CHAPTER 6

LOWER ORDER MODEL FORMULATION OF LINEAR TIME INVARIANT SYSTEMS

6.1 INTRODUCTION

The latest developments in the design of processors have increased the computational speed and accuracy to a large extent. But still there exists increased demands on computational speeds for handling the real time situations that arise in estimating, filtering and control of chemical plants, process industries, air-craft systems and nuclear reactors. Lower order models aid to carry out the process operations in real time by reducing the implementation and computational difficulties involved in the design of optimal and adaptive controller, compensators and observer design for higher order linear time invariant system. A lower order model is preferred because in the analysis and design, simulation of a compensator or controller for stabilization of the output response of the given system, higher order models are much more difficult to handle. A number of methods for deriving such lower order models are available in the literature as discussed in section 1.6 of Chapter 1.

In Chapter 2 and Chapter 3, the stability and design of linear time invariant systems was carried out by extracting lower order polynomials termed as pseudo Routh column polynomials and auxiliary polynomials from the given characteristic equation respectively. The approach of extracting the auxiliary polynomials is found to be simple and direct. Thus, in this Chapter, the concept of formulating auxiliary polynomials from the characteristic equation in Chapter 3 is extended to obtain an initial lower order model from
the given higher order model representing the system. Further, fine tuning of this initial lower order model is performed using stochastic population based optimization technique called as Particle Swarm Optimization (PSO) to obtain a better lower order approximant for the given higher order model. The proposed procedures are illustrated with examples.

6.2 PROBLEM DEFINITION

Consider an \( n^{th} \) order linear time invariant continuous system represented as (Krishnamurthy and Seshadri 1978),

\[
G(s) = \frac{N(s)}{D(s)} = \frac{\sum_{i=0}^{n-1} A_is^i}{\sum_{i=0}^{n} a_is^i} = \frac{A_n-s^{n-1}+A_{n-2}s^{n-2}+...+A_3s^3+A_2s^2+A_1s+A_0}{a_ns^n+a_{n-1}s^{n-1}+...+a_3s^3+a_2s^2+a_1s+a_0} \tag{6.1}
\]

where \( N(s) \) is the numerator polynomial and \( D(s) \) is the denominator polynomial.

For the given original higher order system \( G(s) \) represented in equation (6.1), the problem is to find an \( m^{th} \) lower order model \( R(s) \), where \( m < n \) in the following form represented by equation (6.2), such that the reduced model retains the characteristics of the original system and approximate its response as closely as possible for the same type of inputs with minimum integral square error.

\[
R(s) = \frac{N^m(s)}{D^m(s)} = \frac{\sum_{i=0}^{m-1} B_is^i}{\sum_{i=0}^{m} b_is^i} \tag{6.2}
\]

where \( N^m(s) \) and \( D^m(s) \) are the numerator polynomial and denominator polynomial of the lower order model respectively. Also, \( B_i \) and \( b_i \) represent the constant coefficients of \( s \)-terms of numerator and denominator of \( R(s) \).
The main objective of lower order model formulation is to minimize the integral square error of the unit step time responses of the lower order system and the original higher order system within a specified time interval with the constraint: the transient and steady state gain ratios of the higher order system should be maintained in the designed lower order system.

Mathematically, the integral square error (ISE) (Nagrath and Gopal 2003) can be expressed as,

\[ E = \sum_{t=0}^{\tau} (Y_t - y_t)^2 \]  

where,

- \( Y_t \) is the unit step time response of the given higher order system at the \( t^{th} \) instant in the time interval \( 0 \leq t \leq \tau \)
- \( y_t \) is the unit step time response of the lower order system at the \( t^{th} \) time instant
- \( \tau \) is the time period in seconds over which the integral square error is calculated and is to be chosen.

Minimization of integral square error will ensure a good lower order approximant. The problem defined for linear time invariant continuous system holds the same for linear time invariant discrete system also.

### 6.3 INITIAL LOWER ORDER MODELS USING AUXILIARY POLYNOMIAL APPROACH

In section 3.2 of Chapter 3, the formulation of auxiliary polynomials using the coefficients of the given characteristic equation is dealt. This proposed procedure is extended to obtain the initial lower order models from the given linear time invariant higher order systems represented by their
transfer function in this section. The auxiliary polynomial scheme to extract initial lower order models is as follows:

Consider an \( n \)\(^{th}\) order linear time invariant continuous higher order system represented by its transfer function as given in equation (6.1). Extending the auxiliary polynomial scheme in Chapter 3, the initial lower order model from the given higher order system in equation (6.1) is obtained as follows:

First order: \[
\frac{A_0}{a_j s + a_0}
\]  
(6.4)

Second order: \[
\frac{A_j s + A_0}{a_2 s^2 + a_1 s + a_0}
\]  
(6.5)

Third order: \[
\frac{A_2 s^2 + A_1 s + A_0}{a_3 s^3 + a_2 s^2 + a_1 s + a_0}
\]  
(6.6)

\[\vdots\]

\[\vdots\]

\[\vdots\]

\( (n-1) \)\(^{th}\) order: \[
\frac{A_{n-2} s^{n-2} + A_{n-3} s^{n-3} + \ldots + A_j s + A_0}{a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \ldots + a_1 s + a_0}
\]  
(6.7)

Equations (6.4) through (6.7), give the basic lower order models extracted using the auxiliary polynomial approach from the given higher order system \( G(s) \). Based on the requirement, a suitable lower order model can be selected and operated. It should be noted for a higher order system of order ‘n’, \( (n-1) \) lower order models could be formulated. This method of selection of basic lower order models helps to set the initial values of operating parameters to be used in the particle swarm optimization process. The procedure discussed for extracting initial lower order models using auxiliary polynomials for continuous system holds the same for linear time invariant discrete system also.
6.3.1 Choice of Second Order Model

In this Chapter, for simplicity a second order model as in equation (6.5) is considered for formulation since it will exhibit almost all the characteristics of the original n-th order linear time invariant system. An n-th order system will generally possess four different solutions viz., pure oscillations, damped oscillations, undamped oscillations and exponential decay which can be observed by employing a second order model. Also, second order systems help control system engineers in reducing the computational complexity involved in designing controllers and compensators as well as state variable controllers and observers (Ogata 2004) for a given higher order system.

6.4 PARTICLE SWARM OPTIMIZATION (PSO)

Particle Swarm Optimization is a population based stochastic optimization technique developed by Kennedy and Eberhart (1995) inspired by social behavior of bird flocking or fish schooling. The PSO method is a member of the wide category of Swarm Intelligence methods. In PSO, the system is initialized with a population of random solutions and searches for optima by updating generations. During the process of PSO, the potential solutions, called particles, which are a metaphor of birds in flocks, fly through the problem space by following the current optimum particles. These particles are randomly initialized and freely fly across the multi-dimensional search space. During flight, each particle updates its own velocity and position based on the best experience of its own and the entire population. The updating policy drives the particle swarm to move toward the region with the required objective function value, and eventually all particles will gather around the point with the required objective value. The basic algorithmic operation of Particle Swarm Optimization is as given below:
6.4.1 Particle Swarm Optimization Algorithm

PSO is initialized with a group of random particles (solutions) and then searches for optima by updating generations. In every iteration, each particle is updated by following two “best” values. The first one is the best solution (fitness) it has achieved so far (The fitness value is also stored). This value is called personal best and is called pbest. Another “best” value that is tracked by the particle swarm optimizer is the best value, obtained so far by any particle in the population. This best value is a global best and called gbest. When a particle takes part of the population as its topological neighbors, the best value is a local best and is called lbest. After finding the two best values (pbest and gbest), the particle updates its velocity and positions. The algorithmic steps involved are as follows (Kennedy and Eberhart 2001):

Step 1: Initialization: For each particle initialize its particle values randomly.
Step 2: Fitness Evaluation:
   For each particle,
   i) Calculate the fitness value
   ii) If the fitness value is better than the best fitness value (pbest) in history so far,
       Set current value as the new pbest.
   End
Step 3: Global Best Selection:
   Choose the particle with the best fitness value of all the particles as the gbest.
Step 4: Velocity and Position Updation:
   For each particle
   i) Calculate particle velocity
      \[ v_i = v_i + c_1 R_1 ( p_{i_{best}} - p_i ) + c_2 R_2 ( g_{ibest} - p_i ) \]  \hspace{1cm} (6.8)
where,

\( v_i \) – velocity of particle i

\( p_i \) – position of particle i

\( p_{ibest} \) - position with the ‘best’ fitness value found so far by particle i.

\( g_{ibest} \) - best fitness value obtained so far by any particle in the entire population

\( R_1, R_2 \) - random variables in the range \([0, 1]\)

\( c_1, c_2 \) - learning factors controlling the related weighting of corresponding terms.

The inclusion of random variables endows the PSO with the ability of stochastic searching. The learning factors \( c_1 \) and \( c_2 \), compromise the inevitable trade-off between exploration and exploitation. After updating, \( v_i \) should be checked and secured within a pre-specified range to avoid violent random walking.

ii) Update particle positions

\[
 p_i = p_i + v_i
\]

(6.9)

The position of all particles is updated according to equation (6.9).

Step 5: Termination checking:

The algorithm repeats steps 2 to 4 while maximum iterations or minimum error criterion is not attained. Once terminated, the algorithm reports the values of gbest and its respective fitness value.

The flowchart depicting the process of PSO algorithm is as shown in Figure 6.1.
PSO can be easily implemented and is computationally inexpensive since its memory and CPU speed requirements are low. PSO has proved to be an efficient method for numerous general optimization problems with few parameters to adjust. PSO has been successfully applied to a range of problems, from function optimization to the training of neural networks.

6.5 LOWER ORDER MODEL FORMULATION FOR LINEAR TIME INVARIANT CONTINUOUS SYSTEMS

In this section, the auxiliary polynomial approach is utilized along with Particle Swarm Optimization to obtain lower order models for Single - Input - Single - Output (SISO) and Multi - Input - Multi - Output (MIMO) linear time invariant continuous systems.
6.5.1 Single Input Single Output Linear Time Invariant Continuous Systems

The proposed methodology for lower order model formulation of SISO LTICS is as follows:

Step 1: Consider an n-th order linear time invariant continuous system represented by the transfer function (TF) \( G(s) \) in general form as given in equation (6.1).

