Chapter 6

Quillen-Suslin Theory

D. Quillen-A.A. Suslin independently proved in 1976 the conjecture of J-P. Serre that a finitely generated projective module over a polynomial extension of a field is free. There were two main principles involved in their proofs, viz. a Local Global Principle, and a Monic Inversion Principle. Later in 1977, A.A. Suslin extended these principles to the pair \((\text{Gl}_r(R), E_r(R))\), and also for the symplectic and orthogonal groups. These principles are known as the Quillen-Suslin Theory. We establish the Quillen-Suslin theory for the pair \((\text{SUM}_r(R), \text{EUm}_r(R))\).

As a consequence, we show that for \(r > 1\), \(\text{EUm}_r(R)\) is a normal subgroup of \(\text{SUM}_r(R)\). We also prove that \(\frac{\text{SUM}_r(R)}{\text{EUm}_r(R)}\) is a subgroup of \(\frac{\text{SO}_{2(r+1)}(R)}{\text{EO}_{2(r+1)}(R)}\), for all \(r > 1\).

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By Proposition 5.1.6, if \(r\) is even, all questions concerning the group \(\text{EUm}_r(R)\) can be reduced to the corresponding questions regarding elementary orthog-
quaternion matrices.

For example, one has a Quillen-Suslin theory for the elementary orthogonal groups $\text{EO}_{2r}(R[X])$ due to results of A.A. Suslin - V.I. Kopeiko. Using this we show that both the Local Global Principle and the Monic Inversion Principle of Quillen-Suslin hold for the Elementary Unimodular vector group $\text{EUm}_r(R[X])$. We record this explicitly below:

**Proposition 6.1.1 (Local Global Principle for $\text{EUm}_r(R[X]), r > 1$)**

Let $\alpha(X) \in \text{SUM}_r(R[X]), r > 1$, with $\alpha(0) = I_{2r}$. Suppose that for every $m \in \text{Max}(R)$, $\alpha(X)_m \in \text{EUm}_r(R_m[X])$. Then $\alpha(X) \in \text{EUm}_r(R[X])$.

**Proof:** We prove it when $r$ is even; and a similar argument can be given when $r$ is odd. By above remarks, $T_{\alpha(X)_m} \in \text{EO}_{2(r+1)}(R_m[X])$, for all $m \in \text{Max}(R)$. By ([10], Corollary 4.4), since $T_{\alpha(0)} = I_{2(r+1)}$, $T_{\alpha(X)} \in \text{EO}_{2(r+1)}(R[X])$. By Lemma 5.1.6, $\varphi(\varepsilon(X)) = \varphi(\alpha(X))$, for some $\varepsilon(X) \in \text{EUm}_r(R[X])$.

Now by Lemma 3.3.6, Lemma 4.2.6, and Remark 4.2.7

$$\alpha(X)\varepsilon(X)^{-1} \in \ker(\varphi) \subseteq Z(\text{SUM}_r(R[X])) \subseteq \text{EUm}_r(R[X]).$$

Thus, $\alpha(X) \in \text{EUm}_r(R[X])$. □

For an independent proof of the Local Global Principle for $\text{EUm}_r(R), r > 1$, please refer to [6].

**Corollary 6.1.2 (Normality of $\text{EUm}_r(R), r > 1$)**

The Elementary Unimodular Vector group $\text{EUm}_r(R)$ is a normal subgroup of $\text{SUM}_r(R)$, for $r > 1$.

**Proof:** Consider $S_r(e_1\varepsilon, e_1\varepsilon^T^{-1}) \in \text{EUm}_r(R)$, for some $\varepsilon \in E_{r+1}(R)$. Let $\varepsilon(T) \in E_{r+1}(R[T])$ with $\varepsilon(1) = \varepsilon, \varepsilon(0) = I_{r+1}$. 
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Let $\gamma \in \text{SUM}_r(R)$, and consider

$$\Gamma(T) = \gamma S_r(e_1 \varepsilon(T), e_1 \varepsilon(T)^{T^{-1}}) \gamma^{-1} \in \text{SUM}_r(R[T]).$$

For every $m \in \text{Max}(R)$, $\Gamma(T)_m \in \text{EUm}_r(R_m[T])$. By Proposition 6.1.1, $\Gamma(T) \in \text{EUm}_r(R[T])$. Now put $T = 1$, to get $\gamma S_r(e_1 \varepsilon, e_1 \varepsilon) \gamma^{-1} \in \text{EUm}_r(R)$, as required. \hfill $\Box$

**Proposition 6.1.3 (Monic Inversion Principle for EUm$_r(R[X]), r > 1$)**

Let $\alpha(X) \in \text{SUM}_r(R[X]), r > 1$. Let $f(X) \in R[X]$ be a monic polynomial, and let $\alpha(X)_f(X) \in \text{EUm}_r(R[X, 1/f(X)])$. Then $\alpha(X) \in \text{EUm}_r(R[X])$.

