Chapter 3

A Fundamental Property of Suslin Matrices

In this chapter we study a Fundamental Property of the Suslin matrices and prove the equivalence of the Fundamental Property and the Key lemma. The study of the Fundamental Property leads to an action of the group $\text{SUM}_r(R)$ on the Suslin space of matrices, viz. $\mathbb{S}_r = \{S_r(v, w) : v, w \in M_{1r+1}(R)\}$.

3.1 Fundamental Property

The Suslin matrices satisfy the following Fundamental Property:

Lemma 3.1.1 (Fundamental Property)

Let $v, w, s, t \in M_{1r+1}(R)$. Then

\begin{align*}
S_r(s, t)S_r(v, w)S_r(s, t) &= S_r(v', w'), \quad (3.1) \\
S_r(t, s)S_r(w, v)S_r(t, s) &= S_r(w', v'), \quad (3.2)
\end{align*}
for some \( v', w' \in M_{1r+1}(R) \), which depend linearly on \( v, w \) and quadratically on \( s, t \). Consequently, \( \langle v', w' \rangle = \langle s, t \rangle^2 \langle v, w \rangle \).

**Proof:** Let \( v = (v_0, v_1), w = (w_0, w_1), s = (s_0, s_1), t = (t_0, t_1) \in M_{1r+1}(R) \), where \( v_0, w_0, s_0, t_0 \in R \) and \( v_1, w_1, s_1, t_1 \in M_{1r}(R) \). Let

\[
S_r(s, t) = \begin{pmatrix} s_0 I & S_1 \\ -T_1 & t_0 I \end{pmatrix} \quad \text{and} \quad S_r(v, w) = \begin{pmatrix} v_0 I & S_2 \\ -T_2 & w_0 I \end{pmatrix},
\]

where \( I = I_{2r-1}, S_1 = S_{r-1}(s_1, t_1), T_1 = S_{r-1}(t_1, s_1)^T, S_2 = S_{r-1}(v_1, w_1) \) and \( T_2 = S_{r-1}(w_1, v_1)^T \).

Let \( \gamma \) denote \( S_r(s, t)S_r(v, w)S_r(s, t) \). Then \( \gamma \)

\[
\begin{pmatrix} s_0 I & S_1 \\ -T_1 & t_0 I \end{pmatrix} \begin{pmatrix} v_0 I & S_2 \\ -T_2 & w_0 I \end{pmatrix} \begin{pmatrix} s_0 I & S_1 \\ -T_1 & t_0 I \end{pmatrix} = \begin{pmatrix} (s_0^2 v_0 - s_0 (S_1 T_2 + S_2 T_1) - w_0 S_1 T_1) I & (s_0 v_0 + t_0 w_0) S_1 + s_0 t_0 S_2 - S_1 T_2 S_1 \\ -(s_0 v_0 + t_0 w_0) T_1 - s_0 t_0 T_2 + T_1 S_2 T_1 & (t_0^2 w_0 - t_0 (T_2 S_1 + T_1 S_2) - v_0 T_1 S_1) I \end{pmatrix}
\]

By Lemma 2.4.1,

\[
S_1 T_2 + S_2 T_1 = T_2 S_1 + T_1 S_2 = \{ (s_1, w_1) + (v_1, t_1) \} I_{2r-1},
\]

and by Lemma 1.2.1, \( S_1 T_1 = T_1 S_1 = (s_1, t_1) I_{2r-1} \). Thus,

\[
\gamma_{11} = (s_0^2 v_0 - s_0 ((s_1, w_1) + (v_1, t_1)) - w_0 (s_1, t_1)) I_{2r-1},
\]

\[
\gamma_{12} = (s_0 v_0 + t_0 w_0) S_1 + s_0 t_0 S_2 - S_1 T_2 S_1
\]

\[
= (s_0 v_0 + t_0 w_0) S_1 + s_0 t_0 S_2 - S_1 T_2 S_1 - S_1 T_1 S_2 + S_1 T_1 S_2
\]

\[
= (s_0 v_0 + t_0 w_0 - (T_2 S_1 + T_1 S_2)) S_1 + (s_0 t_0 + S_1 T_1) S_2
\]