Step 2: Calculate the transient gain (TG) and steady state gain (SSG) for the given higher order system in equation (6.1) as follows:

\[
TG = \frac{A_{n-1}}{a_n} \quad (6.10)
\]

\[
SSG = \frac{A_0}{a_0} \quad (6.11)
\]

Step 3: Applying auxiliary polynomial approach in section 6.3 to obtain an initial lower order model of order 2. Thus, the transfer function of basic second order model obtained using auxiliary polynomial approach is as given in equation (6.5) i.e.,

\[
R(s) = \frac{A_1s + A_0}{a_2s^2 + a_1s + a_0} \quad (6.12)
\]

Step 4: Scaling equation (6.12), \( R(s) \) becomes,

\[
R(s) = \frac{s + \left( \frac{A_0}{A_1} \right)}{s^2 + \left( \frac{a_1}{a_2} \right)s + \left( \frac{a_0}{a_2} \right)} \quad (6.13)
\]

Step 5: Equation (6.13) is tuned to maintain the transient and steady state gain obtained in equations (6.10) and (6.11), resulting initial second order model \( R(s) \) is,

\[
R(s) = \frac{(TG)s + \left( SSG \right) \times \left( \frac{a_0}{a_2} \right)}{s^2 + \left( \frac{a_1}{a_2} \right)s + \left( \frac{a_0}{a_2} \right)} \quad (6.14)
\]
Comparing equations (6.12) and (6.14), it can be noted that,

\[ A_i = (TG) \]

and

\[ A_0 = \left( \begin{array}{c} (SSG) \\ \times \end{array} \right) \left( \begin{array}{c} a_0 \\ a_2 \end{array} \right) \]  

(6.15)

Step 6: The coefficients of the initial second order model \( R(s) \) in equation (6.14) are fed as input to particle swarm optimisation process. The main aim of PSO is to minimize the objective function integral square error \( E \) given in equation (6.3). The PSO algorithm in section 6.4.1 is invoked to search for the better values of \( \left( \begin{array}{c} a_0 \\ a_2 \end{array} \right) \) and \( \left( \begin{array}{c} a_1 \\ a_2 \end{array} \right) \) in equation (6.14), so that the characteristics of the formulated second order model matches the given higher order system. The PSO is carried out within the constraint of maintaining the transient and steady state gain of the second order model in accordance with that of the given higher order system calculated in equations (6.10) and (6.11).

Step 7: The second order model corresponding to the minimum integral square error is declared as the best second order approximant for \( G(s) \) in equation (6.1) and is given by,

\[ R^2(s) = \frac{B_1s + B_0}{b_2s^2 + b_1s + b_0} \]  

(6.16)

Step 8: To ensure the effectiveness of the proposed procedure, the unit step response of the given higher order system in equation (6.1) and the response of the proposed second order model in equation (6.16) is compared. Also, the integral square error is computed and compared with that calculated for the second order models obtained by other methods. This formulated second order model \( R(s) \) in equation (6.16) is found to maintain the original characteristics of the given higher order model.

The flowchart for the proposed procedure is as shown in Figure 6.2. The proposed methodology is illustrated with numerical examples.
Read the coefficients of the numerator and denominator from,
\[ G(s) = \frac{A_n s^n + A_{n-1} s^{n-1} + \ldots + A_1 s + A_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \]
which represents the transfer function of the given LTICS.

Calculate transient gain (TG) and steady state gain (SSG) for given \( G(s) \).

Apply the auxiliary polynomial approach to \( G(s) \) and obtain the initial lower order model \( R(s) \), scale it and tune it to maintain the transient and steady state gain of \( G(s) \).

Invoke Particle Swarm Optimization by passing the numerator and denominator coefficients of \( R(s) \).

For each generation

For each particle

If integral square error ‘E’ < best ‘E’ (pbest) so far

Yes

current value = new pbest

No

Choose the particle with the best integral square error of all the particles as the gbest.

Calculate particle velocity

Calculate particle positions

Update memory of each particle

End

A

Figure 6.2 (Continued)
Figure 6.2  Flowchart for lower order model formulation of single input single output linear time invariant continuous systems

Illustration 6.1

An eighth order system is considered and applied with the proposed methodology discussed in section 6.5.1.

Step 1: Consider an eighth order linear time invariant continuous system (Krishnamurthy and Seshadri 1978) represented in the form of transfer function as,

\[
G(s) = \frac{35s^7 + 1086s^6 + 13285s^5 + 82402s^4 + 278376s^3}{s^8 + 33s^7 + 437s^6 + 3017s^5 + 11870s^4 + 27470s^3 + 37492s^2 + 28880s + 9600}
\]

(6.17)

Step 2: Calculating the transient gain (TG) and steady state gain (SSG) for given G(s) in equation (6.17).

\[
TG = \frac{35}{1} = 35
\]

(6.18)

\[
SSG = \frac{194480}{9600} = 20.26
\]
Step 3: Applying auxiliary polynomial approach to \( G(s) \) in equation (6.17) to get basic second order model \( R(s) \),

\[
R(s) = \frac{482964s + 194480}{37492s^2 + 28880s + 9600}
\]  
(6.19)

Step 4: On scaling equation (6.19), \( R(s) \) becomes,

\[
R(s) = \frac{s + 0.4027}{s^2 + 0.7703s + 0.2561}
\]  
(6.20)

Step 5: Tuning equation (6.20) to maintain transient and steady state gain of higher order system obtained in equation (6.18), the initial second order model is,

\[
R(s) = \frac{35s + 5.1887}{s^2 + 0.7703s + 0.2561}
\]  
(6.21)

Step 6: The particle swarm optimisation algorithm is now invoked to search the values of \( s \)-term (0.7703) and constant term (0.2561) of the denominator in \( R(s) \) represented by equation (6.21), so that the characteristics of second order model matches the given higher order system given by equation (6.17). The PSO is performed within the constraint of maintaining the transient and steady state gain of the lower second order model in accordance with that of the given higher order system as in equation (6.18). PSO determines a better second order model for which the integral square error is minimal, using its algorithm in section 6.4.1. The values of various parameters used during the PSO algorithm process are:

- Number of generations = 90
- Number of particles = 80
- Learning factors \( c_1 \) and \( c_2 = 0.5 \)

Step 7: The transfer function of the lower second order model corresponding to minimal integral square error obtained using PSO is,

\[
R^2(s) = \frac{35s + 61.9485}{s^2 + 3.2467s + 3.0577}
\]  
(6.22)
Step 8: The unit step response of the given higher order system, the proposed second order system and that of the second order systems obtained using other known methods are shown in Figure 6.3. The corresponding integral square errors computed are shown in Table 6.1. The final time for simulation is taken as 10 seconds with an interval of 0.1 seconds.

Table 6.1  Comparison of Integral square error of unit step time response for Illustration 6.1

<table>
<thead>
<tr>
<th>Lower order model formulation method</th>
<th>Lower second order model</th>
<th>Integral Square Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chen et al (1979) method</td>
<td>$14.12423s + 5.68755$</td>
<td>$s^2 + 0.844593s + 0.280751$</td>
</tr>
<tr>
<td>Krishnamurthy and Seshadri (1978) method</td>
<td>$16.63852s + 9.664227$</td>
<td>$s^2 + 0.90024s + 0.47705$</td>
</tr>
<tr>
<td>Jayanta Pal et al (1994) method</td>
<td>$33.00792s + 145.2427$</td>
<td>$s^2 + 5.39208s + 7.169529$</td>
</tr>
<tr>
<td>Proposed Method</td>
<td>$35s + 61.9485$</td>
<td>$s^2 + 3.2467s + 3.0577$</td>
</tr>
</tbody>
</table>

Figure 6.3 Comparison of unit step responses for Illustration 6.1
From Table 6.1, it is observed that the proposed methodology yields better value for integral square error with respect to other methods considered for comparison. The unit step responses plotted in Figure 6.3 indicates that the proposed second order approximant is a suitable fit for the given higher order system compared to other methods. The sample program for formulating lower order model of SISO LTICS is provided in Appendix 7(a).

Illustration 6.2

1. Consider an eighth order system (Palaniswami 2001) represented by its transfer function,

\[
G(s) = \frac{35 s^7 + 1086 s^6 + 13285 s^5 + 82402 s^4 + 278376 s^3}{s^8 + 21 s^7 + 220 s^6 + 1558 s^5 + 7669 s^4 + 24469 s^3 + 46350 s^2 + 45952 s + 17760} \quad (6.23)
\]

Using the proposed methodology,

2. The transient gain and steady state gain for given G(s) in equation (6.23) are,

\[
TG = \frac{35}{1} = 35 \\
SSG = \frac{194480}{17760} = 10.95 \quad (6.24)
\]

3. Applying the auxiliary polynomial approach to equation (6.23), the basic second order model is,

\[
R(s) = \frac{482964s + 194480}{46350s^2 + 45952s + 17760} \quad (6.25)
\]

4. Scaling equation (6.25), R(s) becomes,

\[
R(s) = \frac{s + 0.4027}{s^2 + 0.9914s + 0.3832} \quad (6.26)
\]
5. Tuning equation (6.26) to maintain the transient and steady state gain computed in equation (6.24), the initial lower order model \( R(s) \) is,

\[
R(s) = \frac{35s + 4.1960}{s^2 + 0.9914s + 0.3832}
\] (6.27)

6. PSO is invoked to minimize the integral square error by searching for the best values of the denominator coefficients of \( s \)-term (0.9914) and constant term (0.3832) in \( R(s) \) given by equation (6.27) and to maintain the same transient and steady state gain ratio as in equation (6.24). It yields the following transfer function of the lower second order system with minimal integral square error,

\[
R^2(s) = \frac{35s + 397.0754}{s^2 + 1.7664s + 36.2626}
\] (6.28)

7. The unit step time responses of the given higher order system, the proposed second order system and that of the second order systems obtained using other known methods are shown in Figure 6.4. The integral square errors computed are tabulated as shown in Table 6.2.

