CHAPTER 3

FUZZY PSEUDO \((a, p, q)\)-IDEALS AND IMPLICATIVE PSEUDO IDEALS IN PSEUDO-\(BCI\) ALGEBRAS
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CHAPTER 3

Fuzzy pseudo \((a, p, q)\)-ideals and implicative pseudo-ideals in pseudo-

BCI algebras

3.1. Introduction
In this Chapter we generalize several results of Y.B.Jun ([41]) and
Y.L.Li et al ([42]) and we investigate some of their properties.

3.2. Fuzzy implicative pseudo-ideals
In this Section, we give characterizations of fuzzy implicative pseudo-
ideals in pseudo-\(BCK\) algebras and we prove the family of fuzzy im-
pllicative pseudo-ideals is a completely distributive lattice.

Definition 3.2.1 ([39]): A pseudo-\(BCI\) algebra is a structure
\((X, \preceq, *, \ominus, 0)\), where “ \(\preceq\)” is a binary relation on \(X\), “ \(*\)” and “ \(\ominus\)” are binary operations on \(X\) and “ \(0\)” is an element of \(X\), verifying the
axioms:

\[
\begin{align*}
(a1) \quad (x * y) \ominus (x * z) \preceq z * y, & \quad (x \ominus y) * (x \ominus z) \preceq z \ominus y, \\
(a2) \quad x * (x \ominus y) \preceq y, & \quad x \ominus (x * y) \preceq y, \\
(a3) \quad x \preceq x, \\
(a4) \quad x \preceq y, \quad y \preceq x \Rightarrow x = y, \\
(a5) \quad x \preceq y \iff x * y = 0 \iff x \ominus y = 0, \\
(a6) \quad x \preceq 0 \Rightarrow x = 0,
\end{align*}
\]
for all $x, y, z \in X$. If a pseudo-$BCI$ algebra $X$ satisfies the identity 
$0 \preceq x$, for all $x \in X$, then $X$ is called a pseudo-$BCK$ algebra.

If $X$ is a pseudo-$BCI$ algebra satisfying $x \ast y = x \odot y$ for all $x, y \in X$, then $X$ is a $BCI$-algebra. Every pseudo-$BCK$ algebra is a pseudo-$BCI$ algebra.

**Proposition 3.2.2 ([40]):** In a pseudo-$BCI$ algebra $X$, the following holds for all $x, y, z \in X$,

$(p1)$ \hspace{1em} $x \preceq y \Rightarrow z \ast y \preceq z \ast x, \hspace{1em} z \odot y \preceq z \odot x.$

$(p2)$ \hspace{1em} $x \preceq y, \hspace{1em} y \preceq z \Rightarrow x \preceq z.$

$(p3)$ \hspace{1em} $(x \ast y) \odot z = (x \odot z) \ast y.$

$(p4)$ \hspace{1em} $x \ast y \preceq z \iff x \odot z \preceq y.$

$(p5)$ \hspace{1em} $x \preceq y \Rightarrow x \ast z \preceq y \ast z, \hspace{1em} x \odot z \preceq y \odot z.$

$(p6)$ \hspace{1em} $x \ast 0 = x = x \odot 0.$

$(p7)$ \hspace{1em} $x \ast (x \odot (x \ast y)) = x \ast y \hspace{1em}$ and \hspace{1em} $x \odot (x \ast (x \odot y)) = x \odot y.$

**Proposition 3.2.3 ([42]):** In a pseudo-$BCI$ algebra $X$, the following holds for all $x, y \in X$:

$(i)$ \hspace{1em} $0 \ast (x \odot y) \preceq y \odot x,$

$(ii)$ \hspace{1em} $0 \odot (x \ast y) \preceq y \ast x,$

$(iii)$ \hspace{1em} $0 \ast (x \ast y) = (0 \odot x) \odot (0 \ast y),$

$(iv)$ \hspace{1em} $0 \odot (x \odot y) = (0 \ast x) \ast (0 \odot y).$
For any nonempty subset $I$ of $X$ and any element $y$ of $X$, we denote
$$*(y,I) = \{ x \in X \mid x \star y \in I \} \text{ and } \oslash (y,I) = \{ x \in X \mid x \oslash y \in I \}.$$ Note that
$$*(y,I) \cap \oslash (y,I) = \{ x \in X \mid x \star y \in I, \ x \oslash y \in I \}.$$

**Definition 3.2.4 ([40]):** A nonempty subset $I$ of a pseudo-$BCI(BCK)$ algebra $X$ is called a pseudo-ideal if it satisfies:

1. $(PI1) \quad 0 \in I,$
2. $(PI2) \quad \forall y \in I, \quad * (y, I) \subseteq I \text{ and } \oslash (y, I) \subseteq I.$

**Definition 3.2.5 ([41]):** A fuzzy set $\mu : X \to [0,1]$ is called a fuzzy pseudo-ideal of $X$ if for every $t \in Im(\mu)$, $U(\mu, t)$ is pseudo-ideal of $X$.

In what follows, let $X$ be denote a pseudo-$BCK$ algebra unless otherwise specified.

**Theorem 3.2.6 ([41]):** A fuzzy set $\mu$ is a fuzzy pseudo-ideal of $X$ if and only if it satisfies

1. $(i) \quad \mu(0) \geq \mu(x), \ \forall x \in X,$
2. $(ii) \quad \mu(x) \geq \min\{\mu(x \star y), \ \mu(y)\}, \ \forall x, y \in X \text{ and}$
3. $(iii) \quad \mu(a) \geq \min\{\mu(a \oslash b), \ \mu(b)\}, \ \forall a, b \in X.$

**Corollary 3.2.7 ([41]):** Let $\mu$ be a fuzzy pseudo-ideal of $X$. If $x \preceq y$, then $\mu(x) \geq \mu(y)$. 
Example 3.2.8 ([42]): Let $X = [0, \infty)$ and let $\leq$ be the usual order on $X$. Define binary operations $\ast$ and $\odot$ on $X$ by
\[
x \ast y = \begin{cases} 
0 & \text{if } x \leq y, \\
\frac{2x}{\pi} \arctan(\ln\left(\frac{x}{y}\right)) & \text{if } y < x,
\end{cases}
\]
\[
x \odot y = \begin{cases} 
0 & \text{if } x \leq y, \\
x e^{-\tan\left(\frac{\pi x}{2}\right)} & \text{if } y < x,
\end{cases}
\]
for all $x, y \in X$. Then $(X, \preceq, \ast, \odot, 0)$ is a pseudo-$BCK$ algebra, and so a pseudo-$BCI$ algebra.

Theorem 3.2.9: A fuzzy set $\mu$ is a fuzzy pseudo-ideal of $X$ if and only if it satisfies

(i) $\forall x, y, z \in X, x \ast y \preceq z \Rightarrow \mu(x) \geq \min\{\mu(y), \mu(z)\}$ and

(ii) $\forall a, b, c \in X, a \odot b \preceq c \Rightarrow \mu(a) \geq \min\{\mu(b), \mu(c)\}$.