\[
= (s_0 v_0 + t_0 w_0 - (s_1, w_1) - (v_1, t_1)) S_1 + (s, t) S_2,
\]
\[ \gamma_{21} = -(s_0v_0 + t_0w_0)T_1 - s_0t_0T_2 + T_1S_2T_1 \]
\[ = -(s_0v_0 + t_0w_0)T_1 - s_0t_0T_2 + T_1S_2T_1 + T_1S_1T_2 - T_1S_1T_2 \]
\[ = -((s_0v_0 + t_0w_0)T_1 - T_1(S_2T_1 + S_1T_2) + (T_1S_1)T_2 \]
\[ = -(s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)T_1 + \langle s, t \rangle T_2, \]
\[ \gamma_{22} = (t_0^2w_0 - t_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - v_0(s_1, t_1))I_{2r-1}. \]

Then for \( v' = (v'_0, v'_1), w' = (w'_0, w'_1) \in M_{1r+1}(R) \), where
\[ v'_0 = s'_0v_0 - s_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - w_0\langle s_1, t_1 \rangle \in R, \]
\[ v'_1 = (s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)s_1 + (s, t)v_1 \in M_{1r}(R), \]
\[ w'_0 = t'_0w_0 - t_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - v_0\langle s_1, t_1 \rangle \in R, \]
\[ w'_1 = (s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)t_1 + (s, t)w_1 \in M_{1r}(R), \]
\[ \gamma = S_r(v', w') \text{ is a Suslin matrix. The second assertion is got by symmetry. Multiplying both sides of equation (3.1) and the transpose of the equation (3.2) one can derive the consequence via Lemma 1.2.1.} \]

**Lemma 3.1.2** The Fundamental Property of Suslin matrices is a consequence of the Key Lemma.

**Proof:** Let \( \wp \) be a prime ideal of \( R \). Let \( S = S_r(v, w) \), for \( v, w \in R_{r+1}^1 \)
\[ \langle v, w \rangle = 1. \] Note that \( U_{r+1}(R, \wp) = e_1E_{r+1}(R, \wp) \). By Lemma 2.1.12, there is a \( e \in E_{r+1}(R, \wp) \) with \( e_1e = v \), and \( e_1e^{T-1} = w \). One can write \( e \) as a product of elementary generators of type \( E_{1j}(\lambda) \), \( E_{j1}(\mu) \), for \( 2 \leq j \leq r + 1, \lambda, \mu \in R \). Hence, by Key Lemma 2.4.4,
\[ S_r(v, w) = S_r(e_1e, e_1e^{T-1}) = \sigma_{t_1}^{\lambda} \cdots \sigma_{t_i}^{\lambda} S_r(e_1, e_1) \sigma_{t_i}^{\lambda} \cdots \sigma_{t_1}^{\lambda} \]
\[ = \sigma_{t_1}^{\lambda} \cdots \sigma_{t_i}^{\lambda} I_{2r} \sigma_{t_i}^{\lambda} \cdots \sigma_{t_1}^{\lambda} = \sigma_{t_1}^{\lambda} \cdots \sigma_{t_i}^{\lambda} \sigma_{t_i}^{\lambda} \cdots \sigma_{t_1}^{\lambda}, \]
where the $\sigma_{pq}^{tb}$ is of the type $S_r(e_1E_{1k}(\lambda),e_1)^{tb}$ or $S_r(e_1,e_1E_{1k}(\lambda)), \lambda \in R, 2 \leq k \leq r + 1$. Observe that over $R_x$, for $x,y \in R^{r+1}$ with $\langle x,y \rangle = 1$,

$$S_r(v,w)S_r(x,y)S_r(v,w)_\nu = \sigma_{1i}^{tb} \cdots \sigma_{li}^{tb} \sigma_{i1}^{tb} \cdots \sigma_{1l}^{tb} S_r(x,y) \sigma_{1i}^{tb} \cdots \sigma_{li}^{tb} \sigma_{i1}^{tb} \cdots \sigma_{1l}^{tb} = S_r(x',y')_\nu,$$

by Key Lemma 2.4.4, with $\langle x',y' \rangle_\nu = 1$. Thus $S_r(v,w)S_r(x,y)S_r(v,w)$ “locally” looks like a Suslin matrix $S_r(x',y')$ with $\langle x',y' \rangle = 1$.