**Table 6.2** Comparison of integral square error of unit step time response for Illustration 6.2

<table>
<thead>
<tr>
<th>Lower order model formulation method</th>
<th>Lower second order model</th>
<th>Integral Square Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Krishnamurthy and Seshadri (1978) method</td>
<td>( \frac{334828.5s + 194480}{240.52s^2 + 32043.2s + 17760} )</td>
<td>340.9118</td>
</tr>
<tr>
<td>Palaniswami (2001) method</td>
<td>( \frac{30.083369s + 362.846159}{0.859525s^2 + 1.747227s + 33.135272} )</td>
<td>20.6569</td>
</tr>
<tr>
<td><strong>Proposed Method</strong></td>
<td>( \frac{35s + 397.0754}{s^2 + 1.7664s + 36.2626} )</td>
<td><strong>12.0864</strong></td>
</tr>
</tbody>
</table>
From Table 6.2, it is observed that the proposed methodology yields better value for integral square error compared to other methods. The unit step response of the original system and the lower second order systems obtained by different methodologies are shown in Figure 6.4 for comparison.

**6.5.2 Multi Input Multi Output Linear Time Invariant Continuous Systems**

The particle swarm optimization approach for formulating lower order approximant for SISO systems has been established in the previous section. Recently, systems have become complex and the interrelationship of many controlled variables need to be considered in the design process. The computer control systems used to control the fighter aircrafts, fuel injectors and spark timing of automobiles are excellent examples of such multivariable control systems. These MIMO systems are challenging when compared to the analysis of SISO systems. The exact analysis of higher order MIMO systems (Shamash 1975a, 1981) are often difficult due to computational considerations, which further emphasizes the importance of formulating lower order models.
The proposed methodology for lower order model formulation of MIMO LTICS is as follows:

Step 1: Consider the given transfer function matrix of the multivariable systems as,

\[
G(s) = \begin{bmatrix}
G_{11}(s) & G_{12}(s) & \cdots & G_{1r}(s) \\
G_{21}(s) & G_{22}(s) & \cdots & G_{2r}(s) \\
\vdots & \vdots & \ddots & \vdots \\
G_{q1}(s) & G_{q2}(s) & \cdots & G_{qr}(s)
\end{bmatrix}
\]  
(6.29)

where, \( G(s) = \frac{N_{ij}(s)}{D_{ij}(s)} = \frac{\sum_{k=0}^{n-1} A_k(i, j)s^k}{\sum_{k=0}^{n} a_k(i, j)s^k} \) with ‘q’ inputs and ‘r’ outputs. i.e., \( i = 1, 2, \ldots, q \) and \( j = 1, 2, \ldots, r \).

Take common denominator \( D(s) \) for the given \( G(s) \), then the transfer function of the system can be represented in the matrix form as:

\[
G_{ij}(s) = \frac{N_{ij}(s)}{D(s)} = \frac{\sum_{k=0}^{n} A_k(i, j)s^k}{\sum_{k=0}^{n} a_k(i, j)s^k}
\]  
(6.30)

where, \( N_{ij}(s) = \sum_{k=0}^{n-1} A_k(i, j)s^k \) and

\[
D(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_2 s^2 + a_1 s + a_0
\]

Step 2: For each \( G_{ij}(s) \) do the following

Step 2.1: From equation (6.30), get,
\[ G_{ij}(s) = \frac{N_{ij}(s)}{D(s)} \]
\[ A_0(i,j) + A_1(i,j)s + A_2(i,j)s^2 + \ldots + A_{n-1}(i,j)s^{n-1} \]
\[ = \frac{A_{n-1}(i,j)s^{n-1}}{a_0 + a_1s + a_2s^2 + \ldots + a_{n-1}s^{n-1} + a_ns^n} \]

Step 2.2: Compute the transient gain (TG\(_{ij}\)) and steady state gain (SSG\(_{ij}\)) of \( G_{ij}(s) \) in equation (6.31) as follows:
\[ TG_{ij} = \frac{A_{n-1}(i,j)}{a_n} \]  
(6.32)
\[ SSG_{ij} = \frac{A_0(i,j)}{a_0} \]

Step 2.3: Applying the auxiliary polynomial and particle swarm optimization approach discussed in section 6.5.1 step 3 to step 6 for equation (6.31), obtain a lower second order model corresponding to the minimum integral square error of each \( G_{ij}(s) \) with transfer function \( N_{ij}(s)/D(s) \) as,
\[ R_{ij}(s) = \frac{N_{ij}^2(s)}{D_{ij}^2(s)} = \frac{B_{1}(i,j)s + B_{0}(i,j)}{b_2(i,j)s^2 + b_1(i,j)s + b_0(i,j)} \]  
(6.33)
where \( i = 1, 2, \ldots, q \) and \( j = 1, 2, \ldots, r \) with \( B_1(i,j), B_0(i,j), b_2(i,j), b_1(i,j) \) and \( b_0(i,j) \) being constants.

Step 3: Determine the common denominator \( D^2(s) \) as the average of the corresponding coefficients of each \( D_{ij}^2(s) \) from equation (6.33), which is mathematically expressed as,
\[ D^2(s) = b_2s^2 + b_1s + b_0 \]  
(6.34)
where, \( b_k = \frac{\sum_{i=1}^{q} \sum_{j=1}^{r} b_k(i,j)}{Total \ number \ of \ b_k(i,j)} \) and \( k = 0, 1, 2 \) with reference to equation (6.34).

Step 4: Reconstruct the numerators of each \( R_{ij}(s) \) with,
\[ N_{ij}^2(s) = (TG_{ij}s) + (SSG_{ij} \times b_0) \]  
(6.35)
so that the characteristics of the given higher order system are maintained in the proposed second order model.

Step 5: The transfer function matrix of the lower second order system can now be represented as,

\[
\begin{bmatrix}
N_{11}^2(s) & N_{12}^2(s) & \cdots & N_{1r}^2(s) \\
N_{21}^2(s) & N_{22}^2(s) & \cdots & N_{2r}^2(s) \\
\vdots & \vdots & \ddots & \vdots \\
N_{q1}^2(s) & N_{q2}^2(s) & \cdots & N_{qr}^2(s)
\end{bmatrix}
\]

\[
R^2(s) = \frac{N^2(s)}{D^2(s)}
\]

(6.36)

Step 6: To ensure the effectiveness of the proposed procedure, the unit step response of the given higher order system in equation (6.29) and the response of the proposed second order model in equation (6.36) is compared. Also, the integral square error is computed and compared with that calculated for the second order models obtained by other methods.

The proposed methodology is illustrated in the following numerical example.

**Illustration 6.3**

The proposed procedure in section 6.5.2 is applied to linear time invariant multivariable continuous system (Prasad et al 1995) as follows:

Step 1: Consider the sixth order system described by the transfer function matrix,

\[
G(s) = \begin{bmatrix}
\frac{2(s+5)}{(s+1)(s+10)} & \frac{(s+4)}{(s+2)(s+5)} \\
\frac{20}{(s+1)(s+10)} & \frac{(s+6)}{(s+2)(s+3)}
\end{bmatrix}
\]

(6.37)
The common denominator $D(s)$ of sixth order system is,

$$
D(s) = (s + 1)(s + 2)(s + 3)(s + 4)(s + 9)(s + 20)
$$

$$
= s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000
$$

(6.38)

Step 2: For each $G(s)$ do the following,

Step 2.1: $G(s)$ can be represented as,

$$
G(s) = \begin{bmatrix}
G_{11}(s) & G_{12}(s) \\
G_{21}(s) & G_{22}(s)
\end{bmatrix}
$$

(6.39)

where,

$$
G_{12}(s) = \frac{2s^5 + 70s^4 + 762s^3 + 3610s^2 + 7700s + 6000}{D(s)}
$$

(6.40)

$$
G_{12}(s) = \frac{s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400}{D(s)}
$$

(6.41)

$$
G_{22}(s) = \frac{s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000}{D(s)}
$$

(6.42)

$$
G_{22}(s) = \frac{s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000}{D(s)}
$$

(6.43)

Step 2.2: Table 6.3 gives the transient gain and steady state gain ratio computed for $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$ and $G_{22}(s)$ represented in equation (6.40) through (6.43).

**Table 6.3 Transient gain and steady state gain for $G_{11}(s)$, $G_{12}(s)$, $G_{21}(s)$ and $G_{22}(s)$**

<table>
<thead>
<tr>
<th>Higher order system</th>
<th>Transient Gain</th>
<th>Steady state gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_{11}(s)$</td>
<td>$\frac{2}{1}$ = 2</td>
<td>$\frac{6000}{6000}$ = 1</td>
</tr>
<tr>
<td>$G_{12}(s)$</td>
<td>$\frac{1}{1}$ = 1</td>
<td>$\frac{2400}{6000}$ = 0.4</td>
</tr>
<tr>
<td>$G_{21}(s)$</td>
<td>$\frac{1}{1}$ = 1</td>
<td>$\frac{3000}{6000}$ = 0.5</td>
</tr>
<tr>
<td>$G_{22}(s)$</td>
<td>$\frac{1}{1}$ = 1</td>
<td>$\frac{6000}{6000}$ = 1</td>
</tr>
</tbody>
</table>
Step 2.3: Using the proposed auxiliary polynomial and PSO approach for \( G_{11}(s), G_{12}(s), G_{21}(s) \) and \( G_{22}(s) \) represented in equations (6.40) to (6.43), the transfer functions of the respective second order models \( R_{11}(s), R_{12}(s), R_{21}(s) \) and \( R_{22}(s) \) are obtained and presented in Table 6.4.