Proof: Assume that $\mu$ is a fuzzy pseudo-ideal of $X$. Let $x, y, z \in X$ be such that $x \ast y \preceq z$. Then
\[
\mu(x \ast y) \geq \min\{\mu((x \ast y) \odot z), \mu(z)\} = \min\{\mu(0), \mu(z)\} = \mu(z)
\]
and so, by Theorem 3.2.6(iii),
\[
\mu(x) \geq \min\{\mu(x \ast y), \mu(y)\} \geq \min\{\mu(y), \mu(z)\}.
\]
Now assume that $a \odot b \preceq c$ for all $a, b, c \in X$. Then by Theorem 3.2.6(iii)
\[
\mu(a \odot b) \geq \min\{\mu((a \odot b) \ast c), \mu(c)\} = \min\{\mu(0), \mu(c)\} = \mu(c).
\]
Hence
\[
\mu(a) \geq \min\{\mu(a \odot b), \mu(b)\} \geq \min\{\mu(b), \mu(c)\}.
\]

Conversely, let \( \mu \) be a fuzzy set satisfying conditions (i) and (ii). Since
\[
0 \ast x \preceq x \quad \text{and} \quad 0 \odot x \preceq x \quad \text{for all} \quad x \in X,
\]
it follows from (i) and (ii) that
\[
\mu(0) \geq \mu(x) \quad \text{for all} \quad x \in X.
\]
Since \( x \ast (x \odot y) \preceq y \quad \text{for all} \quad x, y \in X, \) we have
\[
\mu(x) \geq \min\{\mu(x \odot y), \mu(y)\}, \quad \text{by (i)}.
\]
Note also that \( a \odot (a \ast b) \preceq b \quad \text{for all} \quad a, b \in X. \) Hence, by (ii), we get
\[
\mu(a) \geq \min\{\mu(a \ast b), \mu(b)\}.
\]
It follows from Theorem 3.2.6 that \( \mu \) is a fuzzy pseudo-ideal of \( X. \) This completes the proof.

\[\square\]

Let \( \mu \) be a fuzzy set in \( X. \) For any \( w \in X, \) we consider the set,
\[
\hat{U}(\mu, w) = \{ x \in X \mid \mu(w) \leq \mu(x) \}.
\]

Obviously, \( w \in \hat{U}(\mu, w). \) If \( \mu \) is a fuzzy pseudo-ideal of \( X, \) then \( 0 \in \hat{U}(\mu, w). \)

**Theorem 3.2.10:** Let \( w \in X. \) If \( \mu \) is a fuzzy pseudo-ideal of \( X, \) then \( \hat{U}(\mu, w) \) is a pseudo-ideal of \( X. \)

**Proof:** Recall that \( 0 \in \hat{U}(\mu, w). \) For every \( y \in \hat{U}(\mu, w), \) let
\[
a \in \ast(y, \hat{U}(\mu, w)) \quad \text{and} \quad b \in \odot(y, \hat{U}(\mu, w)).
\]
Then \( a \ast y \in \hat{U}(\mu, w) \) and \( b \odot y \in \hat{U}(\mu, w), \) further \( \mu(w) \leq \mu(a \ast y), \quad \mu(w) \leq \mu(b \odot y). \)

Since \( \mu \) is a fuzzy pseudo-ideal of \( X, \) it follows that
\[
\mu(a) \geq \min\{\mu(a \ast y), \mu(y)\} \geq \mu(w)
\]
and

\[ \mu(b) \geq \min \{ \mu(b \otimes y), \mu(y) \} \geq \mu(w) \]

and so \( a \in \hat{U}(\mu, w) \) and \( b \in \hat{U}(\mu, w) \). This shows that

\[ \ast(y, \hat{U}(\mu, w)) \subseteq \hat{U}(\mu, w) \text{ and } \ominus(y, \hat{U}(\mu, w)) \subseteq \hat{U}(\mu, w). \]

Therefore \( \hat{U}(\mu, w) \) is a pseudo-ideal of \( X \).

\[ \square \]

**Theorem 3.2.11:** Let \( \mu \) be a fuzzy set in \( X \) and \( w \in X \).

(i) If \( \hat{U}(\mu, w) \) is a pseudo-ideal of \( X \), then \( \mu \) satisfies the following implications:

\[ \forall x, y, z \in X, \ \mu(x) \leq \min \{ \mu(y * z), \mu(z) \} \Rightarrow \mu(x) \leq \mu(y), \]

(ii) If \( \mu \) satisfies Theorem 3.2.6(i) and conditions (1),(2) then \( \hat{U}(\mu, w) \) is a pseudo-ideal of \( X \).

**Proof:** (i) Let \( \hat{U}(\mu, w) \) be a pseudo-ideal of \( X \) for each \( w \in X \). Let \( x, y, z \in X \) be such that \( \mu(x) \leq \min \{ \mu(y * z), \mu(z) \} \). Then

\( y * z \in \hat{U}(\mu, x) \) and hence \( y \in \ast(z, \hat{U}(\mu, x)) \). Since \( \hat{U}(\mu, x) \) is a pseudo-ideal of \( X \), \( y \in \hat{U}(\mu, x) \), that is \( \mu(x) \leq \mu(y) \).

Similarly, if \( \mu(x) \leq \min \{ \mu(y \odot z), \mu(z) \} \), then \( y \odot z \in \hat{U}(\mu, x) \), that is \( y \in \odot(z, \hat{U}(\mu, x)) \). Since \( \hat{U}(\mu, x) \) is a pseudo-ideal of \( X \), \( y \in \hat{U}(\mu, x) \), that is \( \mu(x) \leq \mu(y) \).

(ii) Suppose that \( \mu \) satisfies Theorem 3.2.6(i), (1) and (2).

For every \( y \in \hat{U}(\mu, w) \), let \( a \in \ast(y, \hat{U}(\mu, w)) \) and \( b \in \odot(y, \hat{U}(\mu, w)) \).
Then \( a \ast y \in \widehat{U}(\mu, w) \) and \( b \oslash y \in \widehat{U}(\mu, w) \). Therefore

\[
\mu(a \ast y) \geq \mu(w) \text{ and } \mu(b \oslash y) \geq \mu(w).
\]

Hence

\[
\mu(w) \leq \min\{\mu(a \ast y), \mu(y)\} \text{ and } \mu(w) \leq \min\{\mu(b \oslash y), \mu(y)\}.
\]

Using (1) and (2), we have \( \mu(w) \leq \mu(a) \) and \( \mu(w) \leq \mu(b) \). Then \( a \in \widehat{U}(\mu, w) \), \( b \in \widehat{U}(\mu, w) \). Since \( \mu \) satisfies Theorem 3.2.6(i), it follows that \( 0 \in \widehat{U}(\mu, w) \). Therefore \( \widehat{U}(\mu, w) \) is a pseudo-ideal of \( X \).

\[\square\]

**Definition 3.2.12**: A fuzzy set \( \mu \) is called a **fuzzy implicative pseudo-ideal** of \( X \) if

(M1) \( \mu(x \oslash z) \geq \min\{\mu((x \ast y) \oslash z), \mu(y \oslash z)\}, \forall x, y, z \in X \) and

(M2) \( \mu(a \ast c) \geq \min\{\mu((a \oslash b) \ast c), \mu(b \ast c)\}, \forall a, b, c \in X \).

**Lemma 3.2.13**: Every fuzzy implicative pseudo-ideal is a fuzzy pseudo-ideal of \( X \).

**Proof**: Let \( \mu \) be a fuzzy implicative pseudo-ideal of \( X \) and let \( t \in Im(\mu) \). For every \( y \in U(\mu, t) \), let \( a \in *(y, U(\mu, t)) \) and \( b \in (y, U(\mu, t)) \). Then \( (a \ast y) \oslash 0 = a \ast y \in U(\mu, t) \) and \( (b \oslash y) \ast 0 = b \oslash y \in U(\mu, t) \). Since \( y \oslash 0 = y \in U(\mu, t) \) and \( y \oslash 0 = y \in U(\mu, t) \), it follows that

\[
\mu(a) = \mu(a \oslash 0) \geq \min\{\mu((a \ast y) \oslash 0), \mu(y \oslash 0)\} \geq t
\]

and

\[
\mu(b) = \mu(b \ast 0) \geq \min\{\mu((b \oslash y) \ast 0), \mu(y \ast 0)\} \geq t.
\]
so that \( a, b \in U(\mu, t) \). This shows that

\[(y, U(\mu, t)) \subseteq U(\mu, t) \text{ and } (y, U(\mu, t)) \subseteq U(\mu, t).\]

Hence \( U(\mu, t) \) is a pseudo-ideal of \( X \), and so \( \mu \) is a fuzzy pseudo-ideal of \( X \).

\[\square\]

**Theorem 3.2.14**: If \( \mu \) is a fuzzy pseudo-ideal of \( X \) satisfying,

\[\mu((x \odot z) \ast (y \odot z)) \geq \mu((x \ast y) \odot z), \forall x, y, z \in X \quad (3)\]

and

\[\mu((a * c) \odot (b \ast c)) \geq \mu((a \odot b) \ast c), \forall a, b, c \in X, \quad (4)\]

then \( \mu \) is a fuzzy implicative pseudo-ideal of \( X \).

