CHAPTER 5

STABILITY ANALYSIS OF CERTAIN CLASS OF FUZZY SYSTEMS

5.1 INTRODUCTION

Recently, Fuzzy systems are being used successfully in many practical applications. Despite its success, it has been noted that many basic issues further remain to be addressed. Stability analysis and systematic design are certainly the most important issues for fuzzy systems. Stability analysis of fuzzy systems has been difficult because fuzzy systems are essentially nonlinear systems. Several practical and theoretical methods have been developed for evaluating the stability of fuzzy logic controllers. Braae and Rutherford (1979) proposed a linguistic phase plane trajectory to analyze and improve the stability by exchanging the control rules. Mamdani (1974) have used the describing function method to evaluate the stability of fuzzy control systems. Also, stability of fuzzy systems can be analyzed by applying non-linear stability analysis techniques. Many research works are available in the above aspect (Tanaka 1995, Kawamoto et al 1992, Thathachar and Pramod Viswanath 1997, Teixiera and Zak 1999, Pang and Guu 2003, Liu and Han 2006, Feng 2004, Wai and Chang 2006). Tanaka and Sano (1994) have given a sufficient condition for the asymptotic stability of a class of fuzzy systems employing Lyapunov’s function.

In this Chapter, an algebraic approach is proposed to analyze the stability nature of the given fuzzy system represented by its system matrices
in discrete domain using linear matrix inequality. Also, certain algorithms are proposed for investigating the nature of stability of the fuzzy systems represented in the form of fuzzy relational matrices. Further, using the linearized equations of an inverted pendulum, a fuzzy logic controller (FLC) is designed for stabilizing its motion and the proposed algorithms are applied over pendulum’s output relational matrix for analyzing stability nature of FLC based inverted pendulum. These algebraic procedures are simple in application and are illustrated through examples.

5.2 STABILITY ANALYSIS OF FUZZY SYSTEMS GIVEN BY SYSTEM MATRICES

In this section, an algebraic procedure along with certain sufficient conditions are proposed to analyze the stability nature of given fuzzy system represented by its system matrices in discrete domain using linear matrix inequality.

5.2.1 Fuzzy System Modeling

Takagi and Sugeno (1985) fuzzy modeling is simple and natural wherein the system dynamics are represented by a set of fuzzy implications which characterize local relations in the state space. The main feature of a Takagi-Sugeno model is to describe the local dynamics of each fuzzy rule by a linear system model. The overall fuzzy model of the system is achieved by fuzzy blending of these linear system models.

Takagi-Sugeno fuzzy system is described by fuzzy IF-THEN rules, which locally represent linear input-output relations of a system. Let the fuzzy system in discrete time domain is of the following form (Wang et al 1996):
Rule i: IF $X_1(k)$ is $M_{i1}$ ... and $X_n(k)$ is $M_{in}$

THEN $X(k+1) = A_i X(k) + B_i U(k)$, \quad $i=1,2,...,r$

where,

$$X^T(k) = [X_1(k), X_2(k), \ldots, X_n(k)]$$

$$U^T(k) = [U_1(k), U_2(k), \ldots, U_m(k)]$$

with,

r is the number of IF-THEN rules

$M_{ij}$ are fuzzy sets and

$X(k+1) = A_i X(k) + B_i U(k)$ is the output from the i-th IF-THEN rule.

Given a pair of $(X(k), U(k))$, the final output of the fuzzy system is as follows:

$$X(k+1) = \frac{\sum_{i=1}^{r} w_i(k) (A_i X(k) + B_i U(k))}{\sum_{i=1}^{r} w_i(k)}$$  \hspace{1cm} \text{(5.1)}$$

where,

$$w_i(k) = \prod_{j=1}^{n} M_{ij}(X_j(k))$$

$M_{ij}(X_j(k))$ is the grade of membership of $x_j(k)$ in $M_{ij}$.

The open loop system of equation (5.1) is given by,

$$X(k+1) = \frac{\sum_{i=1}^{r} w_i(k) A_i X(k)}{\sum_{i=1}^{r} w_i(k)}$$  \hspace{1cm} \text{(5.2)}$$

where it is assumed that,

$$\sum_{i=1}^{r} w_i(k) > 0 \text{ and } w_i(k) \geq 0, \quad i=1,2,...,r$$

for all k. Each linear component $A_i X(k)$ is called a subsystem. The subsystem matrix $A_i$ is analyzed for stability in the forthcoming section.
5.2.2 Proposed algebraic approach for stability analysis

The characteristic equation of system matrix $A_i$ is written as,

$$|zI-A_i| = 0$$  \hspace{1cm} (5.3)

where, $z$ is the discrete variable and

$I$ is the unit identity matrix.

Let,

$$|zI-A| = a_0z^n-a_1z^{n-1}+a_2z^{n-2}-a_3z^{n-3}+\ldots+a_{n-1} = 0$$  \hspace{1cm} (5.4)

The roots of equation (5.4) for stability will satisfy,

$$0 < |z| < 1$$  \hspace{1cm} (5.5)

since the membership values exist between 0 and 1. This means that all the roots are positive and lie within the sector region of right half side (RHS) of unit circle.

If every root of equation (5.4) is simple and distinct, aperiodic stability is inferred. This can be ascertained using Fuller’s table while the condition in equation (5.5) is tested employing Marden’s table for asymptotic stability.

Generally, if the roots of $|zI-A|=0$ lie in sector region of left half side (LHS) of unit circle (or) distributed within the unit circle then the fuzzy system response will be unstable.

Without loss of generality and for simplicity, consider a second order fuzzy system. The second order fuzzy system matrix $A_i$ may be written as,

$$A_i = \begin{bmatrix} a_i & b_i \\ I & 0 \end{bmatrix}$$  \hspace{1cm} (5.6)
The characteristic equation for $A_i$ in $z$-domain is,

$$F(z) = z^2 - a_i z + b_i = 0$$  \hspace{1cm} (5.7)

**Case (i) - To ascertain asymptotic stability**

To test for asymptotic stability the Marden’s table (Marden 1949) is formulated for the characteristic equation in equation (5.7) as shown in Table 5.1.