### Table 6.4 Results of auxiliary polynomial and PSO approach for \( G_{11}(s), G_{12}(s), G_{21}(s) \) and \( G_{22}(s) \)

<table>
<thead>
<tr>
<th>System</th>
<th>Basic second order model from auxiliary polynomial approach</th>
<th>Gain adjusted initial second order models for input to PSO</th>
<th>Final second order approximant from PSO</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_{11}(s) )</td>
<td>( \frac{7700 s + 6000}{10060 s^2 + 13100 s + 6000} )</td>
<td>( \frac{2s + 0.5964}{s^2 + 1.3022 s + 0.5964} )</td>
<td>( \frac{2s + 10.0648}{s^2 + 11.0537 s + 10.0648} )</td>
</tr>
<tr>
<td>( R_{12}(s) )</td>
<td>( \frac{4160 s + 2400}{10060 s^2 + 13100 s + 6000} )</td>
<td>( \frac{s + 0.2386}{s^2 + 1.3022 s + 0.5964} )</td>
<td>( \frac{s + 2.5651}{s^2 + 5.3858 s + 6.4128} )</td>
</tr>
<tr>
<td>( R_{21}(s) )</td>
<td>( \frac{3700 s + 3000}{10060 s^2 + 13100 s + 6000} )</td>
<td>( \frac{s + 0.2982}{s^2 + 1.3022 s + 0.5964} )</td>
<td>( \frac{s + 9.8146}{s^2 + 20.6362 s + 19.6293} )</td>
</tr>
<tr>
<td>( R_{22}(s) )</td>
<td>( \frac{9100 s + 6000}{10060 s^2 + 13100 s + 6000} )</td>
<td>( \frac{s + 0.5964}{s^2 + 1.3022 s + 0.5964} )</td>
<td>( \frac{s + 5.9849}{s^2 + 4.9928 s + 5.9849} )</td>
</tr>
</tbody>
</table>

Step 3: From the denominators of the final second order approximants \( R_{11}(s), R_{12}(s), R_{21}(s) \) and \( R_{22}(s) \) in Table 6.4, the common denominator for the second order MIMO model is obtained by computing the average of corresponding coefficients, which is represented as,

\[
D^2(s) = s^2 + 10.5171s + 10.5229
\]  

(6.44)

Step 4: Using the transient gain ratio and steady state gain ratio of \( G_{11}(s), G_{12}(s), G_{21}(s) \) and \( G_{22}(s) \) in Table 6.3 and \( D^2(s) \), the transfer functions \( R_{11}(s), R_{12}(s), R_{21}(s) \) and \( R_{22}(s) \) are reconstructed as,

\[
R_{11}(s) = \frac{2s + 10.5229}{s^2 + 10.5171s + 10.5229}
\]  

(6.45)
\[
R_{12}(s) = \frac{s + 4.20916}{s^2 + 10.5171s + 10.5229}
\] (6.46)

\[
R_{21}(s) = \frac{s + 5.26145}{s^2 + 10.5171s + 10.5229}
\] (6.47)

\[
R_{22}(s) = \frac{s + 10.5229}{s^2 + 10.5171s + 10.5229}
\] (6.48)

Step 5: The second order MIMO model in transfer function matrix is,

\[
R^2(s) = \begin{bmatrix}
R_{11}(s) & R_{12}(s) \\
R_{21}(s) & R_{22}(s)
\end{bmatrix}
= \frac{1}{D^2(s)} \begin{bmatrix}
2s + 10.5229 & s + 4.20916 \\
s + 5.26145 & s + 10.5229
\end{bmatrix}
\] (6.49)

where, \(D^2(s) = s^2 + 10.5171s + 10.5229\).

Step 6: The unit step time responses of the given higher order system, the proposed second order system and that of the second order systems obtained using other known methods are shown in Figure 6.5(a) to 6.5(d). The integral square errors computed using the proposed methodology and that computed for other schemes are tabulated in Table 6.5.

### Table 6.5 Comparison of Integral square error for Illustration 6.3

<table>
<thead>
<tr>
<th>Lower order model formulation methods</th>
<th>(G_{11}(s))</th>
<th>(G_{12}(s))</th>
<th>(G_{21}(s))</th>
<th>(G_{22}(s))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polynomial derivative method (Prasad et al 1995)</td>
<td>0.3068</td>
<td>3.8578</td>
<td>0.7170</td>
<td>0.2168</td>
</tr>
<tr>
<td>Routh approximation method (Prasad 2000)</td>
<td>0.2301</td>
<td>0.0887</td>
<td>0.0468</td>
<td>0.2114</td>
</tr>
<tr>
<td>Improved Routh approximation method (Prasad et al 2003a)</td>
<td>0.1676</td>
<td>0.0955</td>
<td>0.0307</td>
<td>0.1970</td>
</tr>
<tr>
<td><strong>Proposed method</strong></td>
<td><strong>0.0217</strong></td>
<td><strong>0.0564</strong></td>
<td><strong>0.0146</strong></td>
<td><strong>0.1796</strong></td>
</tr>
</tbody>
</table>
From Table 6.5, it is inferred that the proposed procedure yields better value for integral square error with respect to other methods considered for comparison. Also, from the unit step responses shown in Figure 6.5(a) to 6.5(d), it is observed that the characteristics of the proposed second order MIMO model is in close agreement with that of the given higher order MIMO systems. The program for obtaining lower order model of MIMO LTICS is as given in Appendix 7(b).

![Figure 6.5(a) Unit step responses for $G_{11}(s)$ - Illustration 6.3](image)

![Figure 6.5(b) Unit step responses for $G_{12}(s)$ - Illustration 6.3](image)
Figure 6.5(c) Unit step responses for $G_{21}(s)$ - Illustration 6.3

Figure 6.5(d) Unit step responses for $G_{22}(s)$ - Illustration 6.3
6.6 LOWER ORDER MODEL FORMULATION FOR LINEAR TIME INVARIANT DISCRETE SYSTEMS

Recently, many new digital control applications are being simulated by microprocessor technology including control of various aspects of automobiles and household appliances. The control of real time systems with digital computers or microcontrollers is common in these days. Few examples of these systems are, digital computer controlled rolling mill regulating system, digital controllers for generators and turbines and microprocessor controlled systems. These are discrete systems with sampled-data input.

For designing suitable controls for the above discrete systems, it is essential to have a model that describes the behavior of the system. For ease, the basic model considered by most designers is linear time invariant. Thus, analysis of linear time invariant discrete system is of high importance. The order of the system identified varies and it depends upon the complexity of the physical entities involved. Further, analysis of a higher order system for the purpose of designing controllers and compensators is cumbersome with large computational overhead. For reducing the computational complexity involved, often, a lower order system, which retains the dominant characteristics of the higher order system, is of great help.

In this section, the proposed PSO approach for the formulation of lower order approximant for linear time invariant continuous systems has been extended to linear time invariant discrete SISO and MIMO systems in the z-domain.
6.6.1 Single Input Single Output Linear Time Invariant Discrete Systems

The proposed methodology for lower order model formulation of SISO LTIDS is as follows:

Step 1: Consider an $n^{th}$ order linear time invariant discrete system in the $z$-domain represented by its transfer function as,

$$
G(z) = \frac{N(z)}{D(z)} = \frac{A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + \ldots + A_1z + A_0}{a_nz^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1z + a_0} \quad (6.50)
$$

Step 2: Determine the transient gain (TG) and steady state gain (SSG) i.e., the value of $G(z)$ at $z=1$ for equation (6.50) as follows:

$$
TG = \frac{A_{n-1}}{a_n} \quad (6.51)
$$

$$
SSG = G(z) \bigg|_{z=1}
$$

Step 3: Applying auxiliary polynomial approach in section 6.3 to $G(z)$ in equation (6.50) to obtain a basic lower order model of order 2. Its transfer function is given by,

$$
R(z) = \frac{A_1z + A_0}{a_2z^2 + a_1z + a_0} \quad (6.52)
$$

Step 4: Scaling equation (6.52), $R(z)$ becomes,

$$
R(z) = \frac{z + \left(\frac{A_0}{A_1}\right)}{z^2 + \left(\frac{a_1}{a_2}\right)z + \left(\frac{a_0}{a_2}\right)} \quad (6.53)
$$

This equation (6.53) is tuned to maintain the transient gain obtained in equation (6.51) and the resulting initial second order model is,

$$
R(z) = \frac{(TG)z + \left(\frac{A_0}{A_1}\right)}{z^2 + \left(\frac{a_1}{a_2}\right)z + \left(\frac{a_0}{a_2}\right)} \quad (6.54)
$$
Step 5: The coefficients of $R(z)$ in equation (6.54) are fed as input to particle swarm optimization process. PSO aims to minimize the objective function integral square error $E$ given in equation (6.3). PSO algorithm in section 6.4.1 is invoked to search the better values of \[
\begin{pmatrix}
\frac{A_0}{A_1} \\
\frac{a_1}{a_2}
\end{pmatrix}
\] and \[
\frac{a_0}{a_2}
\] in equation (6.54), so that the characteristics of the formulated lower order model matches the given higher order system. The PSO searching is carried out maintaining the steady state constraint i.e., the value of $R(z)$ at $z=1$ in par with that of $G(z)$ at $z=1$ and also in minimizing the integral square error.

Step 6: The transfer function of the lower second order model corresponding to the minimum integral square error using PSO for $G(z)$ in equation (6.50) is given by,

\[
R^2(z) = \frac{B_1z + B_0}{b_2z^2 + b_1z + b_0}
\] (6.55)

Step 7: To ensure the effectiveness of the proposed procedure, the unit step response of the given higher order system in equation (6.50) and the response of the proposed second order model in equation (6.55) is compared. Also, the integral square error is computed and compared with that calculated for the second order models obtained by other methods.

The flowchart for the proposed methodology is as shown in Figure 6.6 and the proposed procedure is illustrated with numerical examples.
Read the coefficients of the numerator and denominator from,
\[ G(z) = \frac{(A_{n-1}z^{n-1} + \ldots + A_1z + A_0)}{(a_{n-1}z^n + a_{n-2}z^{n-1} + \ldots + a_1z + a_0)} \]
which represents the transfer function of the given LTIDS.

Calculate transient gain (TG) and steady state gain (SSG) for given \( G(z) \).

Apply the auxiliary polynomial approach to \( G(z) \) and obtain the initial lower order model \( R(z) \), scale it and tune it to maintain the transient and steady state gain of \( G(z) \).

Invoke Particle Swarm Optimization by passing the numerator and denominator coefficients of \( R(z) \).

For each generation

For each particle

If integral square error ‘E’ < best ‘E’ (pbest) so far

Yes

current value = new pbest

Choose the particle with the best integral square error of all the particles as the gbest.