**Proof**: Let \( \mu \) be a fuzzy set in \( X \) satisfying (3) and (4). Let \( x, y, z \in X \), we have

\[(x \odot z) \odot ((x \odot z) \ast (y \odot z)) \preceq y \odot z\]

and

\[(a \ast c) \ast ((a \ast c) \odot (b \ast c)) \preceq b \ast c.\]

Then

\[\mu(x \odot z) \geq \mu((x \odot z) \ast (y \odot z)) \land \mu(y \odot z)\]

and

\[\mu(a \ast c) \geq \mu((a \ast c) \odot (b \ast c)) \land \mu(b \ast c).\]

By (3) and (4) we have

\[\mu(x \odot z) \geq \mu((x \ast y) \odot z) \land \mu(y \odot z)\]
and
\[ \mu(a \ast c) \geq \mu((a \otimes b) \ast c) \land \mu(b \ast c). \]

Therefore \(\mu\) is a fuzzy pseudo implicative ideal of \(X\).

\(\square\)

As a generalization of Theorem 3.2.9, we have the following results.

**Theorem 3.2.15:** If a fuzzy set \(\mu\) in \(X\) is a fuzzy pseudo-ideal of \(X\), then \(\forall a, x, w_1, w_2, \ldots, w_n \in X\),

\[ x \ast \prod_{i=1}^{n} w_i = 0 \Rightarrow \mu(x) \geq \min\{ \mu(w_i) \mid i = 1, 2, \ldots, n\}, \quad (5) \]

\[ a \otimes \prod_{i=1}^{n} w_i = 0 \Rightarrow \mu(a) \geq \min\{ \mu(w_i) \mid i = 1, 2, \ldots, n\}, \quad (6) \]

where \(x \ast \prod_{i=1}^{n} w_i = (\cdots ((x \ast w_1) \ast w_2) \ast \cdots) \ast w_n\), and

\[ a \otimes \prod_{i=1}^{n} w_i = (\cdots ((a \otimes w_1) \otimes w_2) \otimes \cdots) \otimes w_n. \]

**Proof:** The proof is by induction on \(n\). Let \(\mu\) be a fuzzy pseudo-ideal of \(X\). Theorem 3.2.6 and Corollary 3.2.7 show that the conditions (5) and (6) hold for \(n = k\), that is

\[ x \ast \prod_{i=1}^{k} w_i = 0 \implies \mu(x) \geq \min\{ \mu(w_i) \mid i = 1, 2, \ldots, k\} \]

and

\[ a \otimes \prod_{i=1}^{k} w_i = 0 \implies \mu(a) \geq \min\{ \mu(w_i) \mid i = 1, 2, \ldots, k\}, \]
for all \(a, x, w_1, w_2, \ldots, w_k \in X\). Let \(a, x, w_1, w_2, \ldots, w_k, w_{k+1} \in X\) be such that \(x \ast \prod_{i=1}^{k+1} w_i = 0\), and \(a \ominus \prod_{i=1}^{k+1} w_i = 0\). Then

\[
\mu(x \ast w_1) \geq \min\{ \mu(w_j) \mid j = 2, \cdots, k+1\}
\]

and

\[
\mu(x \ominus w_1) \geq \min\{ \mu(w_j) \mid j = 2, \cdots, k+1\}.
\]

Since \(\mu\) is a fuzzy pseudo-ideal of \(X\), then

\[
\mu(x) \geq \min\{ \mu(x \ast w_1), \mu(w_1)\}
\]

\[
\geq \min\{ \mu(w_1), \min\{\mu(w_j) \mid j = 2, 3, \cdots, k+1\}\}
\]

\[
= \min\{\mu(w_j) \mid j = 1, 2, 3, \cdots, k+1\},
\]

and

\[
\mu(a) \geq \min\{ \mu(x \ominus w_1), \mu(w_1)\}
\]

\[
\geq \min\{ \mu(w_1), \min\{\mu(w_j) \mid j = 2, 3, \cdots, k+1\}\}
\]

\[
= \min\{\mu(w_j) \mid j = 1, 2, 3, \cdots, k+1\}.
\]

This completes the proof.

\(\square\)

We now consider the converse of Theorem 3.2.15.

**Theorem 3.2.16:** Let \(\mu\) be a fuzzy set in \(X\) satisfying the conditions (5) and (6). Then \(\mu\) is a pseudo-ideal of \(X\).

**Proof:** Note that for all \(a, x \in X\),

\[
((\cdots (0 \ast x) \ast x) \ast \cdots) \ast x = 0
\]
and 
\[(\cdots (0 \otimes a) \otimes a) \otimes \cdots) \otimes a = 0.\]

It follows from (5) and (6) that \(\mu(0) \geq \mu(x)\) and \(\mu(0) \geq \mu(a)\). Let \(a, b, c, x, y, z \in X\) be such that \(x \ast y \preceq z\) and \(a \otimes b \preceq c\). Then
\[0 = (x \ast y) \ast z = (\cdots (((x \ast y) \ast z) \ast 0) \ast \cdots) \ast 0\]

and
\[0 = (a \otimes b) \otimes c = (\cdots (((a \otimes b) \otimes c) \otimes 0) \otimes \cdots) \otimes 0.\]

Therefore
\[\mu(x) \geq \min\{\mu(y), \mu(z), \mu(0)\} = \min\{\mu(y), \mu(z)\}\]

and
\[\mu(a) \geq \min\{\mu(b), \mu(c), \mu(0)\} = \min\{\mu(b), \mu(c)\}.\]

Hence by Theorem 3.2.9, we conclude that \(\mu\) is a fuzzy pseudo-ideal of \(X\).

\[\square\]

**Theorem 3.2.17:** If \(\mu\) is a fuzzy implicative pseudo-ideal of \(X\), then
\[((x \odot y) \ast y) \ast u \preceq v \Rightarrow \mu(x \odot y) \geq \min\{\mu(u), \mu(v)\}\]  
(7)

and
\[((a \ast b) \odot b) \odot u \preceq v \Rightarrow \mu(a \ast b) \geq \min\{\mu(u), \mu(v)\},\]  
(8)

for all \(a, b, x, y, u, v \in X\).

**Proof:** Let \(x, y, u, v \in X\) be such that \(((x \odot y) \ast y) \ast u \preceq v\). Using Theorem 3.2.6, we have \(\mu((x \odot y) \ast y) \geq \min\{\mu(u), \mu(v)\}\).
It follows that
\[
\mu(x \ominus y) \geq \min\{\mu((x * y) \ominus y), \mu(y \ominus y)\}
\]
\[
= \min\{\mu((x * y) \ominus y), \mu(0)\}
\]
\[
= \mu((x \ominus y) * y)
\]
\[
\geq \min\{\mu(u), \mu(v)\}.
\]

Now let \(a, b, u, v \in X\) be such that \(((a * b) \ominus b) \ominus u \leq v\). By Theorem 3.2.9, we have \(\mu((a * b) \ominus b) \geq \min\{\mu(u), \mu(v)\}\).

Then
\[
\mu(a * b) \geq \min\{\mu((a \ominus b) * b), \mu(b * b)\}
\]
\[
= \min\{\mu((a \ominus b) * b), \mu(0)\}
\]
\[
= \mu((a \ominus b) * b)
\]
\[
\geq \min\{\mu(u), \mu(v)\}.
\]

This completes the proof.

\[\square\]

**Theorem 3.2.18:** Let \(\mu\) be a fuzzy pseudo ideal of \(X\) then, \(\mu\) is a fuzzy pseudo implicative ideal if and only if it satisfies

(a) \(\mu(x \ominus y) \geq \mu((x * y) \ominus y), \forall x, y \in X\) and

(b) \(\mu(a * b) \geq \mu((a \ominus b) * b), \forall a, b \in X\).