**Table 5.1 Marden’s table for equation (5.7)**

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_2 \left( \frac{1}{z} \right)$</td>
<td>$b_i$</td>
<td>$-a_i$</td>
<td>1</td>
</tr>
<tr>
<td>$F_2(z)$</td>
<td>1</td>
<td>$-a_i$</td>
<td>$b_i$</td>
</tr>
<tr>
<td>$F_1 \left( \frac{1}{z} \right)$</td>
<td>$(-a_i+a_ib_i)$</td>
<td>$(1-b_i^2)$</td>
<td></td>
</tr>
<tr>
<td>$F_1(z)$</td>
<td>$(1-b_i^2)$</td>
<td>$(-a_i+a_ib_i)$</td>
<td></td>
</tr>
<tr>
<td>$F_0 \left( \frac{1}{z} \right)$</td>
<td>$(1-b_i^2)^2 + \left( a_i-a_ib_i \right)^2$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Based on Table 5.1, the inferences are made for stability of the given fuzzy system represented by system matrix using Marden’s criterion. For asymptotic stability,

\begin{align*}
\text{(i)} & \quad (1-b_i^2) > 0 \quad \text{(5.8)} \\
\text{(ii)} & \quad [(1-b_i^2)^2 + (a_i-a_ib_i)^2] > 0 \quad \text{(5.9)}
\end{align*}

On simplifying equations (5.8) and (5.9), certain sufficient conditions for inferring asymptotic stability of given fuzzy system are,
The above conditions from equation (5.10) and equation (5.11) can be directly applied over the fuzzy system matrices and stability nature can be observed.

**Case (ii) - To test for aperiodic stability**

To ascertain aperiodic stability of the fuzzy system, Fuller’s table is formulated for the characteristic equation $F(-z)=0$.

The characteristic equation for $A_i$ in equation (5.7) with $z = -z$ is transformed as,

$$F(-z) = z^2 + a_i z + b_i = 0$$  \hspace{1cm} (5.12)

Fuller’s table (Jury 1974) is formulated for equation (5.12) as shown in Table 5.2.

**Table 5.2 Fuller’s table for equation (5.12)**

<table>
<thead>
<tr>
<th>Column/ Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F(-z) \rightarrow z^4$</td>
<td>1</td>
<td>$a_i$</td>
<td>$b_i$</td>
</tr>
<tr>
<td>$F'(-z) \rightarrow z^3$</td>
<td>2</td>
<td>$a_i$</td>
<td></td>
</tr>
<tr>
<td>$z^2$</td>
<td>$a_i$</td>
<td></td>
<td>$2b_i$</td>
</tr>
<tr>
<td>$z^1$</td>
<td>$(\frac{a_i^2 - 4b_i}{a_i})$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$z^0$</td>
<td>$2b_i$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
For the given fuzzy system to be aperiodically stable, the elements in the first column of Fuller’s table in Table 5.2 should have the same sign. Thus, it is observed that for the given fuzzy system represented by its system matrix in equation (5.6), certain sufficient conditions for aperiodic stability are,

i) \( a_i > 0 \)

ii) \( a_i^2 > 4b_i \) \hspace{1cm} (5.13)

iii) \( b_i > 0 \)

**Case (iii) - To ascertain instability**

If the inferences and sufficient conditions discussed in case (i) are violated then the considered fuzzy system is unstable. If the conditions given in case (ii) represented by equations (5.13) are violated, then the system becomes aperiodically unstable.

The sufficient condition for the given fuzzy system to be unstable is,

\( a_i < 0 \) \hspace{1cm} (5.14)

**Note:**

Further if, \( |b| < 1 \), then the equation (5.7) becomes,

\( (z^2-az) \approx 0 \) \hspace{1cm} (5.15)

which again yields,

\( z(z-a) = 0 \)

Thus, \( z = 0 \) and \( z = a \).

This means that for \( |z| \leq 1 \) (for aperiodic stability),

\( a \leq 1 \) \hspace{1cm} (5.16)
The above proposed procedures can be easily extended for higher order systems. It is inferred from the above three cases that the generalized sufficient conditions for the stability of the given fuzzy system are:

\[
\begin{align*}
\text{i) } & |b_i| < 1 \\
\text{ii) } & a_i > 0
\end{align*}
\]  

\hspace{1cm} (5.17)

The flowchart depicting the entire process of proposed stability analysis is as shown in Figure 5.1. The proposed procedure is illustrated in the following examples.

Figure 5.1 Flowchart for stability analysis of fuzzy systems
5.2.3 Illustrations

The proposed procedure in section 5.2.2 for stability analysis is applied to the following fuzzy systems.

Illustration 5.1

Consider the fuzzy system (Kawamoto et al 1992),

Rule 1: IF $X_2(k)$ is $M_1$ (e.g. small) THEN $X(k+1) = A_1 X(k)$

Rule 2: IF $X_2(k)$ is $M_2$ (e.g. big) THEN $X(k+1) = A_2 X(k)$

where $X(k) = [X_1(k) \ X_2(k)]^T$ and

$$A_1 = \begin{bmatrix} 1.503 & -0.588 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -0.361 \\ 1 & 0 \end{bmatrix}$$

Figure 5.2 shows the membership functions of $M_1$ and $M_2$.