Calculate particle velocity

Calculate particle positions

Update memory of each particle

End

End

Figure 6.6 (Continued)
Illustration 6.4

Step 1: Consider an eighth order linear time invariant discrete system (Prasad 1993) represented in the form of transfer function as,

\[
G(z) = \frac{1.682z^7 + 1.116z^6 - 0.21z^5 + 0.152z^4 - 0.516z^3}{8z^8 - 5.046z^7 - 3.348z^6 + 0.63z^5 - 0.456z^4 + 1.548z^3 + 0.786z^2 - 0.132z + 0.018}
\]  

(6.56)

Step 2: Calculating the transient gain (TG) and the steady state gain (SSG) of \(G(z)\) in equation (6.56),

\[
TG = \frac{1.682}{8} = 0.21025
\]  

(6.57)

\[
SSG = G(z)|_{z=1} = 0.994
\]

Step 3: Applying auxiliary polynomial approach to \(G(z)\) in equation (6.56), to obtain basic second order model \(R(z)\),
\[ R(z) = \frac{0.044z - 0.018}{0.786z^2 - 0.132z + 0.018} \quad (6.58) \]

Step 4: Scaling equation (6.58) and maintaining the transient gain calculated in equation (6.57), \( R(z) \) becomes,
\[ R(z) = \frac{0.21025z - 0.4091}{z^2 - 0.1679z + 0.0229} \quad (6.59) \]

Step 5: Invoking PSO algorithm in section 6.4.1 to search for a better second order model from equation (6.59). The value of constant term in numerator (0.4091), the value of \( z \)-term (0.1679) and constant term (0.0229) in the denominator of \( R(z) \) in equation (6.59) are used as seed values to PSO and are searched for their better values to minimize the integral square error by maintaining the steady state gain of \( R(z) \) in parallel with that of \( G(z) \) given in equation (6.57).

The values of various parameters used during the PSO algorithm process is:
- Number of generations = 150
- Number of particles = 1000
- Learning factors \( c_1 \) and \( c_2 \) = 0.5

Step 6: The transfer function \( R(z) \) corresponding to minimum integral square error obtained using PSO is,
\[ R^2(z) = \frac{0.21025z - 0.1486}{z^2 - 1.789z + 0.8507} \quad (6.60) \]

Step 7: The unit step responses of the given higher order system in equation (6.56), the proposed second order system in equation (6.60) and that of the second order models obtained using other known methods are shown in Figure 6.7. The corresponding integral square errors computed are shown in Table 6.6. The final time for simulation is taken as 60 seconds with a sampling interval of 1 second.
Table 6.6  Comparison of integral square error for Illustration 6.4

<table>
<thead>
<tr>
<th>Lower order model formulation method</th>
<th>Lower second order model</th>
<th>Integral square error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prasad (1993) method</td>
<td>$\frac{0.08516z - 0.03123}{z^2 - 1.7303z + 0.7842}$</td>
<td>2.5700</td>
</tr>
<tr>
<td>Sastry and Srinivasa Reddy (1995a) method</td>
<td>$\frac{0.3167z - 0.2773}{z^2 - 1.6832z + 0.7228}$</td>
<td>0.7591</td>
</tr>
<tr>
<td>Proposed method</td>
<td>$\frac{0.21025z - 0.1486}{z^2 - 1.7890z + 0.8507}$</td>
<td>0.1872</td>
</tr>
</tbody>
</table>

From Table 6.6, it is observed that the proposed methodology yields a lower value for integral square error compared to other methods. In Figure 6.7, it is noted that the response of the proposed second order model closely matches with that of the given higher order system. The response curves of second order models obtained using other methods are also shown for comparison. Appendix 8(a) provides a sample program for formulating lower order models of SISO LTIDS given the higher order models.
Illustration 6.5

1. Consider a fourth order linear time invariant discrete system (Prasad 1993) in z-domain, represented by its transfer function as,

\[
G(z) = \frac{0.3124z^3 - 0.5743z^2 + 0.3879z - 0.0889}{z^4 - 3.233z^3 + 3.9869z^2 - 2.2209z + 0.4723}
\]  

(6.61)

2. The transient gain (TG) and steady state gain (SSG) of G(z) in equation (6.61) are calculated as,

\[
TG = \frac{0.3124}{1} = 0.3124
\]

(6.62)

\[
SSG = G(z) \bigg|_{z=1} = 7.0000
\]

3. Using auxiliary polynomial scheme, the basic second order model R(z) is given by,

\[
R(z) = \frac{0.3879z - 0.0889}{3.9869z^2 - 2.2209z + 0.4723}
\]  

(6.63)

4. Scaling equation (6.63) and maintaining the transient gain (0.3124) of G(z), equation (6.63) becomes,

\[
R(z) = \frac{0.3124z - 0.22918278}{z^2 - 0.55704934z + 0.11846297}
\]  

(6.64)

5. Invoking PSO to search for the better second order approximant using the terms (0.22918278), (0.55704934) and (0.11846297) of initial second order model R(z) in equation (6.64) as its seed values. These values are fine tuned with the constraint of minimizing the integral square error and maintaining the steady state gain of R(z) in par with that of G(z) given in equation (6.62).

6. The transfer function corresponding to minimum integral square error using PSO is,
\[ R^2(z) = \frac{0.3124z - 0.0501}{z^2 - 1.7650z + 0.8032} \] (6.65)

7. The unit step responses of the given higher order system, the proposed second order system and that of the second order system obtained using other methods are as shown in Figure 6.8. The integral square errors computed correspondingly are tabulated in Table 6.7.

**Figure 6.8** Unit step responses for Illustration 6.5

<table>
<thead>
<tr>
<th>Lower order model formulation method</th>
<th>Lower second order model</th>
<th>Integral square error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prasad (1993) method</td>
<td>( \frac{0.27664z - 0.10767}{z^2 - 1.778663z + 0.802801} )</td>
<td>26.6756</td>
</tr>
<tr>
<td>Warwick (1984) method</td>
<td>( \frac{0.3124z - 0.0298}{z^2 - 1.7369z + 0.7773} )</td>
<td>1.4733</td>
</tr>
<tr>
<td>Proposed method</td>
<td>( \frac{0.3124z - 0.0501}{z^2 - 1.7650z + 0.8032} )</td>
<td>1.3666</td>
</tr>
</tbody>
</table>
From Table 6.7, it is observed that the proposed procedure yields better value of integral square error compared to other modeling techniques. The unit step response in Figure 6.8, shows that the characteristics of proposed second order model is in close agreement with that of the given higher order system.

### 6.6.2 Multi Input Multi Output Linear Time Invariant Discrete Systems

The PSO approach for formulating lower order approximant for SISO discrete system in the previous section has been extended for MIMO discrete system in this section. Generally, MIMO systems use measurements of several output variables and involve more than one input variable. The analysis and synthesis of a higher order multivariable system is difficult and possibly not desirable on economic and computational considerations. Due to which, it is necessary to obtain a lower order model so that the obtained lower order models maintain the characteristics of the original system. This helps in minimizing the variations during design and realization of suitable control system components to be attached to the original system.

The proposed methodology for lower order model formulation of MIMO LTIDS is as follows:

Step 1: Consider the given transfer function matrix \( G(z) \) of the higher discrete multivariable systems as,

\[
G(z) = \begin{bmatrix}
G_{11}(z) & G_{12}(z) & \cdots & G_{1r}(z) \\
G_{21}(z) & G_{22}(z) & \cdots & G_{2r}(z) \\
\vdots & \vdots & \ddots & \vdots \\
G_{q1}(z) & G_{q2}(z) & \cdots & G_{qr}(z)
\end{bmatrix}
\]  

(6.66)
where, \( G(z) = \frac{N_{ij}(z)}{D_{ij}(z)} = \frac{\sum_{k=0}^{n-l} A_k(i, j)z^k}{\sum_{k=0}^{n} a_k(i, j)z^k} \) with q inputs and r outputs.

i.e., \( i = 1, 2, \ldots, q \) and \( j = 1, 2, \ldots, r \).

Take common denominator \( D(z) \) for the given \( G(z) \), then the transfer function of the system can be represented in the matrix form as:

\[
G_{ij}(z) = \frac{\begin{bmatrix}
N_{j1}(z) & N_{j2}(z) & \cdots & N_{jr}(z) \\
N_{21}(z) & N_{22}(z) & \cdots & N_{2r}(z) \\
\vdots & \vdots & \ddots & \vdots \\
N_{q1}(z) & N_{q2}(z) & \cdots & N_{qr}(z)
\end{bmatrix}}{D(z)}
\]

where,

\[
N_{ij}(z) = \sum_{k=0}^{n-l} A_k(i, j)z^k
\]

and

\[
D(z) = a_nz^n + a_{n-1}z^{n-1} + \cdots + a_2z^2 + a_1z + a_0
\]

Step 2: For each \( G_{ij}(z) \) do the following

Step 2.1: From equation (6.67), get,

\[
G_{ij}(z) = \frac{N_{ij}(z)}{D(z)} = \frac{A_0(i, j) + A_1(i, j)z + A_2(i, j)z^2 + \cdots + A_{n-l}(i, j)z^{n-l}}{a_0 + a_1z + a_2z^2 + \cdots + a_{n-l}z^{n-l} + a_nz^n}
\]

(6.68)

Step 2.2: Determine the transient gain (\( TG_{ij} \)) and steady state gain (\( SSG_{ij} \)) for equation (6.68) as follows:

\[
TG_{ij} = \frac{A_{n-l}}{a_n}
\]

(6.69)

\[
SSG_{ij} = G(z)|_{z=1}
\]

Step 2.3: Applying the proposed PSO approach from step 3 to step 6 of section 6.6.1 to equation (6.68), compute a lower second order
model corresponding to the minimum integral square error of each \( G_{ij}(z) \) with \( N_{ij}(z)/D(z) \) as,

\[
R_{ij}(z) = \frac{N_{ij}^2(z)}{D_{ij}^2(z)} = \frac{B_i(i,j)z + B_0(i,j)}{b_z(i,j)z^2 + b_1(i,j)z + b_0(i,j)}
\]  \hspace{1cm} (6.70)

where \( i = 1,2,\ldots,q \) and \( j = 1,2,\ldots,r \) with \( B_i(i,j), B_0(i,j), b_2(i,j), b_1(i,j) \) and \( b_0(i,j) \) being constants.