**Proof:** Let \(\mu\) be a fuzzy pseudo implicative ideal of \(X\) we have
\[
\mu(x \ominus z) \geq \mu((x * y) \ominus z) \wedge \mu(y \ominus z), \forall x, y, z \in X
\]
and
\[
\mu(a * c) \geq \mu((a \ominus b) * c) \wedge \mu(b * c), \forall a, b, c \in X.
\]
Putting \( z = y \) and \( c = b \), we have

\[
\mu(x \odot y) \geq \mu((x * y) \odot y), \forall x, y \in X
\]

and

\[
\mu(a * b) \geq \mu((a \odot b) * b), \forall a, b \in X.
\]

Conversely, let \( \mu \) be a fuzzy pseudo ideal of \( X \) satisfying conditions (a) and (b). We prove that \( \mu \) is a fuzzy pseudo implicative ideal of \( X \).

We have

\[
(((x \odot z) * z) * (y \odot z)) * ((x \odot z) * y)
\]

\[
= (((x \odot z) * z) * ((x \odot z) * y)) * (y \odot z)
\]

\[
\leq (y \odot z) * (y \odot z) = 0.
\]

Then

\[
((x \odot z) * z) * (y \odot z) \preceq (x \odot z) * y.
\]

Also

\[
\mu((x \odot z) * z) \geq \mu((x * y) \odot z) \land \mu(y \odot z).
\]

By (a),

\[
\mu(x \odot z) \geq \mu((x * y) \odot z) \land \mu(y \odot z).
\]

Similarly, we have

\[
((a * c) \odot c) \odot (b * c) \preceq (a * c) \odot b
\]

and

\[
\mu((a * c) \odot c) \geq \mu((a \odot b) * z) \land \mu(b * c).
\]

By (b),

\[
\mu(a * c) \geq \mu((a \odot b) * c) \land \mu(b * c).
\]
Therefore $\mu$ is a fuzzy pseudo implicative ideal of $X$. \hfill \Box

We now consider the converse of Theorem 3.2.17.

**Theorem 3.2.19:** Let $\mu$ be a fuzzy set in $X$ satisfying the conditions (7) and (8). Then $\mu$ is a fuzzy pseudo implicative ideal of $X$.

**Proof:** Let $\mu$ be a fuzzy set in $X$. Let $x, y, z \in X$ be such that $x \ast y \preceq z$. Then

\[
(((x \ominus 0) \ast 0) \ast y) \ast z = (x \ast y) \ast z = 0,
\]

that is

\[
((x \ominus 0) \ast 0) \ast y \preceq z.
\]

From (7) it follows that

\[
\mu(x) = \mu(x \ominus 0) \geq \min\{\mu(y), \mu(z)\}.
\]

Now let $a, b, c \in X$ such that $a \ominus b \preceq c$. Then

\[
(((a \ast 0) \ominus 0) \ominus b) \ast c = (a \ominus b) \ast c = 0,
\]

that is

\[
((a \ast 0) \ominus 0) \ominus b \preceq c.
\]

From (8) we have

\[
\mu(a) = \mu(a \ast 0) \geq \min\{\mu(b), \mu(c)\}.
\]

Therefore by Theorem 3.2.9, $\mu$ is a fuzzy pseudo-ideal of $X$.

Note that

\[
(((x \ominus y) \ast y) \ominus ((x \ominus y) \ast y)) \ominus 0 = 0
\]
for all $x, y \in X$. Using (7), we have

$$
\mu(x \odot y) \geq \mu((x \odot y) * y) \land \mu(0)
$$

$$
= \mu((x \odot y) * y)
$$

$$
= \mu((x * y) \odot y).
$$

Since

$$( ((a * b) \odot b) * ((a * b) \odot b)) * 0 = 0$$

for all $x, y \in X$, using (8), we have

$$
\mu(a * b) \geq \mu((a * b) \odot b) \land \mu(0)
$$

$$
= \mu((a * b) \odot b)
$$

$$
= \mu((a \lor b) * b).
$$

Hence $\mu$ is a fuzzy pseudo implicative ideal of $X$ by Theorem 3.2.18.

\[ \square \]

**Definition 3.2.20 ([19]):** An algebra $(L, \lor, \land)$ is called a lattice if and only if $L$ is a nonvoid set, $\land$ and $\lor$ are binary operations on $L$ satisfying the following identities:

- **(L1) Idempotency:** \( a \land a = a, \ a \lor a = a, \)
- **(L2) Commutativity:** \( a \land b = b \land a, \ a \lor b = b \lor a, \)
- **(L3) Associativity:** \( a \land (b \land c) = (a \land b) \land c, \)
  \[ a \lor (b \lor c) = (a \lor b) \lor c, \]
- **(L4) Absorption rules:** \( a \land (a \lor b) = a, \ a \lor (a \land b) = a, \)

for all $a, b, c \in L$. Let $L$ be a lattice. Then $L$ is called complete, if for
any subset $A$ of $L$, $\bigwedge A$ and $\bigvee A$ exist. A lattice $L$ is called distributive if the following condition holds:

$$(a \lor b) \land c = (a \land c) \lor (b \land c)$$

for all $a, b, c \in L$.

A completely istributive lattice is a complete lattice in which arbitrary joins distribute over arbitrary meets.

**Theorem 3.2.21:** The family of fuzzy implicative pseudo-ideals of $X$ is a completely distributive lattice with respect to $\land$ and $\lor$.

**Proof:** Let $\{\mu_i \mid i \in \Omega\}$ be a family of fuzzy implicative pseudo-ideals of $X$. Since $[0, 1]$ is a completely distributive lattice with respect to the usual ordering in $[0, 1]$, it is sufficient to show that $\bigvee_{i \in \Omega} \mu_i$ and $\bigwedge_{i \in \Omega} \mu_i$ are fuzzy implicative pseudo-ideals of $X$. Let $x, y, z \in X$, then

$$\left(\bigvee_{i \in \Omega} \mu_i\right)(x \otimes z)$$

$$= \sup_{i \in \Omega} \mu_i(x \otimes z)$$

$$\geq \sup_{i \in \Omega} \left\{\min_{i \in \Omega}\{\mu_i((x \ast y) \otimes z), \mu_i(y \otimes z)\}\right\}$$

$$\geq \min_{i \in \Omega}\{\sup_{i \in \Omega}\mu_i((x \ast y) \otimes z), \sup_{i \in \Omega}\mu_i(y \otimes z)\}$$

$$= \min_{i \in \Omega}\{\left(\bigvee_{i \in \Omega}\mu_i\right)((x \ast y) \otimes z), \left(\bigvee_{i \in \Omega}\mu_i\right)(y \otimes z)\},$$

$$\text{(2)}$$

And

$$\left(\bigwedge_{i \in \Omega} \mu_i\right)(x \otimes z)$$

$$= \inf_{i \in \Omega} \mu_i(x \otimes z)$$

$$\geq \inf_{i \in \Omega}\left\{\min_{i \in \Omega}\{\mu_i((x \ast y) \otimes z), \mu_i(y \otimes z)\}\right\}$$
\[ \geq \min \{ \inf_{i \in \Omega} \mu_i((x \star y) \ominus z), \inf_{i \in \Omega} \mu_i(y \ominus z) \} \]
\[ = \min\{\left( \bigwedge_{i \in \Omega} \mu_i \right)(x \star y) \ominus z), \left( \bigwedge_{i \in \Omega} \mu_i \right)(y \ominus z)\}. \]

(4)