Compute the characteristic equation of $A_1$, 

$$(zI - A_1) = 0$$

$$F(z) = (z^2 - 1.503z + 0.588) = 0$$

(5.19)
Formulate Marden’s table for equation (5.19) as in Table 5.3,

<table>
<thead>
<tr>
<th>Column/Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>F₂ \left( \frac{I}{z} \right)</td>
<td>0.588</td>
<td>-1.503</td>
<td>1</td>
</tr>
<tr>
<td>F₃(z)</td>
<td>1</td>
<td>-1.503</td>
<td>0.588</td>
</tr>
<tr>
<td>F₁ \left( \frac{I}{z} \right)</td>
<td>-0.6192</td>
<td>0.6543</td>
<td></td>
</tr>
<tr>
<td>F₃(z)</td>
<td>0.6543</td>
<td>-0.6192</td>
<td></td>
</tr>
<tr>
<td>F₀ \left( \frac{I}{z} \right)</td>
<td>0.0447</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 5.3, it is observed that the last element of F₁ \left( \frac{I}{z} \right) row 0.6543>0 and the F₀ \left( \frac{I}{z} \right) element 0.0447>0, which satisfies the Marden’s stability criterion. Thus, the system matrix represented by this characteristic equation is asymptotically stable. Also, the characteristic equation formulated for the other system matrix |zI-A₂| satisfies Marden’s stability conditions, indicating asymptotic stability. The same inference can be obtained by utilizing the sufficient conditions derived in section 5.2.2 case (i) over the A matrices. The sufficient condition observations made over the A matrices is shown in Table 5.4.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>A₁</th>
<th>A₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>|bᵢ|</td>
<td>0.588</td>
<td>0.361</td>
</tr>
<tr>
<td>aᵢ</td>
<td>1.503</td>
<td>1</td>
</tr>
<tr>
<td>(1+|bᵢ|)</td>
<td>1.588</td>
<td>1.361</td>
</tr>
</tbody>
</table>
From Table 5.4, it is observed that for both A matrices, 
\[ |b_i| < 1 \text{ and } a_i < (1+|b_i|) \text{ with } a_i > 0 \]

This substantiates that the given fuzzy system represented by the system matrices in equation (5.18) is asymptotically stable in nature.

To verify the aperiodic nature of the system, Fuller’s stability criterion is applied to the characteristic equation \( F(-z) \) of the system matrices.

The characteristic equation \( F(-z) \) for \( A_1 \) matrix is,
\[
F(z) = (z^2 + 1.503z + 0.588) = 0
\]  
\[ (5.20) \]

Formulate Fuller’s table for equation (5.20) as given in Table 5.5.

**Table 5.5 Fuller’s table for equation (5.20)**

<table>
<thead>
<tr>
<th>Column/ Row</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F(-z) \rightarrow z^4 )</td>
<td>1</td>
<td>1.503</td>
<td>0.588</td>
</tr>
<tr>
<td>( F'(-z) \rightarrow z^3 )</td>
<td>2</td>
<td>1.503</td>
<td></td>
</tr>
<tr>
<td>( z^2 )</td>
<td>0.7515</td>
<td>0.588</td>
<td></td>
</tr>
<tr>
<td>( z^1 )</td>
<td><strong>- 0.0465</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( z^0 )</td>
<td>0.588</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

From Table 5.5, it is noted that there exists sign changes in the first column, indicating Fuller’s stability condition is not satisfied. Thus, the system matrix represented by this characteristic equation is not aperiodically stable. The characteristic equation formulated for the system matrix \( |zI-A_2| \) also do not satisfy Fuller’s stability criterion, indicating the given system is not aperiodic in nature. This inference can also be observed by applying the sufficient condition derived in section 5.2.2 case (ii) over the A matrices. The observations are made based on the tabulation in Table 5.6.
Table 5.6 Sufficient condition for aperiodic stability of equation (5.18)

<table>
<thead>
<tr>
<th>Conditions</th>
<th>A_1</th>
<th>A_2</th>
</tr>
</thead>
<tbody>
<tr>
<td>a_i</td>
<td>1.503</td>
<td>1</td>
</tr>
<tr>
<td>b_i</td>
<td>-0.588</td>
<td>-0.361</td>
</tr>
<tr>
<td>a_i^2 &gt; 4b_i</td>
<td>2.259 &gt; -2.352</td>
<td>1 &gt; -1.444</td>
</tr>
</tbody>
</table>

In Table 5.6, it is noted that the conditions, $a_i > 0$ and $a_i^2 > 4b_i$, are satisfied but the condition, $b_i > 0$, is not satisfied by both $A_1$ and $A_2$ matrices, thus the given fuzzy system is not aperiodically stable.

Also, for both the matrices $A_1$ and $A_2$, the generalized sufficient conditions $|b_i| < 1$ and $a_i > 0$ are satisfied, declaring the given system is stable.

Based on the observations made above, it can be concluded that the given fuzzy system is asymptotically stable but not aperiodic. This conclusion is in agreement with Kawamoto et al (1992). The proposed approach employed simple procedures and analyzing the stability of fuzzy systems involved less computations compared to Kawamoto et al (1992) which used Lyapunov inequalities.

For the chosen initial condition $[0.9 \ -0.7]^T$, the simulation results shown in Figure 5.3 guarantee the above conclusion.

The plot shown in Figure 5.3 where $X(k+1)$ is plotted for different instants of $K$ is obtained using the following procedure:
The elements of $A_1$ matrix in equation (5.18) are written as,

\[
\begin{align*}
a_{11} &= 1.503 \\
a_{12} &= -0.588 \\
a_{21} &= 1 \\
a_{22} &= 0
\end{align*}
\] (5.21)

The elements of $A_2$ matrix in equation (5.18) are given by,

\[
\begin{align*}
b_{11} &= 1 \\
b_{12} &= -0.361 \\
b_{21} &= 1 \\
b_{22} &= 0
\end{align*}
\] (5.22)

The initial conditions are given by,

\[X_1=0.9 \text{ and } X_2=-0.7\] (5.23)

Using Rule 1 and Rule 2, the membership values $M_1$ and $M_2$ are calculated based on $X_2$ value as follows:

\[
\begin{align*}
M_1 &= (-0.5 \times X_2) + 0.5 = (-0.5 \times -0.7) + 0.5 = 0.85 \\
M_2 &= (0.5 \times X_2) + 0.5 = (0.5 \times -0.7) + 0.5 = 0.15
\end{align*}
\] (5.24)

In equation (5.24),

If $(M_1 > 1.0)$ Then $M_1=1.0$ else if $(M_1 < 0.0)$ Then $M_1=0$

If $(M_2 > 1.0)$ Then $M_2=1.0$ else if $(M_2 < 0.0)$ Then $M_2=0$

Also if $M_1$ and $M_2$ lie between 0 to 1, then the value of $M_1$ and $M_2$ is the computed value itself.