Hence, $S_r(v,w)S_r(x,y)S_r(v,w)$ has a certain configuration amongst its entries. Note that if two elements in $R$ are locally same, then they are globally same. Thus one can conclude that $S_r(v,w)S_r(x,y)S_r(v,w) = S_r(x',y')$, for some $x', y'$ with $\langle x',y' \rangle = 1$. Thus the Fundamental Property is obtained from the Key Lemma.

\[ \square \]

**Remark 3.1.3** In Lemma 4.3.16, we show that the Key Lemma is obtained from the Fundamental Property.

### 3.2 Action of $\text{SUM}_r(R)$, $r > 1$, on $S_r$

Let $S_r = \{ S_{r}(v,w) : v,w \in M_{r+1}(R) \}$. Clearly,

$$S_r(v_1,w_1) + S_r(v_2,w_2) = S_r(v_1 + v_2, w_1 + w_2),$$

$$S_r(kv,kw) = kS_r(v,w),$$

where $v_1,v_2 \in M_{r+1}(R), k \in R$. Thus, $S_r$ is a $R$-module. It is clearly a free $R$-module of rank $2(r+1)$ with basis $\{ S_r(e_i,0), S_r(0,e_i) : 1 \leq i \leq r+1 \}$. We shall call this the **Suslin space**. We identify the Suslin space with $R^{2(r+1)}$.

By the Fundamental Property of Suslin matrices, one can define a map

$$\text{SUM}_r(R) \times S_r \rightarrow S_r$$

$$(S_r(v,w), S_{r}(x,y)) \mapsto S_r(v,w)S_{r}(x,y)S_r(v,w).$$
We show that this defines an action of $S U_m(R)$ on $S_r$ when $r$ is even.

Let $\alpha = \prod_{i=1}^n S_i$ be a product of special Suslin matrices $S_i = S_r(v_i, w_i)$, $1 \leq i \leq n$, and let $\alpha^\vee$ (dual) denote $\prod_{i=n}^1 S_i$. (A priori $\alpha^\vee$ will depend on the splitting of $\alpha$).

**Lemma 3.2.1** Let $S_r(v, w)$, $r \geq 2$, be a Suslin matrix. Suppose $S_r(v, w)$ has the property that $S_r(x, y)S_r(v, w) = S_r(p, q)$, for any special Suslin matrix $S_r(x, y)$, then $S_r(v, w) = uI_{2r}$. If $\langle v, w \rangle = 1$, then $u^2 = 1$.

**Proof:** Let $v = (a_0, a_1, v_1)$, $w = (b_0, b_1, w_1)$, where $a_0, a_1, b_0, b_1 \in R$ and $v_1, w_1 \in M_{1, r-1}(R)$. The fact that $S_r(e_1 + e_2, e_1)S_r(v, w)$, $S_r(e_1, e_1 + e_2)S_r(v, w)$ are Suslin matrices, shows that $S_{r-2}(v_1, w_1) = 0$, $S_{r-2}(w_1, v_1)^T = 0$, $a_1 = 0$, $b_1 = 0$, and $a_0 = b_0$. Hence $S_r(v, w) = uI_{2r} = S_r(ue_1, ue_1)$, where $u = a_0$. If $\langle v, w \rangle = 1$, then $u^2 = 1$. □

**Corollary 3.2.2** Let $S_i = S_r(v_i, w_i)$, $1 \leq i \leq n$, be special Suslin matrices. Let $\alpha = S_1 \ldots S_n$, $\alpha^\vee = S_n \ldots S_1$. If $\alpha = I_{2r}$, then $\alpha^\vee = uI_{2r}$, with $u^2 = 1$.

**Proof:** By the Fundamental Property $\alpha \alpha^\vee = S_r(v, w)$ is a special Suslin matrix. Again, by the Fundamental Property, for any special Suslin matrix $S_r(x, y)$, $\alpha S_r(x, y)\alpha^\vee$ is a special Suslin matrix, say, $S_r(p, q)$. But as $\alpha = I_{2r}$, $\alpha^\vee = S_r(v, w)$, and

$$S_r(p, q) = \alpha S_r(x, y)\alpha^\vee = S_r(x, y)\alpha^\vee = S_r(x, y)S_r(v, w),$$

for any special Suslin matrix $S_r(x, y)$. By Lemma 3.2.1, $\alpha^\vee = S_r(v, w) = uI_{2r}$, with $u^2 = 1$. □
Remark 3.2.3 From Corollary 3.2.2, one can conclude the following:

(i) If $r$ is even, then by the Suslin identities,

$$S_r(v, w) = J_r S_r(v, w)^T J_r^{-1}$$

hence $\alpha^\vee = J_r \alpha^T J_r^{-1}$, only depends on $\alpha$ (i.e. it is independent of the splitting of $\alpha$). Thus if $r$ is even and $\alpha = I_2$, then $\alpha^\vee = I_2$. Hence $\alpha \mapsto \alpha^\vee$ is a well defined anti-involution on $\text{SUM}_r(R)$. We shall denote $\alpha^\vee$ by $\alpha^*$ to emphasis that it is independent of the splitting chosen.

Also if $\alpha \in \text{SUM}_r(R)$, then by the Fundamental Property of Suslin matrices $\alpha S_r(x, y) \alpha^* \in S_r$. Thus one defines an action of the group $\text{SUM}_r(R)$ on the Suslin space $S_r$, when $r$ is even.

(ii) In Chapter 7, we show that if $r > 1$ is odd, then for a given unit $u$ in $R$ with $u^2 = 1$, one can find Suslin matrices $S_r(v_1, w_1), \ldots, S_r(v_k, w_k)$, with $\langle v_i, w_i \rangle = 1$, $1 \leq i \leq k$, such that

$$S_r(v_1, w_1) \cdots S_r(v_k, w_k) = I_{2r},$$
$$S_r(v_k, w_k) \cdots S_r(v_1, w_1) = u I_{2r}.$$  

(In fact, $v_i \in e_1 E_{r+1}(R)$, for $1 \leq i \leq k$). Thus the operation $\vee$ can be defined on $\text{SUM}_r(R)$ only upto a unit $u$, with $u^2 = 1$, when $r > 1$ is odd.

Let $\mathbb{P}(S_r) = \{[S_r(v, w)] : v, w \in R^{r+1}\}$, where $[S_r(v, w)] = \{ u S_r(v, w) : u \in R, u^2 = 1 \}$ be the “projective set” associated to the Suslin space $S_r$. One gets an action of $\text{SUM}_r(R)$ on $\mathbb{P}(S_r)$, when $r > 1$ is odd.

Corollary 3.2.4 (Criterion for an element to be $\alpha^\vee$)

Let $\alpha = \prod S_i$, where $S_i = S_r(v_i, w_i)$ are special Suslin matrices. If there exists $\beta \in \text{GL}_r(R)$ such that $\alpha S_r(v, w) \beta$ is a special Suslin matrix for all special Suslin matrices $S_r(v, w)$, then $\beta = u \alpha^\vee$, with $u^2 = 1$. 

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Proof: We can write $\alpha S_r(v,w)\beta = \alpha S_r(v,w)\alpha^\vee(\alpha^{-1}\beta)$. By the Fundamental Property, $\alpha S_r(v,w)\alpha^\vee$ is a special Suslin matrix, say $S_r(x,y)$ for some $x,y \in R^{r+1}$. Therefore, $S_r(x,y)\alpha^\vee\beta$ is a special Suslin matrix. Take $x = e_1$, $y = e_1$, we get $\alpha^{-1}\beta$ is a special Suslin matrix say $S_r(p,q)$. Moreover, $\langle p,q \rangle = 1$. Thus, $S_r(x,y)S_r(p,q)$ is a Suslin matrix for any special Suslin matrix $S_r(x,y)$. Hence, by Lemma 3.2.1, $S_r(p,q) = uI_{2r}$, $u^2 = 1$. Thus $\beta = u\alpha^\vee$, with $u^2 = 1$. □

Corollary 3.2.5 (Description of $Z(\text{SUM}_r(R)), r > 1$)

$$Z(\text{SUM}_r(R)) = \begin{cases} 
\{uI_{2r} : u \in R, u^2 = 1\}, & \text{if } r \text{ odd} \\
\{uI_{2r} : u \in R, u^4 = 1\}, & \text{if } r \text{ even.}
\end{cases}$$

Proof: Let $\alpha \in Z(\text{SUM}_r(R))$. Then by Proposition 2.3.2, $\alpha = uI_{2r}$ for some $u \in R$. If $r$ is odd, then by the Suslin identities (see Lemma 1.2.12), $uI_{2r}J_r(uI_{2r})^T = J_r$. Hence $u^2 = 1$ and $Z(\text{SUM}_r(R)) = \{uI_{2r} : u \in R, u^2 = 1\}$.

