrm{j} = 1.....\mathrm{N_P}) \tag{3.57}$$

$$\mathbf{y}^{\mathrm{L}} \le \hat{\boldsymbol{y}}(k+j|k) \le \mathbf{y}^{\mathrm{U}} \qquad (\mathrm{j} = 1.....\mathrm{N_P}) \tag{3.58}$$

$$\mathbf{u}^{\mathrm{L}} \le \boldsymbol{u}(k+j|k) \le \mathbf{u}^{\mathrm{U}} \qquad (\mathrm{j} = 0....\mathrm{N_c}\text{ -1}) \tag{3.59}$$

$$\Delta u^{\mathrm{L}} \le \Delta\boldsymbol{u}(k+j|k) \le \Delta u^{\mathrm{U}} \quad (\mathrm{j} = 0....\mathrm{N_C}\text{ -1}) \tag{3.60}$$

where $\mathbf{E}(\mathrm{k+j|k}) = \mathbf{y}_{\mathrm{r}}(\mathrm{k+j|k}) - \hat{\mathbf{y}}(\mathrm{k+j|k})$ and $\mathbf{\Delta u}(\mathrm{k+j|k}) = \mathbf{u}(\mathrm{k+j|k}) - \mathbf{u}(\mathrm{k+j\text{-}1|k})$. $\mathbf{y}_{\mathrm{r}}(\mathrm{k+j|k})$ represents the future set point trajectory.

As discussed in the previous subsection, the closed loop performance of the proposed NMPC scheme using fuzzy Kalman filter can be achieved by judiciously selecting the prediction horizon $\mathrm{N_P}$, control horizon $\mathrm{N_c}$, the error weighting matrix (W_{E}) and the input move weight matrix (W_{U}). The NMPC scheme is implemented in a moving horizon framework. That is only the first move **u**(k/k) is implemented on the plant and the optimization problem is reformulated at the next sampling instant based on the updated information from the plant.

3.5.4 NMPC formulation using Augmented State Fuzzy Kalman Filter (ASFKF)

The FKF based NMPC formulation described in the section 3.5.3 provides biased state estimates in the presence of step like disturbance at the output, slow drifts or step like disturbances at the input or states. Thus there is a need to introduce some mechanism to allow for the plant-model mismatch, which in turn introduces integral action in the controller formulation. This can be achieved by augmenting the state space model with the artificially introduced input and output disturbance variables ($\boldsymbol{\beta}$ and $\boldsymbol{\eta}$). The resulting augmented model for each fuzzy rule can be written as:

$$\boldsymbol{x}_i(k) = \boldsymbol{\Phi}_i(\mathbf{x}_i(k-1) - \bar{x}_i) + \boldsymbol{\Gamma}_i(\mathbf{u}(k-1) - \bar{u}_i) + \boldsymbol{\Psi}_i w(k-1) + \boldsymbol{\Upsilon}_i \boldsymbol{\beta}_i(k-1) \quad (3.61)$$

$$\boldsymbol{\beta}_i(k) = \boldsymbol{\beta}_i(k-1) + \boldsymbol{w}_\beta(k-1) \quad (3.62)$$

$$\boldsymbol{\eta}_i(k) = \boldsymbol{\eta}_i(k-1) + \boldsymbol{w}_\eta(k-1) \quad (3.63)$$

$$\boldsymbol{y}_\mathrm{i}(\mathrm{k}) = \boldsymbol{C}\boldsymbol{x}_\mathrm{i}(\mathrm{k}) + \boldsymbol{C}_\eta \boldsymbol{\eta}_\mathrm{i}(\mathrm{k}) + \boldsymbol{v}_\mathrm{i}(\mathrm{k}) \quad (3.64)$$

where, $\mathbf{w}_\beta$ and $\mathbf{w}_\eta$ are zero mean white noise sequences with covariances $\mathbf{Q}_\beta$ and $\mathbf{Q}_\eta$ respectively ($\mathbf{w}_\beta \in R^s$ and $\mathbf{w}_\eta \in R^t$). The above set of equations can be combined into an augmented state space model and is as follows:

$$\boldsymbol{x}_i^{'}(k) = \boldsymbol{\Phi}_i^{'}(\boldsymbol{x}_i^{'}(k-1) - \bar{\mathbf{x}}_i^{'}) + \boldsymbol{\Gamma}_i^{'}(\boldsymbol{u}(k-1) - \bar{\boldsymbol{u}}_i) + \boldsymbol{\Psi}_i^{'} \boldsymbol{w}(k-1) \quad (3.65)$$