These $M_1$ and $M_2$ values from equation (5.24) are utilized for computing $X(k+1)$ instants as given below:
\[\begin{align*}
X_{t1} &= (a_{11}X_1 + a_{12}X_2) \times M_1 + (b_{11}X_1 + b_{12}X_2) \times M_2 \\
&= ((1.503 \times 0.9) + (-0.588 \times 0.7)) \times 0.85 + \\
&\quad ((1 \times 0.9) + (-0.361 \times -0.7)) \times 0.15 \\
X_{t1} &= 1.6726 \quad (5.25) \\
\end{align*}\]

\[\begin{align*}
X_{t2} &= (a_{21}X_1 + a_{22}X_2) \times M_1 + (b_{21}X_1 + b_{22}X_2) \times M_2 \\
&= ((1 \times 0.9) + (0 \times -0.7)) \times 0.85 + \\
&\quad ((1 \times 0.9) + (0 \times -0.7)) \times 0.15 \\
X_{t2} &= 0.9 \quad (5.26) \\
\end{align*}\]

The value of \(X_{t1}\) is the \(X(k+1)\) computed at this instant. For the next instant computation, \(X_{t1}\) and \(X_{t2}\) from equations (5.25) and (5.26) are used as initial conditions. i.e.,

\[\begin{align*}
X_{i_1} &= X_{t1} \\
X_{i_2} &= X_{t2} \quad (5.27) \\
\end{align*}\]

The process is repeated for 20 instants of \('k'\) and \(X(k+1)\) for respective initial positions are as shown in Table 5.7.

**Table 5.7 X(k+1) at different instants of k for Illustration 5.1**

<table>
<thead>
<tr>
<th>Time Index k</th>
<th>(X(k+1))</th>
<th>Time Index k</th>
<th>(X(k+1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9</td>
<td>11</td>
<td>-0.0321</td>
</tr>
<tr>
<td>1</td>
<td>1.6726</td>
<td>12</td>
<td>-0.0094</td>
</tr>
<tr>
<td>2</td>
<td>1.3795</td>
<td>13</td>
<td>0.0035</td>
</tr>
<tr>
<td>3</td>
<td>0.7757</td>
<td>14</td>
<td>0.0089</td>
</tr>
<tr>
<td>4</td>
<td>0.2777</td>
<td>15</td>
<td>0.0094</td>
</tr>
<tr>
<td>5</td>
<td>-0.0064</td>
<td>16</td>
<td>0.0076</td>
</tr>
<tr>
<td>6</td>
<td>-0.1306</td>
<td>17</td>
<td>0.0050</td>
</tr>
<tr>
<td>7</td>
<td>-0.1606</td>
<td>18</td>
<td>0.0027</td>
</tr>
<tr>
<td>8</td>
<td>-0.1424</td>
<td>19</td>
<td>0.0010</td>
</tr>
<tr>
<td>9</td>
<td>-0.1048</td>
<td>20</td>
<td>-0.0001</td>
</tr>
<tr>
<td>10</td>
<td>-0.0650</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The plot points in Figure 5.3 are the $X(k+1)$ values at different instants as shown in Table 5.7.

![Figure 5.3 Behavior of fuzzy system for Illustration 5.1](image)

**Figure 5.3 Behavior of fuzzy system for Illustration 5.1**

Figure 5.3 shows that the given fuzzy system is asymptotically stable and it agrees with the conclusion obtained using the proposed approach. Appendix 3 provides a program to compute the output response of the fuzzy system and also to obtain a graphical plot based on the output computed.

**Illustration 5.2**

Consider the fuzzy system (Wang et al 1996):

Rule 1: IF $X_2(k)$ is $M_1$ (e.g. small) THEN $X(k+1) = A_1X(k)$

Rule 2: IF $X_2(k)$ is $M_2$ (e.g. big) THEN $X(k+1) = A_2X(k)$

where $X(k) = [X_1(k) \ X_2(k)]^T$ and
\[ A_1 = \begin{bmatrix} 1 & -0.5 \\ 1 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & -0.5 \\ 1 & 0 \end{bmatrix} \] (5.28)

Figure 5.2 shows the membership functions of \(M_1\) and \(M_2\).

For the stability analysis of the given fuzzy system represented by its system matrices in equation (5.28), applying the proposed sufficient condition given in equation (5.14) and equation (5.17) to equation (5.28), the value of \(a_i\) in matrix \(A_2\) is found to be less than zero, declaring the given system is unstable in nature.

This conclusion is also verified graphically, which is as shown in Figure 5.4, for the initial conditions \(X(k) = 0.9, X(k-1) = -0.7\) and Table 5.8 shows the value of \(X(k+1)\) for respective initial positions. The result obtained is in agreement with Wang et al (1996).

**Table 5.8 X(k+1) at different instants of k for Illustration 5.2**

<table>
<thead>
<tr>
<th>Time Index k</th>
<th>X(k+1)</th>
<th>Time Index k</th>
<th>X(k+1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.9</td>
<td>11</td>
<td>-2.5569</td>
</tr>
<tr>
<td>1</td>
<td>0.98</td>
<td>12</td>
<td>1.5455</td>
</tr>
<tr>
<td>2</td>
<td>-1.332</td>
<td>13</td>
<td>2.8239</td>
</tr>
<tr>
<td>3</td>
<td>0.8154</td>
<td>14</td>
<td>-3.5967</td>
</tr>
<tr>
<td>4</td>
<td>1.4814</td>
<td>15</td>
<td>2.1847</td>
</tr>
<tr>
<td>5</td>
<td>-1.6155</td>
<td>16</td>
<td>3.9831</td>
</tr>
<tr>
<td>6</td>
<td>0.8748</td>
<td>17</td>
<td>-5.0754</td>
</tr>
<tr>
<td>7</td>
<td>1.6826</td>
<td>18</td>
<td>3.0839</td>
</tr>
<tr>
<td>8</td>
<td>-1.9094</td>
<td>19</td>
<td>5.6216</td>
</tr>
<tr>
<td>9</td>
<td>1.0681</td>
<td>20</td>
<td>-7.1635</td>
</tr>
<tr>
<td>10</td>
<td>2.0228</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.3 STABILITY ANALYSIS OF FUZZY SYSTEMS REPRESENTED BY RELATIONAL MATRICES

In this section, certain algorithms are proposed to perform the stability analysis of given fuzzy system (Ross 1997) represented by its relational matrix. The given relational matrix depicting the fuzzy system is transformed into compositional matrices and operations are performed over these matrices to analyze the stability nature with minimal computations.