Step 3: Determine the common denominator \( D^2(z) \) as the average of the corresponding coefficients of each \( D_{ij}^2(z) \) from equation (6.70), which is mathematically expressed as,

\[
D^2(z) = b_2z^2 + b_1z + b_0
\]  \hspace{1cm} (6.71)

where,

\[
b_k = \frac{\sum_{i=1}^{q} \sum_{j=1}^{r} b_k(i,j)}{\text{Total number of } b_k(i,j)}
\]  \hspace{1cm} (6.72)

and \( k = 0,1,2 \) with reference to equation (6.71).

Step 4: To maintain the steady state gain of the second order model to be same as that of higher order systems calculated in equation (6.69), the constant term of the numerator polynomial of lower second order model in equation (6.70) is adjusted along with the commonized denominator in equation (6.71) and the final second order approximant is obtained.

Step 5: The transfer function matrix \( R(z) \) of the lower second order model can now be represented as,

\[
R^2(z) = \begin{bmatrix}
R_{11}(z) & R_{12}(z) & \ldots & R_{1r}(z) \\
R_{21}(z) & R_{22}(z) & \ldots & R_{2r}(z) \\
\vdots & \vdots & \ddots & \vdots \\
R_{q1}(z) & R_{q2}(z) & \ldots & R_{qr}(z)
\end{bmatrix}
\]  \hspace{1cm} (6.73)
Step 6: To ensure the effectiveness of the proposed methodology, the unit step response of the given higher order system in equation (6.67) and the response of proposed second order model in equation (6.74) is compared. Further, the integral square error is computed and compared with that calculated for the second order models obtained by other methods.

The proposed methodology is applied to the following numerical example.

**Illustration 6.6**

Step 1: Consider a sixth order reachable and observable system (Bistritz and Shaked 1984) described in z-domain by the transfer function matrix as,

\[
G(z) = \begin{bmatrix}
N_{11}(z) & N_{12}(z) & \ldots & N_{1r}(z) \\
N_{21}(z) & N_{22}(z) & \ldots & N_{2r}(z) \\
\vdots & \vdots & \ddots & \vdots \\
N_{q1}(z) & N_{q2}(z) & \ldots & N_{qr}(z)
\end{bmatrix}
= \frac{D^2(z)}{D(z)}
\]

(6.74)

The common denominator \(D(z)\) of given sixth order system represented by \(G(z)\) is,

\[
D(z) = (z - 0.95)(z - 0.5)(z - 0.9)(z - 0.75)(z - 0.3)
\]

(6.76)

\[
D(z) = (z - 0.95)(z - 0.5)(z - 0.9)(z - 0.75)(z - 0.3)
= z^6 - 3.75z^5 + 5.6625z^4 - 4.390625z^3 + 1.8382125z^2
- 0.39346875z + 0.033665625
\]
Step 2: For each G(z) do the following,

Step 2.1: G(z) can be represented as,

\[
G(z) = \begin{bmatrix} G_{11}(z) & G_{12}(z) \\ G_{21}(z) & G_{22}(z) \end{bmatrix}
\] (6.77)

where,

\[
G_{11}(z) = \frac{2.25z^5 - 6.8625z^4 + 8.0494z^3 - 4.5031z^2 + 1.1922z - 0.1196}{D(z)}
\] (6.78)

\[
G_{12}(z) = \frac{1.5z^5 - 4.35z^4 + 4.80375z^3 - 2.5185z^2 + 0.62801z - 0.05985}{D(z)}
\] (6.79)

\[
G_{21}(z) = \frac{1.04z^5 - 3.276z^4 + 4.0326z^3 - 2.4207z^2 + 0.70792z - 0.0808625}{D(z)}
\] (6.80)

\[
G_{22}(z) = \frac{z^5 - 3.2z^4 + 3.9725z^3 - 2.38075z^2 + 0.684375z - 0.0748125}{D(z)}
\] (6.81)

with D(z) as in equation (6.76).

Step 2.2: The transient gain and steady state gain computed for G_{11}(z), G_{12}(z), G_{21}(z) and G_{22}(z) represented in equations (6.78) through (6.81) are shown in Table 6.8.

<table>
<thead>
<tr>
<th>Table 6.8</th>
<th>Transient gain and Steady state gain for G_{11}(z), G_{12}(z), G_{21}(z) and G_{22}(z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Higher order system</td>
<td>Transient Gain</td>
</tr>
<tr>
<td>G_{11}(z)</td>
<td>( \frac{2.25}{l} = 2.25 )</td>
</tr>
<tr>
<td>G_{12}(z)</td>
<td>( \frac{1.5}{l} = 1.5 )</td>
</tr>
<tr>
<td>G_{21}(z)</td>
<td>( \frac{1.04}{l} = 1.04 )</td>
</tr>
<tr>
<td>G_{22}(z)</td>
<td>( \frac{l}{l} = 1 )</td>
</tr>
</tbody>
</table>
Step 2.3: Using the proposed PSO approach for $G_{11}(z)$, $G_{12}(z)$, $G_{21}(z)$ and $G_{22}(z)$ represented in equations (6.78) through (6.81), the transfer functions of the respective second order models $R_{11}(z)$, $R_{12}(z)$, $R_{21}(z)$ and $R_{22}(z)$ are obtained. The trace of the auxiliary polynomial and PSO algorithm is shown in Table 6.9.

<table>
<thead>
<tr>
<th>System</th>
<th>Transient gain adjusted initial second order models from auxiliary polynomial approach for input to PSO</th>
<th>Second order approximant from PSO</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_{11}(z)$</td>
<td>$\frac{2.25z - 0.1003}{z^2 - 0.2140z + 0.0183}$</td>
<td>$\frac{2.25z + 0.1738}{z^2 - 0.06023z - 0.8288}$</td>
</tr>
<tr>
<td>$R_{12}(z)$</td>
<td>$\frac{1.5z - 0.0953}{z^2 - 0.2140z + 0.0183}$</td>
<td>$\frac{1.5z + 1.133}{z^2 - 0.02448z - 0.7549}$</td>
</tr>
<tr>
<td>$R_{21}(z)$</td>
<td>$\frac{1.04z - 0.1142}{z^2 - 0.2140z + 0.0183}$</td>
<td>$\frac{1.04z + 0.1643}{z^2 - 0.0488z - 0.9267}$</td>
</tr>
<tr>
<td>$R_{22}(z)$</td>
<td>$\frac{z - 0.1093}{z^2 - 0.2140z + 0.0183}$</td>
<td>$\frac{z - 0.3318}{z^2 - 0.6534z - 0.2015}$</td>
</tr>
</tbody>
</table>

Step 3: Using the denominators of the second order approximants obtained from PSO approach of $R_{11}(z)$, $R_{12}(z)$, $R_{21}(z)$ and $R_{22}(z)$ in Table 6.9, the common denominator for the second order MIMO model is obtained by evaluating the average of corresponding coefficients, which is given by,

$$D^2(z) = \frac{1 + 1 + 1 + 1}{4} z^2 = \left(\frac{0.06023 + 0.02448 + 0.0488 + 0.6534}{4}\right) z^2 - \left(\frac{0.8288 + 0.7549 + 0.9267 + 0.2015}{4}\right)$$

$$D^2(z) = z^2 - 0.1967z - 0.6780 \quad (6.82)$$
Step 4: With the commonized denominator in equation (6.82), the second order approximants from PSO approach in Table 6.9 becomes,

\[
R_{11}(z) = \frac{2.25z + 0.1738}{z^2 - 0.1967z - 0.6780}
\]  
(6.83)

\[
R_{12}(z) = \frac{1.5z + 1.133}{z^2 - 0.1967z - 0.6780}
\]  
(6.84)

\[
R_{21}(z) = \frac{1.04z + 0.1643}{z^2 - 0.1967z - 0.6780}
\]  
(6.85)

\[
R_{22}(z) = \frac{z - 0.3318}{z^2 - 0.1967z - 0.6780}
\]  
(6.86)

The constant terms of the numerator polynomial of \(R_{11}(z)\), \(R_{12}(z)\), \(R_{21}(z)\) and \(R_{22}(z)\) in equation (6.83) through (6.86) are adjusted to maintain the same steady state gain of the given higher order system calculated in Table 6.8. This process gives the final second order approximant of the given MIMO higher order system as,

\[
R_{11}(z) = \frac{2.25z + 0.56925}{z^2 - 0.1967z - 0.6780}
\]  
(6.87)

\[
R_{12}(z) = \frac{1.5z + 0.036}{z^2 - 0.1967z - 0.6780}
\]  
(6.88)

\[
R_{21}(z) = \frac{1.04z + 0.2631}{z^2 - 0.1967z - 0.6780}
\]  
(6.89)

\[
R_{22}(z) = \frac{z - 0.4217}{z^2 - 0.1967z - 0.6780}
\]  
(6.90)

Step 5: From equations (6.87) through (6.90), the transfer function matrix of the second order MIMO model is,

\[
R^2(z) = \begin{bmatrix}
R_{11}(z) & R_{12}(z) \\
R_{21}(z) & R_{22}(z)
\end{bmatrix}
\]

\[
= \frac{1}{D^2(z)} \begin{bmatrix}
2.25z + 0.56925 & 1.5z + 0.036 \\
1.04z + 0.2631 & z - 0.4217
\end{bmatrix}
\]  
(6.91)

where, \(D^2(z) = z^2 - 0.1967z - 0.6780\).
Step 6: The unit step responses of the given higher order system in equation (6.78) through (6.81) and that of the proposed second order systems in equations (6.87) through (6.90) are as shown in Figures 6.9(a) to 6.9(d). Table 6.10 shows the integral square error computed for the proposed second order model and that calculated for the second order models obtained using other methods.