Let $r$ be even. Then by the Suslin identities $\alpha^\vee = J_r\alpha J_r^{-1} = uI_{2r}$. By the Fundamental Property $\alpha\alpha^\vee$ is a special Suslin matrix. But $\alpha\alpha^\vee = u^2I_{2r} = S_r(u^2e_1, u^2e_1)$. Hence $\langle u^2, u^2 \rangle = 1$, i.e. $u^4 = 1$. Hence $Z(\text{SUM}_r(R)) \subseteq \{uI_{2r} : u \in R, u^4 = 1\}$. In Chapter 7, Proposition 7.2.11 we shall prove the reverse inclusion. □

Corollary 3.2.6 The operation $\lor$ induces an anti-involution $\ast$ on $\text{SUM}_r(R)$, if $r$ even, and an anti-involution $\ast$ on $\text{SUM}_r(R)/Z(\text{SUM}_r(R))$, if $r > 1$ odd.

Proof: By Remark 3.2.3 (i), $\ast$ is an anti-involution on $\text{SUM}_r(R)$, if $r$ is even. Let $r$ be odd. If $\prod_{i=1}^{k} S_r(v_i, w_i) = I_{2r}$ then by Corollary 3.2.2,

$$\prod_{i=k}^{1} S_r(v_i, w_i) = uI_{2r},$$
for some unit $u \in R$, with $u^2 = 1$. Hence if

$$S_r(v_1, w_1) \cdots S_r(v_k, w_k) = S_r(v'_1, w'_1) \cdots S_r(v'_k, w'_k),$$

then

$$S_r(v_k, w_k) \cdots S_r(v_1, w_1) = u S_r(v'_k, w'_k) \cdots S_r(v'_1, w'_1),$$

for some unit $u \in R$, with $u^2 = 1$. Hence by Corollary 3.2.5,

$$S_r(v_k, w_k) \cdots S_r(v_1, w_1) \mapsto S_r(v'_k, w'_k) \cdots S_r(v'_1, w'_1),$$

is an anti-involution on $\text{SUM}_r(R)/\mathbb{Z}$. □

### 3.3 Suslin Matrices and Reflections

The anti-involution $\ast$, via the Fundamental Property of Suslin matrices, enables one to define an action of the group $\text{SUM}_r(R)$ on the Suslin space $\mathbb{S}_r$, if $r$ is even. The operation $\vee$ on $\text{SUM}_r(R), r > 1$, enables one to associate a map $T_{S_r(v, w)} : \mathbb{S}_r \to \mathbb{S}_r$ for $S_r(v, w) \in \text{SUM}_r(R)$, via $T_{S_r(v, w)}(x, y) = (x', y')$, where $S_r(v, w)S_r(x, y)S_r(v, w) = S_r(x', y')$.

**Lemma 3.3.1** For each $S_r(v, w) \in \text{SUM}_r(R), T_{S_r(v, w)}$ is a linear transformation on the Suslin space $\mathbb{S}_r$. Moreover, if $g = S_r(v_1, w_1) \cdots S_r(v_k, w_k) \in \text{SUM}_r(R), r > 1$, where $\langle v_i, w_i \rangle = 1, 1 \leq i \leq k$, then $T_g \in O_{2(r+1)}(R)$ (defined w.r.t. the above splitting if $r$ odd), i.e.

$$\langle T_g(v, w), T_g(s, t) \rangle = \langle (v, w), (s, t) \rangle = v \cdot t^T + w \cdot s^T.$$

**Proof:** It is easy to check that $T_g$ is a linear transformation on the Suslin space $\mathbb{S}_r$. The fact that it is an orthogonal transformation is evident from
the Fundamental Property, as one has for \( g = S_r(v, w) \), that

\[
\langle T_g(x, y), T_g(x, y) \rangle = \langle (x', y'), (x', y') \rangle = \langle x', y' \rangle + \langle y', x' \rangle \\
= \langle x', y' \rangle + \langle x', y' \rangle = 2\langle x', y' \rangle = 2\langle v, w \rangle^2 \langle x, y \rangle \\
= \langle v, w \rangle^2 (\langle x, y \rangle + \langle x, y \rangle) = \langle v, w \rangle^2 (\langle x, y \rangle + \langle y, x \rangle) \\
= \langle v, w \rangle^2 (\langle x, y \rangle, (x, y)) = 2 \langle v, w \rangle^2 (\langle x, y \rangle, (x, y)).
\]