5.3.1 Basic Pre-requisites

The basic equation of a fuzzy system shown in Figure 5.5 is of the form,

\[ X_{k+1} = X_k \circ U_k \circ R, \quad k=0,1,2,3,\ldots \]  

(5.29)
where,  
\( X_k \) is the fuzzy set of the states at the \( k^{th} \) time instant  
\( X_{k+1} \) is the fuzzy set of the states at the \((k+1)^{th}\) time instant  
\( U_k \) is the input at the \( k^{th} \) instant and  
\( R \) is a fuzzy relation describing the fuzzy system.

\[
\begin{align*}
U_k & \quad \rightarrow \quad X_k \circ R \quad \rightarrow \quad X_{k+1}
\end{align*}
\]

Figure 5.5 A Fuzzy System

The states and control are denoted by means of membership functions given by,

\[
\begin{align*}
X_k, X_{k+1} \in F(X), & \quad \mu_{X_k}, \mu_{X_{k+1}} : X \rightarrow [0,1] \\
U_k \in F(U), & \quad \mu_{U_k} : U \rightarrow [0,1]
\end{align*}
\]  

(5.30)

where, \( F(\cdot) \) is a family of fuzzy sets defined on a proper space  
\( F(x) = \{x | \mu_x : x \rightarrow [0,1] \} \).

The fuzzy relation \( R \) in equation (5.29) describing the given fuzzy system is defined on the Cartesian product of \( X \times U \times X \):

\[
R \in F(X \times U \times X), \quad \mu_R : X \times U \times X \rightarrow [0,1]
\]  

(5.31)

The operator \( \circ \) in equation (5.29) stands for max-min composition. As a result equation (5.29) can be rewritten as (Zadeh 1973):

\[
\mu_{X_{k+1}}(x) = \max_{u \in U} \max_{x \in X} \min \left[ \mu_{U_k}(u), \mu_{X_k}(x), \mu_R(x, u, x) \right], \quad k = 0, 1, 2, 3, \ldots
\]  

(5.32)
The fuzzy system given in equation (5.29) is called free or unforced fuzzy system if there is no input; that is, \( U_k = \text{zero} \) for all \( k \), where zero is given by,

\[
\mu_{U_k}(u) = \begin{cases} 
1 & u = u_i( k ) \\
0 & \text{otherwise}
\end{cases}
\]

for all \( k \). Let,

\[
P = U_k \circ R, \quad P \in F(X \times X)
\]

(5.34)

for \( U_k = \text{zero} \) with \( k=0,1,2,3,\ldots \)
is called the characteristic or relational matrix describing the given fuzzy system.

Then the governing equation of a free fuzzy dynamic system has the form,

\[
X_{k+1} = X_k \circ P \quad k=0,1,2,3,\ldots
\]

(5.35)

The forthcoming sections discuss the stability investigations of free fuzzy systems based on its relational matrix \( P \) and propose certain algorithms for the same. The proposed algorithms are illustrated with examples.

### 5.3.2 Max-Min Composition

The Max-Min composition (Sivanandam et al 2007) is performed as given below:

If \( R \) is a relation that relates elements from universe \( X \) to universe \( Y \) and \( S \) is a relation that relates elements from universe \( Y \) to universe \( Z \), then a relation \( T \) is to be formed between the relation \( R \) and \( S \) that relates the
elements from universe X to universe Z. The relational matrix T is evaluated using Max-Min composition and is defined by,

\[ T = R \circ S \]

\[ T(X, Z) = \text{Max} (\text{Min}(X, Y), \text{Min}(Y, Z)) \]  

For instance, consider P as a fuzzy relational matrix defined by,

\[ P = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]  

\[ P^2 = P \circ P \]

\[ = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \circ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \]  

Then, \[ P^2 = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \]  

where, \[ b_{11} = \text{max} \{ \text{min} (a_{11}, a_{11}), \text{min} (a_{12}, a_{21}) \} \]  

\[ b_{12} = \text{max} \{ \text{min} (a_{11}, a_{12}), \text{min} (a_{12}, a_{22}) \} \]  

\[ b_{21} = \text{max} \{ \text{min} (a_{21}, a_{11}), \text{min} (a_{22}, a_{21}) \} \]  

\[ b_{22} = \text{max} \{ \text{min} (a_{21}, a_{12}), \text{min} (a_{22}, a_{22}) \} \]  

This procedure can be extended for constructing other \( P^3 \) to \( P^n \) compositional matrices:

\[ P^3 = P^2 \circ P \]

\[ P^4 = P^3 \circ P \]

\[ \vdots \]

\[ P^n = P^{n-1} \circ P \]  

The program for performing the max-min composition given the fuzzy relational matrices is as given in Appendix 4.
5.3.3 Proposed Algorithm 1 using Step Responses of Compositional Matrices

The stability investigation in this algorithm is performed with the help of characteristic polynomial having its characteristic roots within unit circle. Using the given fuzzy relational matrix, the characteristic polynomial is formed and its corresponding all pole system is tested for unit step input. The output response is analyzed for nature of stability. The various steps in the algorithm are as follows:

Step 1: Read the given fuzzy relational matrix \( P \) representing the fuzzy system and normalize it based on its dimension (i.e., product of the number of rows and columns). Let it be \( P_s \).

Step 2: Formulate the characteristic polynomial using \( P_s \). It is given by,

\[
C(z) = |zI - P_s|
\]  \hspace{1cm} (5.41)

where, \( z \) is the discrete variable with \(|z| < 1\) and 

\( I \) is the identity matrix

Step 3: Assume an open loop all pole fuzzy system,

\[
Y(z) = \frac{1}{C(z)}
\]  \hspace{1cm} (5.42)

Step 4: Obtain the unit step response of \( Y(z) \) and observe its peak amplitude value \( A(P_s) \).

Step 5: Repeat the steps 2 to 4 for all compositions obtained using \( P_s \). The compositional operation is performed as given in section 5.3.2

Step 6: Declare the nature of stability of the given fuzzy system based on the observations made over the compositional matrices i.e.,

i) If the output response amplitude values remain same for all the compositional matrices, then the system is aperiodically stable.

ii) If the output response amplitude values alter between any two values for the compositional matrices symmetrically, then the system is periodically stable i.e., stable with oscillations.
iii) If the output response amplitude values vary drastically between compositional matrices, the given fuzzy system represented by its fuzzy relational matrix is unstable.