Table 6.10  Comparison of Integral square error for Illustration 6.6

<table>
<thead>
<tr>
<th>Lower order model formulation methods</th>
<th>Integral square error (E) for 80 seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$G_{11}(z)$</td>
</tr>
<tr>
<td>Prasad and Devi (2001) method</td>
<td>1247.546</td>
</tr>
<tr>
<td>Bistritz and Shaked (1984) method</td>
<td>216.1736</td>
</tr>
<tr>
<td>Proposed method</td>
<td><strong>150.0432</strong></td>
</tr>
</tbody>
</table>

From Table 6.10, it can be observed that the integral square error obtained using the proposed method is less than that of Bistritz and Prasad methods. Also, Figures 6.9(a) to 6.9(d) shows that the response of the proposed second order models closely matches with that of the given higher order MIMO models. The program for obtaining lower order models of MIMO LTIDS is as given in Appendix 8(b).
Figure 6.9(a) Unit step response for $G_{11}(z)$ - Illustration 6.6

Figure 6.9(b) Unit step response for $G_{12}(z)$ - Illustration 6.6
Figure 6.9(c) Unit step response for $G_{21}(z)$ - Illustration 6.6

Figure 6.9(d) Unit step response for $G_{22}(z)$ - Illustration 6.6
6.7 LOWER ORDER MODEL FORMULATION FOR LINEAR TIME INVARIANT DISCRETE SYSTEMS WITH TRANSFORMATION

In this section, the proposed PSO approach for the formulation of lower order approximant for linear time invariant continuous systems has been extended to linear time invariant discrete SISO and MIMO systems using transformation. A linear transformation (LT) is applied to the given higher order discrete system and converted into an equivalent continuous system of the same order. The proposed approach is used to form a lower order approximant in the continuous domain. The lower order approximant is transformed back to the discrete domain using the reverse transformation.

6.7.1 Single Input Single Output Linear Time Invariant Discrete Systems

The proposed methodology for lower order model formulation of SISO LTIDS using transformation is as follows:

Step 1: Consider an n-th order linear time invariant discrete system in the discrete domain (z-domain) represented by its transfer function as in equation (6.50).

Step 2: Applying linear transformation (Shamash 1974) $z = p+1$ in equation (6.50), the transfer function of an equivalent continuous system in the continuous domain (p-domain) is,

$$G(p) = \frac{N(p)}{D(p)} = \frac{A_{n-1}(p+1)^{n-1} + A_{n-2}(p+1)^{n-2} + ... + A_2(p+1)^2 + A_1(p+1) + A_0}{a_n(p+1)^n + a_{n-1}(p+1)^{n-1} + ... + a_2(p+1)^2 + a_1(p+1) + a_0}$$

(6.92)
Equation (6.92) can be further simplified as,

\[
G(p) = \frac{\alpha_{n-1}p^n + \alpha_{n-2}p^{n-1} + \ldots + \alpha_2 p^2 + \alpha_1 p + \alpha_0}{\beta_n p^n + \beta_{n-1}p^{n-1} + \ldots + \beta_2 p^2 + \beta_1 p + \beta_0}
\]  

(6.93)

Step 3: Applying the proposed PSO approach from step 2 to step 7 of section 6.5.1, a second order approximant for the system represented by equation (6.93) is obtained as,

\[
R^2(p) = \frac{\gamma_1 p + \gamma_0}{\mu_2 p^2 + \mu_1 p + \mu_0}
\]  

(6.94)

Step 4: Using the reverse transformation \( p = z - 1 \) in equation (6.94), the desired lower second order approximant of \( G(z) \) in \( z \)-domain is obtained as,

\[
R^2(z) = \frac{\gamma_1(z - 1) + \gamma_0}{\mu_2(z - 1)^2 + \mu_1(z - 1) + \mu_0}
\]  

(6.95)

This can be further simplified as,

\[
R^2(z) = \frac{B_j z + B_0}{b_2 z^2 + b_1 z + b_0}
\]  

(6.96)

Step 5: To ensure the effectiveness of the proposed procedure, the unit step response of the given higher order system represented by equation (6.50) is compared with that of the lower second order approximant represented by equation (6.96) for a suitable sampling time. The integral square error is calculated over a specified period of time and is compared with that computed for the lower second order models obtained by other methods.

The proposed methodology is illustrated with numerical examples and the flowchart depicting the proposed procedure is as shown in Figure 6.10.
Figure 6.10 Flowchart for model formulation of single input single output linear time invariant discrete systems using transformation

Start

Read the coefficients of numerator and denominator from,
\[ G(z) = \frac{N(z)}{D(z)} = \frac{A_{n-1}z^{n-1} + A_{n-2}z^{n-2} + \ldots + A_1z + A_0}{a_nz^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \ldots + a_1z + a_0} \]
representing the transfer function of the given higher order LTIDS

Applying linear transformation \( z = p+1 \), the transfer function in p-domain is,
\[ G(p) = \frac{\alpha_{n-1}p^{n-1} + \alpha_{n-2}p^{n-2} + \ldots + \alpha_2p^2 + \alpha_1p + \alpha_0}{\beta_n p^n + \beta_{n-1}p^{n-1} + \ldots + \beta_2p^2 + \beta_1p + \beta_0} \]

Perform the operations given in Figure 6.2 to obtain the lower order model in p-domain

The computed lower second order approximant using the proposed approach in p-domain is,
\[ R^2(p) = \frac{\gamma_1p + \gamma_0}{\mu_2p^2 + \mu_1p + \mu_0} \]

Using reverse linear transformation \( p = z-1 \), the second order approximant in z-domain is,
\[ R^2(z) = \frac{B_z + B_0}{b_2z^2 + b_1z + b_0} \]

Compute Integral square error (E) for proposed second order model with that of given higher order system and compare it with second order models obtained by other methods.

Stop
Illustration 6.7

Step 1: Consider an 8-th order linear time invariant discrete system (Prasad 1993) represented in the z-domain by its transfer function as in equation (6.56) of Illustration 6.4.

Step 2: Substituting $z = p+1$ in equation (6.56), the transfer function of the equivalent linear time invariant continuous system in the $p$-domain is,

$$G(p) = \frac{1.682 p^7 + 12.89 p^6 + 41.808 p^5 + 74.712 p^4 + 79.182 p^3 + 49.064 p^2 + 16 p + 1.988}{8 p^8 + 58.954 p^7 + 185.33 p^6 + 322.576 p^5 + 335.864 p^4 + 210.454 p^3 + 76.808 p^2 + 16 p + 2}$$  \hspace{1cm} (6.97)

Step 3: Applying the proposed PSO approach, the transfer function of a second order approximant is obtained as,

$$R^2(\,p\,) = \frac{0.2103 p + 0.07876}{p^2 + 0.2286 p + 0.0792}$$  \hspace{1cm} (6.98)

Step 4: Substituting $p = z-1$ in equation (6.98), the corresponding second order approximant in the $z$-domain is computed as,

$$R^2(\,z\,) = \frac{0.2103 z - 0.13149}{z^2 - 1.7714 z + 0.8506}$$  \hspace{1cm} (6.99)

Step 5: The unit step response of the given higher order LTIDS represented by the equation (6.56) and that of the proposed discrete second order approximant represented by equation (6.99) are shown in Figure 6.11. The unit step response of equivalent second order models obtained by other methodologies is also shown for comparison. The integral square error of the second order approximant obtained by the proposed method and other available methods are tabulated in Table 6.11.
Table 6.11 Comparison of integral square error of unit step time response for Illustration 6.7

<table>
<thead>
<tr>
<th>Lower order model formulation method</th>
<th>Lower second order model</th>
<th>Integral square error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prasad (1993) method</td>
<td>$\frac{0.08516z - 0.03123}{z^2 - 1.7303z + 0.7842}$</td>
<td>2.5700</td>
</tr>
<tr>
<td>Sastry and Srinivasa Reddy (1995a) method</td>
<td>$\frac{0.3167z - 0.2773}{z^2 - 1.6832z + 0.7228}$</td>
<td>0.7591</td>
</tr>
<tr>
<td>Proposed method</td>
<td>$\frac{0.2103z - 0.13149}{z^2 - 1.7714z + 0.8506}$</td>
<td>0.0181</td>
</tr>
</tbody>
</table>

Figure 6.11 Unit step response for Illustration 6.7

The unit step response of the proposed second order system is observed to be in match with that of the original higher order system as shown in Figure 6.11. Table 6.11 shows that the integral square error obtained using proposed approach is less when compared to the other methods. The sample program for lower order model formulation of SISO LTIDS with transformation is given in Appendix 9(a).
Illustration 6.8

Step 1: Consider a 4-th order linear time invariant discrete system (Prasad 1993) represented in the $z$-domain by its transfer function as in equation (6.61) of Illustration 6.5.

Step 2: Applying linear transformation $z = p + 1$ to equation (6.61), the transfer function of the equivalent linear time invariant continuous system in the $p$-domain is,

$$G(p) = \frac{0.3124 p^3 + 0.3629 p^2 + 0.1765 p + 0.0371}{p^4 + 0.767 p^3 + 0.2879 p^2 + 0.0539 p + 0.0053} \quad (6.100)$$

Step 3: Using the proposed PSO approach, the transfer function of a second order approximant in $p$-domain for equation (6.100) is obtained as,

$$R^2(p) = \frac{0.3124 p + 0.2622}{p^2 + 0.2341 p + 0.0375} \quad (6.101)$$

Step 4: Substituting reverse transformation $p = z - 1$ in equation (6.101), the corresponding second order approximant in the $z$-domain is computed as,

$$R^2(z) = \frac{0.3124 z - 0.0502}{z^2 - 1.7659 z + 0.8034} \quad (6.102)$$

Step 5: The unit step response of the given higher order linear time invariant discrete system represented by the equation (6.61) and that of the proposed discrete second order approximant represented by equation (6.102) are shown in Figure 6.12. The unit step responses of equivalent second order models obtained by other methodologies are also shown for comparison. The integral square error comparison is shown in Table 6.12.
Table 6.12 Comparison of integral square error for Illustration 6.8

<table>
<thead>
<tr>
<th>Lower order model formulation method</th>
<th>Lower second order model</th>
<th>Integral square error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Prasad (1993) method</td>
<td>( \frac{0.27664z - 0.10767}{z^2 - 1.778663z + 0.802801} )</td>
<td>26.6756</td>
</tr>
<tr>
<td>Warwick (1984) method</td>
<td>( \frac{0.3124z - 0.0298}{z^2 - 1.7369z + 0.7773} )</td>
<td>1.4733</td>
</tr>
<tr>
<td><strong>Proposed method</strong></td>
<td>( \frac{0.3124z - 0.0502}{z^2 - 1.7659z + 0.8034} )</td>
<td><strong>0.9604</strong></td>
</tr>
</tbody>
</table>

The unit step response of the proposed second order system is observed to be in close match with that of the original higher order system as shown in Figure 6.12. Table 6.12 indicates that the integral square error is less when compared to other methods.
6.7.2 Multi Input Multi Output Linear Time Invariant Discrete Systems

The proposed methodology for lower order model formulation of MIMO linear time invariant discrete system using linear transformation is as follows:

Step 1: Consider the transfer function matrix of the given higher order discrete multivariable system as in equation (6.66). Taking the common denominator of given $G(z)$ the equation (6.67) is obtained.