Now let \( a, b, c \in X \), then

\[ \left( \bigvee_{i \in \Omega} \mu_i \right)(a \star c) = \sup_{i \in \Omega} \mu_i(a \star c) \]
\[ \geq \sup_{i \in \Omega} \{ \min_{i \in \Omega} \{ \mu_i((a \ominus b) \star c), \mu_i(b \star c) \} \} \]
\[ \geq \min_{i \in \Omega} \{ \sup_{i \in \Omega} \{ \mu_i((a \ominus b) \star c), \sup_{i \in \Omega} \mu_i(b \star c) \} \} \]
\[ = \min\{\left( \bigvee_{i \in \Omega} \mu_i \right)((a \ominus b) \star c), \left( \bigvee_{i \in \Omega} \mu_i \right)(b \star c)\}, \]

(6)

And

\[ \left( \bigwedge_{i \in \Omega} \mu_i \right)(a \star c) = \inf_{i \in \Omega} \mu_i(a \star c) \]
\[ \geq \inf_{i \in \Omega} \{ \min_{i \in \Omega} \{ \mu_i((a \ominus b) \star c), \mu_i(b \star c) \} \} \]
\[ \geq \min_{i \in \Omega} \{ \inf_{i \in \Omega} \mu_i((a \ominus b) \star c), \inf_{i \in \Omega} \mu_i(b \star c) \} \]
\[ = \min\{\left( \bigwedge_{i \in \Omega} \mu_i \right)((a \ominus b) \star c), \left( \bigwedge_{i \in \Omega} \mu_i \right)(b \star c)\}. \]

(8)
Hence the family of fuzzy implicative pseudo-ideals of $X$ is a completely distributive lattice with respect to $\land$ and $\lor$. This completes the proof. \qed
3.3. On pseudo \((a,p,q)\)-ideals of pseudo-BCI algebras

In this section, we define fuzzy pseudo \(q\)-ideals and fuzzy pseudo \(a\)-ideals in pseudo-BCI algebras. We give several characterizations and the extensive theorems about fuzzy pseudo \(q\)-ideals and fuzzy pseudo \(a\)-ideals.

**Definition 3.3.1:** A fuzzy set \(\mu\) is called a fuzzy pseudo \(q\)-ideal in \(X\) if

\[
(QI1) \quad \mu(x \odot z) \geq \min\{\mu(x \odot (y * z)), \mu(y)\}, \forall x, y, z \in X.
\]

\[
(QI2) \quad \mu(a * c) \geq \min\{\mu(a * (b \odot c)), \mu(b)\}, \forall a, b, c \in X.
\]

**Lemma 3.3.2:** Every fuzzy pseudo \(q\)-ideal is a fuzzy pseudo-ideal.

**Proof:** Let \(\mu\) be a fuzzy pseudo \(q\)-ideal of \(X\) and let \(t \in \text{Im}(\mu)\).

For every \(y \in U(\mu, t)\), let \(a \in *(y, U(\mu, t))\) and \(b \in \odot(y, U(\mu, t))\).

Then

\[
a * (y \odot 0) = a * y \in U(\mu, t)
\]

and

\[
b \odot (y * 0) = b \odot y \in U(\mu, t).
\]

Since \(y \in U(\mu, t)\), it follows that

\[
\mu(b) = \mu(b \odot 0) \geq \min\{\mu(b \odot (y * 0)), \mu(y)\} \geq t
\]

and

\[
\mu(a) = \mu(a * 0) \geq \min\{\mu(a * (y \odot 0)), \mu(y)\} \geq t,
\]
so that \( a, b \in U(\mu, t) \). This shows that
\[
*(y, U(\mu, t)) \subseteq U(\mu, t) \quad \text{and} \quad \odot (y, U(\mu, t)) \subseteq U(\mu, t).
\]
Hence \( U(\mu, t) \) is a pseudo-ideal of \( X \), and so \( \mu \) is a fuzzy pseudo-ideal of \( X \).

\[\square\]

**Definition 3.3.3**: A fuzzy set \( \mu \) is a fuzzy pseudo subalgebra of \( X \) if:

(i) \( \mu(x \odot y) \geq \min\{ \mu(x), \mu(y) \} \), \( \forall \) \( x, y \in X \) and

(ii) \( \mu(a \ast b) \geq \min\{ \mu(a), \mu(b) \} \), \( \forall \) \( a, b \in X \).

**Definition 3.3.4**: A fuzzy set \( \mu \) is called a fuzzy pseudo closed ideal in \( X \) if,

(CI1) \( \mu(0 \odot x) \geq \mu(x) \), \( \mu(0 \ast a) \geq \mu(a) \), \( \forall \) \( a, x \in X \), and

(CI2) \( \mu(x) \geq \mu(x \odot y) \land \mu(y) \), \( \mu(a) \geq \mu(a \ast b) \land \mu(b) \),

for all \( a, b, x, y \in X \).

**Theorem 3.3.5**: Every fuzzy pseudo \( q \)-ideal of \( X \) is a fuzzy pseudo subalgebra of \( X \).

**Proof**: If \( \mu \) be a fuzzy pseudo \( q \)-ideal, then putting \( z = y \) in \( (QI1) \) and \( b = c \) in \( (QI2) \), we have

\[
\mu(x \odot y) \geq \min\{ \mu(x), \mu(y) \}, \quad \mu(a \ast b) \geq \min\{ \mu(a), \mu(b) \}.
\]

This completes the proof. \[\square\]
**Theorem 3.3.6**: Let $\mu$ be a fuzzy pseudo-ideal of $X$. Then the following are equivalent:

(i) $\mu$ is a fuzzy pseudo $q$-ideal of $X$.

(ii) $\mu((a \ominus b) \ast c) \geq \mu((a \ominus b) \ast (0 \ominus c))$ and

$$\mu((x \ast y) \ominus z) \geq \mu((x \ast y) \ominus (0 \ast z)).$$

(iii) $\mu(x \ominus y) \geq \mu(x \ominus (0 \ast y))$ and $\mu(a \ast b) \geq \mu(a \ast (0 \ominus b))$.

**Proof**: (i) $\Rightarrow$ (ii). Since $\mu$ is a fuzzy pseudo $q$-ideal of $X$, we have

$$\mu((a \ominus b) \ast c) \geq \min \{ \mu((a \ominus b) \ast (0 \ominus c)), \mu(0) \} = \mu((a \ominus b) \ast (0 \ominus c)).$$

Similarly, since $\mu$ is a fuzzy pseudo $q$-ideal, we have

$$\mu((x \ast y) \ominus z) \geq \min \{ \mu((x \ast y) \ominus (0 \ast z)), \mu(0) \} = \mu((x \ast y) \ominus (0 \ast z)).$$

(ii) $\Rightarrow$ (iii). Letting $y = 0$ and $z = y$ in (QI1) and $b = 0$ and $c = b$ in (QI2), we get the required implication.

(iii) $\Rightarrow$ (i). We have

$$(x \ominus (0 \ast y)) \ast (x \ominus (z \ast y)) \preceq (z \ast y) \ominus (0 \ast y) \preceq z$$

and

$$(a \ast (0 \ominus b)) \ominus (a \ast (c \ominus b)) \preceq (c \ominus b) \ast (0 \ominus b) \preceq c.$$ 

Then we have

$$\mu(x \ominus (0 \ast y)) \geq \min \{ \mu(x \ominus (z \ast y)), \mu(z) \}.$$
On pseudo \((a, p, q)\)-ideals of pseudo-BCI algebras

and

\[ \mu(a \ast (0 \varpi b)) \geq \min\{ \mu(a \ast (c \varpi b)), \mu(c) \}. \]

Therefore by hypothesis

\[ \mu(x \varpi y) \geq \mu(x \varpi (0 \ast y)) \geq \min\{ \mu(x \varpi (z \ast y)), \mu(z) \} \]

and

\[ \mu(a \ast b) \geq \mu(a \ast (0 \varpi b)) \geq \min\{ \mu(a \ast (c \varpi b)), \mu(c) \}. \]

Hence \(\mu\) is a fuzzy pseudo \(q\)-ideal of \(X\).

\[ \square \]

**Theorem 3.3.7:** Let \(\mu\) and \(\nu\) be fuzzy pseudo-ideals of \(X\), such that \(\mu \subseteq \nu\) and \(\mu(0) = \nu(0)\). If \(\mu\) is a fuzzy pseudo \(q\)-ideal of \(X\), then so \(\nu\).