Step 7: Stop.

The flowchart depicting the process of operation of the above algorithm is as shown in Figure 5.6. The above scheme is applied for the following Illustrations.

![Flowchart for proposed algorithm 1](image)

Figure 5.6 Flowchart for proposed algorithm 1
**Illustration 5.3**

Let us investigate the stability of a free fuzzy system (Kiszka et al. 1985) represented by equation (5.35), where the fuzzy relation \( P \) is given by,

\[
P = \begin{bmatrix} 0.1 & 0.5 \\ 0.3 & 0.2 \end{bmatrix}
\]  

(5.43)

Applying the proposed algorithm in section 5.3.2,

Step 1: The relational matrix \( P \) in equation (5.43) is read and normalized using the product of number of rows and number of columns (i.e., 4), \( P_s \) is obtained as,

\[
P_s = \begin{bmatrix} 0.025 & 0.125 \\ 0.750 & 0.050 \end{bmatrix}
\]  

(5.44)

Step 2: Computing the characteristic polynomial for \( P_s \) in equation (5.44),

\[
C(z) = |zI - P_s|
\]

\[
C(z) = (z^2 - 0.075z - 0.008125)
\]  

(5.45)

Step 3: Assuming an open loop all pole fuzzy system,

\[
Y(z)|_{P_s} = \frac{1}{C(z)} = \frac{1}{z^2 - 0.075z - 0.008125}
\]  

(5.46)

Step 4: Obtaining step response for \( Y(z) \) in equation (5.46) as shown in Figure 5.7 and observing its peak amplitude,

\[
A(P_s) = 1.0907
\]  

(5.47)

Repeating the above steps for the compositional matrix \( P_s^2 \) obtained using \( P_s \) via max-min compositional approach,

\[
P_s^2 = P_s \circ P_s
\]

\[
P_s^2 = \begin{bmatrix} 0.075 & 0.05 \\ 0.05 & 0.075 \end{bmatrix}
\]  

(5.48)
The characteristic polynomial using $P_s^2$ is,

$$ C(z) = \left| zI - P_s^2 \right| $$

$$ C(z) = z^2 - 0.15z + 0.003125 \quad (5.49) $$

The open loop all pole fuzzy system for equation (5.49) is,

$$ Y(z)\big|_{P_s^2} = \frac{I}{C(z)} = \frac{I}{z^2 - 0.15z + 0.003125} \quad (5.50) $$

The step response for equation (5.50) is as shown in Figure 5.8 and its peak amplitude is,

$$ A(P_s^2) = 1.1722 \quad (5.51) $$

**Figure 5.7** Step response for $Y(z)$ in equation (5.46)
The same process is repeated for further compositional matrices until required observation is met and the peak amplitudes of response curves for all compositional matrices are tabulated as shown in Table 5.9.

**Table 5.9  Peak amplitude for compositional matrices of fuzzy relational matrix in Illustration 5.3**

<table>
<thead>
<tr>
<th>Peak Amplitude of output response for unit step input</th>
<th>A(P₃)</th>
<th>A(P₅²)</th>
<th>A(P₅³)</th>
<th>A(P₅⁴)</th>
<th>A(P₅⁵)</th>
<th>A(P₅⁶)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.0907</td>
<td>1.1722</td>
<td>1.1150</td>
<td>1.1722</td>
<td>1.1150</td>
<td>1.1722</td>
<td>…</td>
</tr>
</tbody>
</table>

From Table 5.9, it can be observed that the given fuzzy system possesses stable oscillations i.e.,
\[
A(P_s^2) = A(P_s^4) = A(P_s^6) = \ldots = 1.1722
\]
\[
A(P_s^3) = A(P_s^5) = A(P_s^7) = \ldots = 1.1150
\]

Thus, it can be concluded that the fuzzy system represented by its relational matrix in equation (5.43) of Illustration 5.3, is stable with periodic in nature. This conclusion is in agreement with Kiszka et al (1985). The proposed algorithm is simple and easy to apply for the given fuzzy system with minimal computations compared to Kiszka et al (1985), which employed energetistic stability method involving more computations.

**Illustration 5.4**

Consider a fuzzy system (Ibrahim 1999), where fuzzy relation matrix \( P \) is given by,

\[
P = \begin{bmatrix}
0.1 & 0.3 & 0.3 \\
0.7 & 0.2 & 0.5 \\
0.8 & 0.1 & 0.4
\end{bmatrix}
\]

Applying the proposed algorithm in section 5.3.2, \( P \) in equation (5.53) is normalized using the product of number of columns and number of rows to get \( P_s \),

\[
P_s = \begin{bmatrix}
0.0111 & 0.0333 & 0.0333 \\
0.0778 & 0.0222 & 0.0556 \\
0.0889 & 0.0111 & 0.0444
\end{bmatrix}
\]

Computing the characteristic polynomial using \( P_s \),

\[
C(z) = |zI - P_s|
\]

\[
C(z) = z^3 - 0.0778z^2 - 0.0044z - 1.6461 \times 10^{-5}
\]
The all pole fuzzy system of equation (5.55) is,

\[ Y(z)|_{P_s} = \frac{I}{C(z)} = \frac{I}{z^3 - 0.0778z^2 - 0.0044z - 1.6461 \times 10^{-5}} \]  

(5.56)

The step response for \( Y(z) \) in equation (5.56) is as shown in Figure 5.9 with its peak amplitude as,

\[ A(P_s) = 1.0896 \]  

(5.57)

![Figure 5.9: Step response for Y(z) in equation (5.56)]

Performing the above steps for the first compositional matrix computed using \( P_s \),

\[ P_s^2 = P_s \circ P_s \]

\[ P_s^2 = \begin{bmatrix}
0.0333 & 0.0222 & 0.0333 \\
0.0556 & 0.0333 & 0.0444 \\
0.0444 & 0.0333 & 0.0444
\end{bmatrix} \]  

(5.58)
The characteristic polynomial of equation (5.58) is,

\[ C(z) = |zI - P^2_z| \]

\[ C(z) = z^3 - 0.1111z^2 - 0.00012z - 1.3717 \times 10^{-6} \]  \hspace{1cm} (5.59)