**Proof:** We prove the result when $r$ is even, a similar argument can be given when $r$ is odd. For a commutative ring $A$ with 1, one has an exact sequence, with commutative squares,

$$\text{EUM}_r(A) = \text{SUM}_r(A) \xrightarrow{\varphi} \text{EO}_{2(r+1)}(A) \rightarrow 0$$

$$1 \rightarrow \{u \in R : u^2 = 1\} \rightarrow \text{SUM}_r(A) \xrightarrow{\varphi} \text{SO}_{2(r+1)}(A).$$

Take $A = R[X, 1/f(X)]$. Then $\varphi(\alpha(X)_f(X)) \in \text{EO}_{2(r+1)}(A)$. By ([10], Theorem 6.9), $\varphi(\alpha(X)) \in \text{EO}_{2(r+1)}(R[X])$. By Proposition 5.1.6, $\varphi(\alpha(X)) = \varphi(\varepsilon(X))$, for some $\varepsilon(X) \in \text{EUm}_r(R[X])$.

Then by Lemma 3.3.6, Lemma 4.2.6, and Remark 4.2.7

$$\alpha(X) \varepsilon(X)^{-1} \in \ker(\varphi) \subseteq Z(\text{SUM}_r(R[X])) \subseteq \text{EUm}_r(R[X]).$$

Thus, $\alpha(X) \in \text{EUm}_r(R[X])$, as required. \hfill $\Box$

### 6.2 SUM$_r(R)/EUM$_r(R) is Nilpotent

In this section we prove that SUM$_r(R)/EUM$_r(R) is a subgroup of the special orthogonal quotient group SO$_{2(r+1)}(R)/EO_{2(r+1)}(R)$, and hence is nilpotent.
of class $\leq d$, for $r \geq 2$.

**Theorem 6.2.1** For all $r \geq 2$, one has an injective homomorphism

$$
\varphi : \text{SUM}_r(R) \to \text{SO}_{2(r+1)}(R) \to \text{EO}_{2(r+1)}(R).
$$

(This is induced by $\varphi : \text{SUM}_r(R) \to \text{SO}_{2(r+1)}(R)$, when $r$ is even.)

**Proof:** Let $r$ be odd. Let $g \in \text{SUM}_r(R)$. By Corollary 3.3.1, $T_g$ (defined w.r.t. some splitting of $g$) is an orthogonal transformation. By Corollary 3.3.1, one has $T_g \in \text{SO}_{2(r+1)}(R)$. By Proposition 5.1.4, if $g \in \text{EU}_r(R)$ then $T_g \in \text{EO}_{2(r+1)}(R)$ w.r.t. a splitting of $g$ as a product of elementary generators in $\text{EU}_r(R)$ of the type $E(c)(\lambda)^b$. Note that if one computes $T_g$ w.r.t. any other splitting of $g$, then due to Corollary 3.2.2, the matrix of the linear transformation atmost differs by a multiple of a unit $u$, with $u^2 = 1$. But $uI_{2(r+1)} \in \text{EO}_{2(r+1)}(R)$. Hence, $\varphi : g \mapsto T_g \in \text{SO}_{2(r+1)}(R)/\text{EO}_{2(r+1)}(R)$ is a well-defined homomorphism.

The injectivity of $\varphi$: If $\varphi(g) = 1$, then $T_g = I_{2(r+1)}$. Therefore, w.r.t. some splitting of $g$, there exists a unit $u \in R$, with $u^2 = 1$, such that

$$
gS_r(v, w)g^\vee = uS_r(v, w),
$$

for any special Suslin matrix $S_r(v, w)$.

We shall now avail of a result Corollary 7.2.9 which is proved in the next chapter, viz. for any unit $u \in R$ with $u^2 = 1$, $uI_{2r} = \varepsilon_1 \ldots \varepsilon_k$, for some $\varepsilon_1, \ldots, \varepsilon_k \in \text{EU}_r(R)$, $\varepsilon_i = S_r(v_i, w_i)$, $\langle v_i, w_i \rangle = 1$, $v_i \in e_1E_{r+1}(R)$, and with $\varepsilon_k \ldots \varepsilon_1 = I_{2r}$. Then $uS_r(v, w) = \varepsilon_1 \ldots \varepsilon_kS_r(v, w)\varepsilon_k \ldots \varepsilon_1$. Thus,

$$
\varepsilon_k^{-1} \ldots \varepsilon_1^{-1} gS_r(v, w)g^\vee \varepsilon_1^{-1} \ldots \varepsilon_k^{-1} = S_r(v, w).
$$

Therefore, by Lemma 3.3.5, $\varepsilon_k^{-1} \ldots \varepsilon_1^{-1} g \in Z(\text{SUM}_r(R))$ and by Lemma 4.2.6, $\varepsilon_k^{-1} \ldots \varepsilon_1^{-1} g \in \text{EU}_r(R)$. Hence, $g \in \text{EU}_r(R)$. The case when $r$ is even is proved similarly. $\square$
Corollary 6.2.2 Let $R$ be a commutative noetherian ring with 1 and let $\dim R = d$. Then the group $\text{SUM}_r(R)/\text{EU}_r(R)$ is nilpotent of class $\leq d$, for $r \geq 2$.

Proof: By the main theorems of [4], [3], (see also [1]) the orthogonal quotient group $\text{SO}_{2(r+1)}(R)/\text{EO}_{2(r+1)}(R)$, $r > 1$, is nilpotent of class $d$, and so its subgroup $\text{SUM}_r(R)/\text{EU}_r(R)$ is nilpotent of class $\leq d$. □