Hence, \( T_g \) is orthogonal if \( \langle v, w \rangle = v \cdot w^T = 1 \).

\[\square\]

**Remark 3.3.2** One can show that \( T_g \) is orthogonal if we take the inner product \( \langle (v, w), (s, t) \rangle = v \cdot w^T + s \cdot t^T \). We have

\[
\langle T_g(x, y), T_g(x, y) \rangle = \langle (x', y'), (x', y') \rangle = \langle x', y' \rangle + \langle x', y' \rangle \\
= 2\langle x', y' \rangle = 2\langle v, w \rangle^2 \langle x, y \rangle \\
= \langle v, w \rangle^2 (\langle x, y \rangle + \langle x, y \rangle) = \langle v, w \rangle^2 (\langle x, y \rangle, (x, y)).
\]

**Definition 3.3.3** Let \( \pi \) denote the permutation \((1 \ r + 1) \ldots (r \ 2r)\) corresponding to the form \( I_r \cap I_r \). We denote by \( \text{SO}_{2r}(R) \), the **Special Orthogonal group** with respect to the above form.

**Lemma 3.3.4** If \( r \) is even, then the anti-involution \( * \) induces a canonical homomorphism \( \varphi : \text{SU}_{n}(R) \to \text{SO}_{2r+1}(R) \), by \( g \mapsto T_g \). Moreover,

\[
T_{S_r(s, t)} = \tau(s, t) \circ \tau(e_1, e_1),
\]

where \( \tau(s, t) \) is the standard reflection with respect to the vector \( (s, t) \in R^{2(r+1)} \) (of length one) given by the formula

\[
\tau(s, t)(v, w) = (s, t)(v, w) - (\langle s, w \rangle + \langle t, v \rangle)(s, t).
\]

This formula is also true when \( r \) is odd.
Proof: $\varphi$ is well-defined via Corollary 3.2.6. $\varphi$ is clearly a homomorphism as $\ast$ is an anti-involution: $\varphi(g_1g_2) = T_{g_1g_2} = T_{g_1} \circ T_{g_2}$ as

$$T_{g_1} \circ T_{g_2}(x, y) = T_{g_1}(g_2S_r(x, y)g_2^*) = g_1(g_2S_r(x, y)g_2^*)g_1^* = (g_1g_2)S_r(x, y)(g_1g_2)^* = T_{g_1g_2}(x, y) \forall (x, y) \in R^{+1} \times R^{+1}$$

Let $v = (v_0, v_1), w = (w_0, w_1), s = (s_0, s_1), t = (t_0, t_1) \in M_{1r+1}(R)$, where $v_0, w_0, s_0, t_0 \in R$ and $v_1, w_1, s_1, t_1 \in M_{1r}(R)$. From the proof of Fundamental Property (Lemma 3.1.1),

$$T_{S_r(s, t)}(v, w) = (v', w') = ((v'_0, v'_1), (w'_0, w'_1)),$$

where

$$v'_0 = s_0^2v_0 - s_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - w_0\langle s_1, t_1 \rangle \in R,$$

$$v'_1 = (s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)s_1 + \langle s, t \rangle v_1 \in M_{1r}(R),$$

$$w'_0 = t_0^2w_0 - t_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - v_0\langle s_1, t_1 \rangle \in R,$$

$$w'_1 = (s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)t_1 + \langle s, t \rangle w_1 \in M_{1r}(R).$$