For the equation (5.59), the all pole fuzzy system is,

\[ Y(z) \big|_{P^2} = \frac{I}{C(z)} = \frac{I}{z^3 - 0.1111z^2 - 0.00012z - 1.3717 \times 10^{-6}} \]  \hspace{1cm} (5.60)

The step response for \( Y(z) \) in equation (5.60) is obtained as shown in Figure 5.10 and its peak amplitude is noted to be,

\[ A(P^2_z) = 1.1252 \]  \hspace{1cm} (5.61)

Figure 5.10  Step response for \( Y(z) \) in equation (5.60)

The steps are repeated for other compositional matrices and their peak amplitudes of response curves are tabulated in Table 5.10.
Table 5.10 Peak amplitude for compositional matrices of fuzzy relational matrix in Illustration 5.4

<table>
<thead>
<tr>
<th>Peak Amplitude of output response for unit step input</th>
<th>A(Pₚ)</th>
<th>A(Pₛ²)</th>
<th>A(Pₛ³)</th>
<th>A(Pₛ⁴)</th>
<th>A(Pₛ⁵)</th>
<th>A(Pₛ⁶)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1.0896</td>
<td>1.1252</td>
<td>1.1255</td>
<td>1.1255</td>
<td>1.1255</td>
<td>1.1255</td>
</tr>
</tbody>
</table>

From Table 5.10, it is observed that the given fuzzy system with \( P \) as in equation (5.53) possesses aperiodic stability (with no oscillations), viz.,

\[
A(Pₚ) = A(Pₛ²) = A(Pₛ³) = A(Pₛ⁴) = A(Pₛ⁵) = \cdots = 1.1255
\] (5.62)

Thus, it can be concluded that the given fuzzy system is stable aperiodic nature without oscillations. The compositions are performed until the required observations are met. The conclusion obtained using the proposed approach is in agreement with Ibrahim (1999) and simple to apply with minimal computations.

### 5.3.4 Proposed Algorithm 2 using Necessary Conditions of Compositional Matrices

In this algorithm, a fuzzy system represented in the form of relation matrix and its corresponding compositional matrices are analyzed for stability by suitably transforming it into characteristic equation forms. For assessing the nature of stability, the characteristic equations of compositional matrices are evaluated for the chosen three points \( z = 0, 1 \) and \(-1\) in discrete domain. The algorithmic steps involved for analyzing the stability nature is as follows:
Step 1: Read the given relational matrix \( P \) representing the fuzzy system and normalize it based on the product of number of rows and number of columns. Let it be \( P_s \).

Step 2: Formulate the characteristic polynomial using \( P_s \). It is given by,

\[
C(z) = | zI - P_s |
\]

where, \( I \) is the identity matrix and 
\( z \) is the discrete variable with \(|z|<1\)

Step 3: Evaluate \( C(z) \) in equation (5.63) at three points \( z = 0, 1 \) and \(-1\)

Step 4: Repeat steps 2 and 3 for all composition \( P_s^2, P_s^3, P_s^4, \ldots \) obtained using \( P_s \). The compositional operation is performed as given in section 5.3.2.

Step 5: Based on the observations over compositional matrices, the nature of stability is declared as follows:

At \( z=0, 1 \) and \(-1\),

if \( C(z)|_{P_s} = C(z)|_{P_s^2} = C(z)|_{P_s^3} = C(z)|_{P_s^4} = \ldots \) then stable aperiodic

elseif

\[ C(z)|_{P_s} = C(z)|_{P_s^2} = C(z)|_{P_s^3} = C(z)|_{P_s^4} = \ldots \text{and} \]

\[ C(z)|_{P_s^2} = C(z)|_{P_s^3} = C(z)|_{P_s^4} = \ldots \]

then stable periodic

else the system is unstable.

Step 6: Stop

The algorithm is represented by the flowchart as shown in Figure 5.11. The proposed algorithm is applied to the illustrative examples discussed in section 5.3.3.
Figure 5.11 Flowchart for proposed algorithm 2

Illustration 5.5

The proposed algorithm 2 is applied to a fuzzy system (Kiszka et al 1985) used in Illustration 5.3 of section 5.3.3.
From Illustration 5.3, considering the characteristic polynomial in equation (5.45),

\[ C(z)_{|_{P_s}} = z^2 - 0.075z - 0.008125 \]  

(5.64)

As per step 3 of proposed algorithm 2, evaluating \( C(z) \) in equation (5.64) at \( z = 0, 1 \) and \(-1\),

For \( P_s \),

\[
\begin{align*}
C(0) &= -0.008125 \\
C(1) &= 0.9169 \\
C(-1) &= 1.0669
\end{align*}
\]  

(5.65)

Performing the above steps for the compositional matrix computed using \( P_s \), the characteristic polynomial for \( P_s^2 \) is obtained from equation (5.49) of Illustration 5.3,

\[ C(z)_{|_{P_s^2}} = z^2 - 0.15z + 0.003125 \]  

(5.66)

Evaluating equation (5.66) at \( z = 0, 1 \) and \(-1\) as per proposed algorithm 2,

For \( P_s^2 \),

\[
\begin{align*}
C(0) &= 0.003125 \\
C(1) &= 0.8531 \\
C(-1) &= 1.1531
\end{align*}
\]  

(5.67)

The characteristic polynomials are computed for other compositional matrices and are evaluated at three points \( z = 0, 1 \) and \(-1\) as shown in Table 5.11.
Table 5.11 C(z) at z = 0, 1 and -1 for compositional matrices of Illustration 5.5

<table>
<thead>
<tr>
<th>Compositional Matrices</th>
<th>C(0)</th>
<th>C(1)</th>
<th>C(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_s$</td>
<td>-0.008125</td>
<td>0.91688</td>
<td>1.0669</td>
</tr>
<tr>
<td>$P_s^2$</td>
<td>0.003125</td>
<td>0.85313</td>
<td>1.1531</td>
</tr>
<tr>
<td>$P_s^3$</td>
<td>-0.003125</td>
<td>0.89687</td>
<td>1.0969</td>
</tr>
<tr>
<td>$P_s^4$</td>
<td>0.003125</td>
<td>0.85313</td>
<td>1.1531</td>
</tr>
<tr>
<td>$P_s^5$</td>
<td>-0.003125</td>
<td>0.89687</td>
<td>1.0969</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
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</tr>
</tbody>
</table>

From Table 5.11, it can be observed that C(z) at z = 0, 1 and -1 for $P_s^2, P_s^3, P_s^6, ...$ are equal and $P_s^3, P_s^5, P_s^7, ...$ are equal. This infers that the considered fuzzy system for analysis is periodically stable with oscillations. The inference obtained using the proposed algorithm is in agreement with Kiszka et al (1985) and is computationally simpler than the energetistic stability criterion proposed by Kiszka et al (1985).