**Proof:** We want to show that by Theorem 3.3.6(iii), for any \(x, y, a, b \in X\),

\[ \nu(x \varpi y) \geq \nu(x \varpi (0 \ast y)) \quad \text{and} \quad \nu(a \ast b) \geq \nu(a \ast (0 \varpi b)). \]

Putting, \(s = x \varpi (0 \ast y)\), then \((x \ast s) \varpi (0 \ast y) = 0\), hence

\[ \mu((x \ast s) \varpi (0 \ast y)) = \mu(0) = \nu(0). \]

Since \(\mu\) is a fuzzy pseudo \(q\)-ideal of \(X\) and using Theorem 3.3.6(iii),

\[ \mu((x \ast s) \varpi y) \geq \mu((x \ast s) \varpi (0 \ast y)) = \nu(0). \]

Thus

\[ \nu((x \ast s) \varpi y) \geq \mu((x \ast s) \varpi y) \geq \nu(0) \geq \nu(s). \]
Since $\nu$ is a fuzzy pseudo-ideal we have

$$\nu(x \odot y) \geq \min\{ \nu((x \odot y) \ast s), \nu(s)\} = \nu(s) = \nu(x \odot (0 \ast y)). \quad (1)$$

Putting $t = a \ast (0 \odot b)$ then $(a \odot t) \ast (0 \odot b) = 0$. Hence,

$$\mu((a \odot t) \ast (0 \odot b)) = \mu(0) = \nu(0).$$

Since $\mu$ is a fuzzy pseudo $q$-ideal of $X$ and using Theorem 3.3.6(iii),

$$\mu((a \odot t) \ast b) \geq \mu((a \odot t) \ast (0 \odot b)) = \nu(0).$$

Thus

$$\nu((a \odot t) \ast b) \geq \mu((a \odot t) \ast b) \geq \nu(0) \geq \nu(t).$$

Since $\nu$ is a fuzzy pseudo-ideal we have

$$\nu(a \ast b) \geq \min\{ \nu((a \ast b) \odot t), \nu(t)\} = \nu(t) = \nu(a \ast (0 \odot b)). \quad (2)$$

From (1), (2) and Theorem 3.3.6(iii), $\nu$ is a fuzzy pseudo $q$-ideal of $X$. This completes the proof.

\[ \square \]

**Definition 3.3.8:** A fuzzy set $\mu$ is called a **fuzzy pseudo $a$-ideal of $X$** if,

\[ (AI1) \quad \mu(y \odot x) \geq \min\{ \mu((x \ast z) \odot (0 \ast y)), \mu(z)\}, \forall x, y, z \in X \]  
\[ (AI2) \quad \mu(b \ast a) \geq \min\{ \mu((a \odot c) \ast (0 \odot b)), \mu(c)\}, \forall a, b, c \in X. \]
Theorem 3.3.9: Let $\mu$ be a fuzzy pseudo-ideal of $X$. Then the following are equivalent:

(i) $\mu$ is a fuzzy pseudo $a$-ideal of $X$.
(ii) $\mu(b \ast (a \odot c)) \geq \mu((a \odot c) \ast (0 \odot b))$ and

$$\mu(y \odot (x \ast z)) \geq \mu((x \ast z) \odot (0 \ast y)).$$

(iii) $\mu(y \odot x) \geq \mu(x \odot (0 \ast y))$ and $\mu(b \ast a) \geq \mu(a \ast (0 \odot b))$.

Proof: $(i) \Rightarrow (ii)$. Suppose that $\mu$ is a fuzzy pseudo $a$-ideal of $X$, we have

$$\mu(y \odot (x \ast z)) \geq \min\{\mu(((x \ast z) \ast s) \odot (0 \ast y)), \mu(s)\}$$

and

$$\mu(b \ast (a \odot c)) \geq \min\{\mu(((a \odot c) \odot t) \ast (0 \odot b)), \mu(t)\}.$$ 

We write $s = (x \ast z) \odot (0 \ast y)$ and $t = (a \odot c) \ast (0 \odot b)$. Since

$$((x \ast z) \ast s) \odot (0 \ast y) = ((x \ast z) \odot (0 \ast y)) \ast s = 0$$

and

$$((a \odot c) \odot t) \ast (0 \odot b) = ((a \odot c) \ast (0 \odot b)) \odot t = 0.$$ 

Then we have

$$\mu(y \odot (x \ast z)) \geq \min\{ \mu(0), \mu(s) \} = \mu(s) = \mu((x \ast z) \odot (0 \ast y))$$

and

$$\mu(b \ast (a \odot c)) \geq \min\{ \mu(0), \mu(t) \} = \mu(t) = \mu((a \odot c) \ast (0 \odot y)).$$
(ii) ⇒ (iii). Letting \( z = 0 \) and \( c = 0 \) in (ii), we obtain
\[
\mu(y \oplus x) \geq \mu(x \oplus (0 \ast y)) \quad \text{and} \quad \mu(b \ast a) \geq \mu(a \ast (0 \oplus y)).
\]

(iii) ⇒ (i). Note that
\[
(x \oplus (0 \ast y)) \oplus ((x \ast z) \oplus (0 \ast y)) \leq x \oplus (x \ast z) \leq z
\]
and
\[
(a \ast (0 \oplus b)) \ast ((a \ominus c) \ast (0 \oplus b)) \leq a \ast (a \ominus c) \leq c,
\]
we have
\[
\mu(x \oplus (0 \ast y)) \geq \min\{\mu((x \ast z) \oplus (0 \ast y)), \mu(z)\}
\]
and
\[
\mu(a \ast (0 \oplus b)) \geq \min\{\mu((a \ominus c) \ast (0 \oplus b)), \mu(c)\}.
\]
From condition (iii), we have
\[
\mu(y \oplus x) \geq \mu(x \oplus (0 \ast y)) \geq \min\{\mu((x \ast z) \oplus (0 \ast y)), \mu(z)\}
\]
and
\[
\mu(b \ast a) \geq \mu(a \ast (0 \oplus b)) \geq \min\{\mu((a \ominus c) \ast (0 \oplus b)), \mu(c)\}.
\]
Hence \( \mu \) is a fuzzy pseudo \( a \)-ideal of \( X \).
Theorem 3.3.10: Let $\mu$ be a fuzzy pseudo-ideal of $X$, then $\mu$ is a fuzzy pseudo $a$-ideal of $X$ if and only if it satisfies the following conditions:

(i) $\mu(y) \geq \mu(0 \ominus (0 * y))$ and $\mu(b) \geq \mu(0 * (0 \ominus b)) \quad \forall y, b \in X$

(ii) $\mu(x \ominus y) \geq \mu(x \ominus (0 * y))$ and $\mu(a * b) \geq \mu(a * (0 \ominus b))$

for all $x, y, a, b \in X$.