Thus

$$v' = (v'_0, v'_1)$$

$$= ((s_0^2v_0 - s_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - w_0\langle s_1, t_1 \rangle),$$

$$(s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)s_1 + \langle s, t \rangle v_1))$$

$$= ((s_0^2v_0 + s_0t_0w_0 - s_0(\langle s_1, w_1 \rangle + \langle v_1, t_1 \rangle) - w_0\langle s, t \rangle),$$

$$(s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle v_1, t_1 \rangle)s_1 + \langle s, t \rangle v_1))$$

$$= (-\langle s, t \rangle w_0, \langle s, t \rangle v_1) + (s_0v_0 + t_0w_0 - \langle s_1, w_1 \rangle - \langle t_1, v_1 \rangle) s$$

$$= \langle s, t \rangle (-w_0, v_1) - (\langle s, (-v_0, w_1) \rangle + \langle t, (-w_0, v_1) \rangle) s.$$
Similarly one can show that
\[ w' = (w'_0, w'_1) = \langle s, t \rangle (-v_0, w_1) - (\langle s, (-v_0, w_1) \rangle + \langle t, (-w_0, v_1) \rangle) t. \]
Thus \( T_{S_r(s, t)}(v, w) = (v', w') \)
\[ = (s, t)((-w_0, v_1), (-v_0, w_1)) - (s, (-v_0, w_1)) (s, t) \]
\[ = \tau_{(s, t)}((-w_0, v_1), (-v_0, w_1)) \]
\[ = \tau_{(s, t)}((v, w) - (v_0 + w_0)(e_1, e_1)) = \tau_{(s, t)} \circ \tau_{(e_1, e_1)}(v, w). \]
Hence \( T_{S_r(s, t)} = \tau_{(s, t)} \circ \tau_{(e_1, e_1)}. \)

We next compute \( \ker(\varphi) \). We need the following preliminary observation.

**Lemma 3.3.5** Let \( \alpha = S_r(v_1, w_1) \cdots S_r(v_k, w_k) \in \text{SUM}_r(R), r > 1, \) with \( \langle v_i, w_i \rangle = 1, 1 \leq i \leq k. \) If \( \alpha S_r(v, w) \alpha^\vee = S_r(v, w) \), and \( \alpha S_r(v, w)^2 \alpha^\vee = S_r(v, w)^2 \), then \( [\alpha, S_r(v, w)] = 1. \) Consequently, if \( \alpha S_r(v, w) \alpha^\vee = S_r(v, w) \), for all \( S_r(v, w) \in \text{SUM}_r(R) \), then \( \alpha \in \text{Z}(\text{SUM}_r(R)) \); and so is a scalar \( uI_{2r} \) for some unit \( u \in R. \)

**Proof:** By the Fundamental Property \( S_r(x, y)^2 = S_r(p, q) \), for some \( p, q \in R^{r+1} \) with \( \langle x, y \rangle^2 = \langle p, q \rangle \). Now
\[ S_r(v, w)^2 = \alpha S_r(v, w)^2 \alpha^\vee = \alpha S_r(v, w) \alpha^{-1} \alpha S_r(v, w) \alpha^\vee \]
\[ = \alpha S_r(v, w) \alpha^{-1} S_r(v, w), \]
by the hypothesis. Multiply both sides on the right by \( S_r(w, v)^T \), then via Lemma 1.2.1, one gets
\[ S_r(v, w) \langle v, w \rangle = \alpha S_r(v, w) \alpha^{-1} \langle v, w \rangle, \]
whence \( S_r(v, w) = \alpha S_r(v, w) \alpha^{-1} \) as \( \langle v, w \rangle = 1. \) Thus \( [\alpha, S_r(v, w)] = 1. \) If \( \alpha S_r(v, w) \alpha^\vee = S_r(v, w) \), for all \( S_r(v, w) \in \text{SUM}_r(R) \), then \( \alpha S_r(v, w)^2 \alpha^\vee = S_r(v, w) \), as \( S_r(v, w)^2 \) is a special Suslin matrix. The rest follows easily. \( \Box \)
Lemma 3.3.6 \( \ker(\varphi) = \{ uI_{2r} : u \in R, u^2 = 1 \} \subseteq Z(\text{SUM}_r(R)) \).

Proof: By definition, and via Lemma 3.3.5,

\[
\ker \varphi = \{ g \in \text{SUM}_r(R) : T_g = \text{Id} \} \\
= \{ g \in \text{SUM}_r(R) : gS_r(x, y)g^\vee = S_r(x, y) \forall S_r(x, y) \text{ with } \langle x, y \rangle = 1 \} \\
\subseteq Z(\text{SUM}_r(R)) = \{ uI_{2r} : u \text{ unit} \}.
\]

Clearly, \( Tu_{I_{2r}} = u^2I_{2(r+1)} \). Hence, \( \ker(\varphi) = \{ uI_{2r} : u \in R, u^2 = 1 \} \). \( \square \)