Step 2: From equation (6.67), the individual transfer functions of given MIMO higher order system is obtained as shown in equation (6.68).

Step 3: Applying linear transformation $z = p+1$ in equation (6.68), the transfer function of an equivalent continuous system is given by,

$$
G_{ij}(p) = \frac{\begin{bmatrix}
N_{i1}(p) & N_{i2}(p) & \ldots & N_{ir}(p) \\
N_{21}(p) & N_{22}(p) & \ldots & N_{2r}(p) \\
\vdots & \vdots & \ddots & \vdots \\
N_{q1}(p) & N_{q2}(p) & \ldots & N_{qr}(p) \\
\end{bmatrix}}{D(p)}
$$

(6.103)

where,

$$
N_{ij}(p) = \sum_{k=0}^{n-1} A_k(i, j) p^k
$$

with $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, r$,

$$
D(p) = a_n p^n + a_{n-1} p^{n-1} + \ldots + a_2 p^2 + a_1 p + a_0
$$

Step 4: For each $G_{ij}(p)$, do step 2 to step 4 of section 6.5.2 and obtain the transfer function matrix of the lower second order system as,
Step 5: Applying reverse linear transformation \( p = z^{-1} \) to equation (6.104), the transfer function of the lower second order model in \( z \)-domain as given in equation (6.74) is obtained.

Step 6: The unit step response of the given higher order system and that of the proposed second order model are obtained and compared to check the effectiveness of the proposed procedure. Also, the integral square error is computed and compared with that calculated for the second order models obtained by other methods.

The proposed methodology is applied to the following numerical Illustration.

**Illustration 6.9**

1. Consider the \( z \)-transfer function matrix of a reachable and observable system (Bistritz and Shaked 1984) described by its transfer function matrix as given in equation (6.75) of Illustration 6.6.

2. The common denominator \( D(z) \) of sixth order system is obtained as given in equation (6.76).

3. The individual transfer functions of the given MIMO higher order system is obtained as shown in equations (6.78) to (6.81) using the common denominator from equation (6.76).
4. Using the linear transformation, \( z = p^1 \) the equations (6.78) through (6.81) can be transformed as,

\[
G_{11}(p) = \frac{2.25 p^5 + 4.3875 p^4 + 3.099375 p^3 + 0.97003125 p^2 + 0.13415625 p + 0.0063984375}{D(p)} \tag{6.105}
\]

\[
G_{12}(p) = \frac{1.5 p^5 + 3.15 p^4 + 2.40375 p^3 + 0.79275 p^2 + 0.1022625 p + 0.0034125}{D(p)} \tag{6.106}
\]

\[
G_{21}(p) = \frac{1.04 p^5 + 1.924 p^4 + 1.3286 p^3 + 0.42211 p^2 + 0.06032 p + 0.0029575}{D(p)} \tag{6.107}
\]

\[
G_{22}(p) = \frac{p^5 + 1.8 p^4 + 1.1725 p^3 + 0.33675 p^2 + 0.040375 p}{D(p)} + 0.0013125 \tag{6.108}
\]

where,

\[
D(p) = p^6 + 2.25 p^5 + 1.9125 p^4 + 0.759375 p^3 + 0.1413375 p^2 + 0.01108125 p + 0.000284375 \tag{6.109}
\]

5. The proposed auxiliary polynomial and PSO approach is applied for \( G_{11}(p) \), \( G_{12}(p) \), \( G_{21}(p) \) and \( G_{22}(p) \) represented in equations (6.105) to (6.108), and the transfer functions of the corresponding second order approximants \( R_{11}(p) \), \( R_{12}(p) \), \( R_{21}(p) \) and \( R_{22}(p) \) obtained are given below:

\[
R_{11}(p) = \frac{2.25 p + 0.7498}{p^2 + 0.6951 p + 0.0333} \tag{6.110}
\]

\[
R_{12}(p) = \frac{1.5 p + 0.41108}{p^2 + 0.4332 p + 0.03426} \tag{6.111}
\]

\[
R_{21}(p) = \frac{1.04 p + 0.3655}{p^2 + 0.7545 p + 0.0351} \tag{6.112}
\]

\[
R_{22}(p) = \frac{p + 0.3036}{p^2 + 0.7627 p + 0.0658} \tag{6.113}
\]
6. From the denominators of $R_{11}(p)$, $R_{12}(p)$, $R_{21}(p)$ and $R_{22}(p)$ represented in equations (6.110) to (6.113), the common denominator of the second order MIMO model is obtained by computing the average of the corresponding coefficients, which is given by:

$$D^2(p) = p^2 + 0.6614p + 0.0421 \quad (6.114)$$

7. Using the transient gain and steady state gain of $G_{11}(p)$, $G_{12}(p)$, $G_{21}(p)$ and $G_{22}(p)$ represented in equations (6.105) to (6.108) and $D^2(p)$, the transfer functions in equation (6.110) to (6.113) are reconstructed as:

$$R_{11}(p) = \frac{2.25p + 0.9473}{p^2 + 0.6614p + 0.0421} \quad (6.115)$$

$$R_{12}(p) = \frac{1.5p + 0.5052}{p^2 + 0.6614p + 0.0421} \quad (6.116)$$

$$R_{21}(p) = \frac{1.04p + 0.4378}{p^2 + 0.6614p + 0.0421} \quad (6.117)$$

$$R_{22}(p) = \frac{p + 0.1943}{p^2 + 0.6614p + 0.0421} \quad (6.118)$$

8. Applying the reverse transformation $p = z-1$, the equivalent $z$-transfer function of equations (6.115) to (6.118) are obtained as:

$$R_{11}(z) = \frac{2.25z - 1.3027}{z^2 - 1.3386z + 0.3807} \quad (6.119)$$

$$R_{12}(z) = \frac{1.5z - 0.9948}{z^2 - 1.3386z + 0.3807} \quad (6.120)$$

$$R_{21}(z) = \frac{1.04z - 0.6022}{z^2 - 1.3386z + 0.3807} \quad (6.121)$$

$$R_{22}(z) = \frac{z - 0.8057}{z^2 - 1.3386z + 0.3807} \quad (6.122)$$
9. The second order discrete MIMO model in transfer function matrix form is:

\[
R^2(z) = \begin{bmatrix}
R_{11}(z) & R_{12}(z) \\
R_{21}(z) & R_{22}(z)
\end{bmatrix}
\]

\[
= \frac{1}{D^2(z)} \begin{bmatrix}
2.25z - 1.3027 & 1.5z - 0.9948 \\
1.04z - 0.6022 & z - 0.8057
\end{bmatrix}
\]  

(6.123)

where, \(D^2(z) = z^2 - 1.3386z + 0.3807\).

10. The unit step response of the given higher order system in equation (6.75) and that of the proposed second order model given in equation (6.123) are shown in Figures 6.13 (a) to 6.13 (d). The integral square errors computed for the proposed second order model and that calculated for the second order models obtained by other methods are shown in Table 6.13.

### Table 6.13 Comparison of integral square error for Illustration 6.9

<table>
<thead>
<tr>
<th>Lower order model formulation methods</th>
<th>Integral Square error E for 80 seconds</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>G_{11}(z)</td>
</tr>
<tr>
<td>Prasad and Devi (2001) method</td>
<td>1247.5459</td>
</tr>
<tr>
<td>Bistritz and Shaked (1984) method</td>
<td>216.714</td>
</tr>
<tr>
<td>Proposed Method</td>
<td>163.9547</td>
</tr>
</tbody>
</table>

From Table 6.13, it is noted that the integral square error computed using the proposed methodology is less than that of Bistritz and Shaked (1984) and Prasad and Devi (2001) method. From Figures 6.13 (a) to 6.13 (d), it is inferred that the characteristics of the second order model almost matches with that of the given higher order system. Appendix 9(b) provides the program for obtaining lower order MIMO LTIDS models with transformation.
Figure 6.13(a) Unit step response for $G_{11}(z)$ - Illustration 6.9

Figure 6.13(b) Unit step response for $G_{12}(z)$ - Illustration 6.9
Figure 6.13(c) Unit step response for $G_{21}(z)$ - Illustration 6.9

Figure 6.13(d) Unit step response for $G_{22}(z)$ - Illustration 6.9
6.8 SUMMARY

An auxiliary polynomial and particle swarm optimization based approach has been developed for formulating a lower order approximant for a given higher order linear time invariant system in this Chapter. Unlike the other available algebraic and graphical model formulation techniques, the proposed approach iteratively evolves a lower order approximant that maintains the characteristics of the given higher order system. This approach formulates a stable lower order model, provided the given higher order system is absolutely stable. The proposed procedure is simple and straightforward in application and is applicable for both continuous and discrete linear time invariant systems. The approach has been applied to SISO and MIMO, continuous and discrete linear time invariant systems to formulate the lower order models. The validity of the proposed methodology is illustrated with higher order system taken from the literature. The effectiveness of the approach has been well established by comparing the results with other schemes. The sample program for comparison of various methods using step responses for LTIS is provided in Appendix 10.