Proof: Assume that $\mu$ is a fuzzy pseudo $a$-ideal of $X$. Setting $x = 0$ and $a = 0$ in Theorem 3.3.9(iii), we get

$$
\mu(y) \geq \mu(0 \ominus (0 * y)) \quad \text{and} \quad \mu(b) \geq \mu(0 * (0 \ominus b)).
$$

Note that

$$
(0 * (0 \ominus (y \ominus (0 * x)))) * (x \ominus (0 * y))
= (0 * ((0 * y) * (0 \ominus (0 * x)))) * (x \ominus (0 * y))
= ((0 \ominus (0 * y)) \ominus (0 * (0 \ominus (0 * x)))) * (x \ominus (0 * y))
= ((0 \ominus (0 * y)) \ominus (0 * x)) * (x \ominus (0 * y))
= ((0 \ominus (0 * y)) * (x \ominus (0 * y))) \ominus (0 * x)
= (0 * x) \ominus (0 * x) = 0 * (x * x) = 0.
$$

Then

$$
\mu(0 * (0 \ominus (y \ominus (0 * x)))) \geq \min\{\mu(x \ominus (0 * y)), \mu(0)\} = \mu(x \ominus (0 * y)),
$$

hence

$$
\mu(y \ominus (0 * x)) \geq \mu(0 * (0 \ominus (y \ominus (0 * x)))) \geq \mu(x \ominus (0 * y)).
$$

Applying Theorem 3.3.9(iii), we have

$$
\mu(x \ominus y) \geq \mu(y \ominus (0 * x)) \geq \mu(x \ominus (0 * y)).
$$
Similarly, we have
\[(0 \odot (0 * (b * (0 \odot a)))) \odot (a * (0 \odot b))\]
\[= (0 \odot ((0 \odot b) \odot (0 * (0 \odot a)))) \odot (a * (0 \odot b))\]
\[= ((0 * (0 \odot b)) * (0 \odot (0 * (0 \odot a)))) \odot (a * (0 \odot b))\]
\[= ((0 * (0 \odot b)) * (0 \odot a)) * (a * (0 \odot b))\]
\[= ((0 * (0 \odot b)) \odot (a * (0 \odot b))) * (0 \odot a)\]
\[\preceq (0 * a) * (0 \odot a) = 0 \odot (a \odot a) = 0.\]

We have
\[\mu(0 \odot (0 * (b * (0 \odot a)))) \geq \min\{ \mu(a * (0 \odot b)), \mu(0) \} = \mu(a * (0 \odot b)),\]
hence
\[\mu(b * (0 \odot a)) \geq \mu(0 \odot (0 * (b * (0 \odot a)))) \geq \mu(a * (0 \odot b)).\]

Applying Theorem 3.3.9(iii), we have
\[\mu(a * b) \geq \mu(b * (0 \odot a)) \geq \mu(a * (0 \odot b)).\]

Conversely, suppose \(\mu\) is a fuzzy pseudo-ideal satisfying (i) and (ii). In order to prove that \(\mu\) is a fuzzy pseudo \(a\)-ideal of \(X\), from Theorem 3.3.9, we show that
\[\mu(y \odot x) \geq \mu(x \odot (0 * y)) \quad \text{and} \quad \mu(b * a) \geq \mu(a * (0 \odot b))\]
for all \(a, b, x, y \in X\). By (ii), we have \(\mu(x \odot y) \geq \mu(x \odot (0 * y))\).

Since
\[0 * (y \odot x) \preceq x \odot y,\]
by Theorem 3.3.9, we get
\[\mu(0 * (y \odot x)) \geq \mu(x \odot y) \geq \mu(x \odot (0 * y)),\]
thus
\[\mu(0 * (0 * (y \odot x))) \geq \mu(0 * (y \odot x)) \geq \mu(x \odot (0 * y)).\]
Applying (i), we get
\[ \mu(y \odot x) \geq \mu(0 \ast (0 \ast (y \odot x))) \geq \mu(x \odot (0 \ast y)). \]

Similarly, we can get \( \mu(b \ast a) \geq \mu(a \ast (0 \odot b)) \). Therefore \( \mu \) is a fuzzy pseudo \( a \)-ideal of \( X \). This completes the proof.

Definition 3.3.11: A fuzzy set \( \mu \) is called a fuzzy pseudo \( p \)-ideal in \( X \) if,

\[
(P1) \quad \mu(x) \geq \min\{\mu((x \ast z) \odot (y \ast z)), \mu(y)\}, \quad \forall \, x, y, z \in X
\]

\[
(P2) \quad \mu(a) \geq \min\{\mu((a \odot c) \ast (b \odot c)), \mu(b)\}, \quad \forall \, a, b, c \in X.
\]

Theorem 3.3.12: Every fuzzy pseudo \( p \)-ideal is a fuzzy pseudo-ideal.

**Proof:** Let \( \mu \) be a fuzzy pseudo \( p \)-ideal of \( X \) and let \( t \in \text{Im}(\mu) \). For every \( y \in U(\mu, t) \), let \( a \in \ast(y, U(\mu, t)) \) and \( b \in \odot(y, U(\mu, t)) \). Then

\[ (b \ast 0) \odot (y \ast 0) = b \odot y \in U(\mu, t) \]

and

\[ (a \odot 0) \ast (y \odot 0) = a \ast y \in U(\mu, t). \]

Since \( y \in U(\mu, t) \) and \( \mu \) is a pseudo \( p \)-ideal we have

\[ \mu(a) \geq \min\{\mu((a \odot 0) \ast (y \odot 0)), \mu(y)\} \geq t \]

and

\[ \mu(b) \geq \min\{\mu((b \ast 0) \odot (y \ast 0)), \mu(y)\} \geq t, \]
so that \( a, b \in U(\mu, t) \). This shows that

\[ *(y, U(\mu, t)) \subseteq U(\mu, t) \quad \text{and} \quad \ominus (y, U(\mu, t)) \subseteq U(\mu, t). \]

Hence \( U(\mu, t) \) is a pseudo-ideal of \( X \), and so \( \mu \) is a fuzzy pseudo-ideal of \( X \).

\[ \square \]

**Lemma 3.3.13:** A fuzzy pseudo-ideal of \( X \) is a fuzzy pseudo \( p \)-ideal if and only if

\[ \mu(x) \geq \mu(0 \ominus (0 \ast x)) \quad \text{and} \quad \mu(a) \geq \mu(0 \ast (0 \ominus a)) \]

for all \( x, a \in X \).

**Proof:** Let \( \mu \) be fuzzy pseudo \( p \)-ideal then for all \( x, y, z, a, b, c \) in \( X \) we have

\[ \mu(x) \geq \mu((x \ast z) \ominus (y \ast z)) \land \mu(y) \]

and

\[ \mu(a) \geq \mu((a \ominus c) \ast (b \ominus c)) \land \mu(b). \]

Letting \( x = z, \ y = 0 \) and \( a = c, \ b = 0 \), we have

\[ \mu(x) \geq \mu(0 \ominus (0 \ast x)) \quad \text{and} \quad \mu(a) \geq \mu(0 \ast (0 \ominus a)) \]

for all \( x, a \in X \).

Conversely, suppose \( \mu \) satisfies

\[ \mu(x) \geq \mu(0 \ominus (0 \ast x)) \quad \text{and} \quad \mu(a) \geq \mu(0 \ast (0 \ominus a)) \]

for all \( x, a \in X \).
We have
\[(0 \ast (0 \odot a)) \ast ((a \odot c) \ast (b \odot c))\]
\[= (((b \odot c) \ast (b \odot c)) \ast (0 \ast a)) \ast ((a \odot c) \ast (b \odot c))\]
\[= (((b \odot c) \ast (0 \odot a)) \ast (b \odot c)) \ast ((a \odot c) \ast (b \odot c))\]
\[\preceq ((b \odot c) \ast (0 \odot a)) \ast (a \odot c)\]
\[= ((b \odot c) \ast (a \odot c)) \ast (0 \odot a)\]
\[\preceq (b \odot a) \ast (0 \odot a) \preceq b.\]

For all \(x, y, z \in X\) we have,
\[(0 \odot (0 \ast x)) \odot ((x \ast z) \odot (y \ast z))\]
\[= (((y \ast z) \odot (y \ast z)) \odot (0 \ast x)) \odot ((x \ast z) \odot (y \ast z))\]
\[= (((y \ast z) \odot (0 \ast x)) \odot (y \ast z)) \odot ((x \ast z) \odot (y \ast z))\]
\[\preceq ((y \ast z) \odot (0 \ast x)) \odot (x \ast z)\]
\[= ((y \ast z) \odot (x \ast z)) \odot (0 \ast x)\]
\[\preceq (y \ast x) \odot (0 \ast x) \preceq y.\]