$$\boldsymbol{y}_i(k) = \boldsymbol{C}^{'} \boldsymbol{x}_i^{'}(k) + \boldsymbol{v}_i(k) \quad (3.66)$$

$$\boldsymbol{\Phi}_i^{'} = \begin{bmatrix} \boldsymbol{\Phi}_i & \boldsymbol{\Upsilon}_i \\ 0 & \boldsymbol{I} \end{bmatrix} \quad \boldsymbol{\Gamma}_i^{'} = \begin{bmatrix} \boldsymbol{\Gamma}_i \\ 0 \end{bmatrix} \quad \boldsymbol{C}^{'} = \begin{bmatrix} \boldsymbol{C}\, C_\eta \end{bmatrix} \quad \Psi_i^{'} = \begin{bmatrix} \Psi_i \\ 0 \end{bmatrix}$$

where

$$\boldsymbol{x}_i^{'}(k) = \begin{bmatrix} \boldsymbol{x}_i(k) \\ \boldsymbol{\beta}_i(k) \\ \boldsymbol{\eta}_i(k) \end{bmatrix} \quad \textit{and} \quad \boldsymbol{w}(k) = \begin{bmatrix} \boldsymbol{w}(k) \\ \boldsymbol{w}_\beta(k) \\ \boldsymbol{w}_\eta(k) \end{bmatrix}$$

This augmented state space model can be used to design an ASFKF using Equations (3.32-3.39). The ASFKF based MPC is defined as a constrained optimization problem where the future manipulated inputs are determined by minimizing the objective function

$$\min_{\boldsymbol{u}(k/k)\ldots\ldots\boldsymbol{u}(k+N_p-1/k)} \sum_{j=1}^{N_P} \boldsymbol{E}(k+j/k)^T \boldsymbol{W_E} \boldsymbol{E}(k+j/k) + \sum_{j=0}^{N_C-1} \boldsymbol{\Delta u}(k+j/k) \boldsymbol{W_U} \boldsymbol{\Delta u}(k+j/k) \qquad (3.67)$$

Subject to the following constraints:

$$\boldsymbol{x}'(k+j+1/k) = \sum_{i=1}^{N} h_i(\mathbf{z}(k))[\boldsymbol{\Phi}_i'(\boldsymbol{x}'(k+j/k)-\overline{\mathbf{x}}_i') + \boldsymbol{\Gamma}_i'(\boldsymbol{u}(k+j/k)-\overline{\boldsymbol{u}}_i) + \overline{\mathbf{x}}_i'] \qquad (3.68)$$

$$\hat{\boldsymbol{y}}(k+j+1/k) = \boldsymbol{C}'\boldsymbol{x}'(k+j+1/k) \quad j = 0....N_p-1 \qquad (3.69)$$

$$\boldsymbol{u}(k+N_c/k) = \boldsymbol{u}(k+N_c+1/k) = \boldsymbol{u}(k+N_p-1/k) = u(k+N_c-1/k) \qquad (3.70)$$

$$\mathbf{x}^L \le \boldsymbol{x}'(k+j/k) \le \mathbf{x}^U \qquad (j = 1.....N_p) \qquad (3.71)$$

$$\mathbf{y}^L \le \hat{\boldsymbol{y}}(k+j/k) \le \mathbf{y}^U \qquad (j = 1.....N_p) \qquad (3.72)$$

$$\mathbf{u}^L \le \boldsymbol{u}(k+j/k) \le \mathbf{u}^U \qquad (j = 0.....N_c - 1) \qquad (3.73)$$

$$\boldsymbol{\Delta}\mathbf{u}^L \le \Delta\boldsymbol{u}(k+j/k) \le \boldsymbol{\Delta}\mathbf{u}^U \qquad (j = 0...N_C - 1) \qquad (3.74)$$

As discussed in the subsection 3.5.2 and 3.5.3, the desired closed loop performance of the proposed NMPC scheme using ASFKF can be achieved by appropriately selecting the prediction horizon N_P, control horizon N_c, the error weighting matrix (W_E) and the input move weight matrix (W_U). Also, the NMPC schemes have been implemented in a moving horizon framework i.e. only the first move u(k/k) is implemented on the plant and the optimization problem is reformulated at the next sampling instant based on the updated information from the plant.