Then we have
\[\mu(0 \odot (0 \ast x)) \geq \mu((x \ast z) \odot (y \ast z)) \land \mu(y)\]

and
\[\mu(0 \ast (0 \odot a)) \geq \mu((a \odot c) \ast (b \odot c)) \land \mu(b).\]

Then
\[\mu(x) \geq \mu((x \ast z) \odot (y \ast z)) \land \mu(y)\]

and
\[\mu(a) \geq \mu((a \odot c) \ast (b \odot c)) \land \mu(b).\]
Hence $\mu$ is a fuzzy pseudo $p$-ideal of $X$. \qed

**Theorem 3.3.14:** Any fuzzy pseudo $a$-ideal is a fuzzy pseudo $p$-ideal.

**Proof:** Let $\mu$ be a fuzzy pseudo $a$-ideal of $X$. Then $\mu$ is a fuzzy pseudo-ideal. Setting $x = z = 0$ and $a = c = 0$ in Theorem 3.3.9 (ii), then we have

$$\mu(y) \geq \mu(0 \odot (0 \ast y)) \quad \text{and} \quad \mu(b) \geq \mu(0 \ast (0 \odot b)).$$

From Lemma 3.3.13, $\mu$ is a fuzzy pseudo $p$-ideal. \qed

**Theorem 3.3.15:** Any fuzzy pseudo $a$-ideal is a fuzzy pseudo $q$-ideal.

**Proof:** Let $\mu$ be a fuzzy pseudo $a$-ideal of $X$. Then $\mu$ is a fuzzy pseudo ideal. In order to prove that $\mu$ is a fuzzy pseudo $q$-ideal, from Theorem 3.3.6(iii), it is sufficient to show that

$$\mu(x \odot y) \geq \mu(x \odot (0 \ast y)) \quad \text{and} \quad \mu(a \ast b) \geq \mu(a \ast (0 \odot b)).$$

We have

$$
\begin{align*}
(0 \ast (0 \odot (y \odot (0 \ast x)))) \ast (x \odot (0 \ast y)) & \\
= (0 \ast ((0 \ast y) \ast (0 \odot (0 \ast x)))) \ast (x \odot (0 \ast y)) & \\
= ((0 \odot (0 \ast y)) \odot (0 \ast (0 \odot (0 \ast x)))) \ast (x \odot (0 \ast y)) & \\
= ((0 \odot (0 \ast y)) \odot (0 \ast x)) \ast (x \odot (0 \ast y)) & \\
= ((0 \odot (0 \ast y)) \ast (x \odot (0 \ast y))) \odot (0 \ast x) & \\
\leq (0 \ast x) \odot (0 \ast x) = 0 \ast (x \ast x) & = 0.
\end{align*}
$$
Hence
\[ \mu(0 \ast (0 \odot (y \odot (0 \ast x)))) \geq \mu(x \odot (0 \ast y)) \tag{6} \]
and we have
\[
(0 \odot (0 \ast (b \ast (0 \odot a)))) \odot (a \ast (0 \odot b)) \\
= (0 \odot (0 \odot b) \odot (0 \ast (0 \odot a))) \odot (a \ast (0 \odot b)) \\
= ((0 \ast (0 \odot b)) \ast (0 \odot (0 \ast (0 \odot a)))) \odot (a \ast (0 \odot b)) \\
= ((0 \ast (0 \odot b)) \ast (0 \odot a)) \ast (a \ast (0 \odot b)) \\
= ((0 \ast (0 \odot b)) \odot (a \ast (0 \odot b))) \ast (0 \odot a) \\
\leq (0 \ast a) \ast (0 \odot a) = 0 \odot (a \odot a) = 0.
\]
Then
\[ \mu(0 \odot (0 \ast (b \ast (0 \odot a)))) \geq \mu(a \ast (0 \odot b)). \tag{7} \]
Since, \( \mu \) is a fuzzy pseudo \( p \)-ideal by Lemma 3.3.13, we have
\[ \mu(y \odot (0 \ast x)) \geq \mu(0 \odot (0 \ast (y \odot (0 \ast x)))) \]
and
\[ \mu(b \ast (0 \odot a)) \geq \mu(0 \ast (0 \odot (b \ast (0 \odot a)))). \]
By Theorem 3.3.9(iii)
\[ \mu(x \odot y) \geq \mu(y \odot (0 \ast x)) \tag{8} \]
and
\[ \mu(a \ast b) \geq \mu(b \ast (0 \odot a)). \tag{9} \]
From (6), (7) and (8), (9), we have
\[ \mu(x \odot y) \geq \mu(y \odot (0 \ast x)) \geq \mu(0 \odot (0 \ast (y \odot (0 \ast x)))) \geq \mu(x \odot (0 \ast y)) \]
and

\[ \mu(a \ast b) \geq \mu(b \ast (0 \ominus b)) \geq \mu(0 \ast (0 \ast (b \ast (0 \ominus a)))) \geq \mu(a \ast (0 \ominus b)). \]

Therefore \( \mu \) is a fuzzy pseudo \( q \)-ideal of \( X \).

\[ \square \]

**Theorem 3.3.16:** Let \( \mu \) be a fuzzy pseudo-ideal of \( X \), \( \mu \) is a fuzzy pseudo \( a \)-ideal if and only if it is both a fuzzy pseudo \( p \)-ideal and a fuzzy pseudo \( q \)-ideal.

**Proof:** If \( \mu \) is a fuzzy pseudo \( a \)-ideal, then \( \mu \) is a fuzzy pseudo \( p \)-ideal and a fuzzy pseudo \( q \)-ideal by Theorem 3.3.15 and Theorem 3.3.16.

Conversely, let \( \mu \) be both a fuzzy \( p \)-ideal and a \( q \)-ideal, we want to show

\[ \mu(y \ominus x) \geq \mu(x \ominus (0 \ast y)) \quad \text{and} \quad \mu(b \ast a) \geq \mu(a \ast (0 \ominus b)). \]

By Theorem 3.3.9(iii),

\[ \mu(x \ominus y) \geq \mu(x \ominus (0 \ast y)) \quad \text{and} \quad \mu((a \ast b) \geq \mu(a \ast (0 \ominus b)). \]

Hence

\[ \mu(0 \ominus (y \ominus x)) \geq \mu(x \ominus y) \geq \mu(x \ominus (0 \ast y)) \]

and

\[ \mu(0 \ast (b \ast a)) \geq \mu(a \ast b) \geq \mu(a \ast (0 \ominus b)). \]

Since \( \mu \) is a fuzzy pseudo \( p \)-ideal by Lemma 3.3.13

\[ \mu(y \ominus x) \geq \mu(0 \ominus (0 \ast (y \ominus x))) \quad \text{and} \quad \mu(b \ast a) \geq \mu(0 \ast (0 \ominus (b \ast a))). \]
Also $\mu$ is a fuzzy pseudo $q$-ideal, therefore

$$\mu(0 \ominus (y \ominus x)) \leq \mu(0 \ominus (0 \ast (y \ominus x)))$$

and

$$\mu(0 \ast (b \ast a)) \leq \mu(0 \ominus (0 \ominus (b \ast a))).$$

We have

$$\mu(y \ominus x) \geq \mu(0 \ominus (0 \ast (y \ominus x))) \geq \mu(0 \ominus (y \ominus x)) \geq \mu(x \ominus y) \geq \mu(x \ominus (0 \ast y))$$

and

$$\mu(b \ast a) \geq \mu(0 \ast (0 \ominus (b \ast a))) \geq \mu(0 \ast (b \ast a)) \geq \mu(a \ast b) \geq \mu(a \ast (0 \ominus b)).$$

Thus, $\mu$ is a fuzzy pseudo $a$-ideal, by Theorem 3.3.9(iii), completing the proof.

$\square$