Illustration 5.6

The proposed algorithm 2 is applied to a fuzzy system (Ibrahim 1999) in Illustration 5.4 of section 5.3.3.

Evaluating C(z) in equation (5.55) at z = 0, 1 and -1 as per proposed algorithm 2, the values are,

\[
\begin{align*}
C(0) &= -1.6461 \times 10^{-5} \\
C(1) &= 0.9178 \\
C(-1) &= -1.0733
\end{align*}
\]
Similarly computing $C(z)$ obtained for the first compositional matrix $P_s^2$ in equation (5.59) at $z = 0$, 1 and -1, the values are,

$C(0) = -1.3717\times10^{-6}$
$C(1) = 0.8888$ \hspace{1cm} (5.69)
$C(-1) = -1.1110$

The above operations are performed for the other compositional matrices and the results are tabulated as shown in Table 5.12.

**Table 5.12**  $C(z)$ at $z = 0$, 1 and -1 for compositional matrices of Illustration 5.6

<table>
<thead>
<tr>
<th>Compositional Matrices</th>
<th>C(0)</th>
<th>C(1)</th>
<th>C(-1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_s$</td>
<td>-1.6461\times10^{-5}</td>
<td>0.91776</td>
<td>-1.0733</td>
</tr>
<tr>
<td>$P_s^2$</td>
<td>-1.3717\times10^{-6}</td>
<td>0.88876</td>
<td>-1.111</td>
</tr>
<tr>
<td>$P_s^3$</td>
<td>6.4249\times10^{-22}</td>
<td>0.88852</td>
<td>-1.1107</td>
</tr>
<tr>
<td>$P_s^4$</td>
<td>6.4249\times10^{-22}</td>
<td>0.88852</td>
<td>-1.1107</td>
</tr>
<tr>
<td>$P_s^5$</td>
<td>6.4249\times10^{-22}</td>
<td>0.88852</td>
<td>-1.1107</td>
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</table>

From Table 5.12, it can be observed from $P_s^3$ onwards, the values of $C(0)$, $C(1)$ and $C(-1)$ settles at a particular value without alternating. Hence the fuzzy system represented by the relational matrix in equation (5.53) is stable and aperiodic in nature. This conclusion is in agreement with Ibrahim (1999). The proposed algorithm is simple involving less computation compared to the energetistic stability utilized by Ibrahim (1999).
5.3.5 Proposed Algorithm 3 using Trace and Determinant of Compositional Matrices

The stability investigation in this algorithm is performed by calculating the trace and determinant of the given fuzzy relational matrix and its corresponding compositional matrices. The steps involved in the algorithm are as follows:

Step 1: Read the given relational matrix P.
Step 2: Compute the trace of matrix P. The trace is the sum of diagonal elements of the given matrix.
Step 3: Calculate the determinant of matrix P
Step 4: Repeat the steps 2 and 3 for compositional matrices $P^2, P^3, P^4, P^5, \ldots$ obtained using $P$. The max-min composition is performed as in section 5.3.2.
Step 5: Declare the nature of stability based on the observations over compositional matrices. For instance, for periodic oscillatory nature, the values of trace and determinant of respective compositional matrices will be alternating and for aperiodic nature, all the values will be settled at one particular value. On the other hand, if the above mentioned criterion is not satisfied, then the system is unstable.

The flowchart depicting the algorithmic process is shown in Figure 5.12. The proposed algorithm 3 is applied to the illustrative examples discussed in section 5.3.3.
Figure 5.12  Flowchart for proposed algorithm 3
Illustration 5.7

The proposed algorithm 3 is applied to a fuzzy system (Kiszka et al. 1985) in Illustration 5.3 of section 5.3.3.

From equation (5.43), the fuzzy relational matrix $P$ for the given fuzzy system in Illustration 5.3 is,

$$
P = \begin{bmatrix}
0.1 & 0.5 \\
0.3 & 0.2
\end{bmatrix}
$$  \hspace{1cm} (5.70)

Using proposed algorithm 3 to $P$ and computing its trace and determinant,

Trace of $P = T(P) = 0.3$
Determinant of $P = D(P) = -0.13$  \hspace{1cm} (5.71)

Performing composition of $P$ matrix in equation (5.70),

$$
P^2 = P \circ P = \begin{bmatrix}
0.3 & 0.2 \\
0.2 & 0.3
\end{bmatrix}
$$  \hspace{1cm} (5.72)

Trace and determinant of $P^2$ is obtained as,

$T(P^2) = 0.6$
$D(P^2) = 0.05$  \hspace{1cm} (5.73)

As per the proposed algorithm 3, the trace and determinant are computed for other compositional matrices and the results are tabulated as shown in Table 5.13.
Table 5.13 Trace and determinant values of compositional matrices in Illustration 5.7

<table>
<thead>
<tr>
<th>Compositional Matrices</th>
<th>Trace</th>
<th>Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>0.3</td>
<td>-0.13</td>
</tr>
<tr>
<td>P^2</td>
<td>0.6</td>
<td>0.05</td>
</tr>
<tr>
<td>P^3</td>
<td>0.4</td>
<td>-0.05</td>
</tr>
<tr>
<td>P^4</td>
<td>0.6</td>
<td>0.05</td>
</tr>
<tr>
<td>P^5</td>
<td>0.4</td>
<td>-0.05</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td></td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

From Table 5.13, it can be observed that,

\[ T(P^3) = T(P^5) = T(P^7) = \ldots = 0.4 \]
\[ T(P^2) = T(P^4) = T(P^6) = \ldots = 0.6 \]

and

\[ D(P^3) = D(P^5) = D(P^7) = \ldots = -0.05 \]
\[ D(P^2) = D(P^4) = D(P^6) = \ldots = 0.05 \] (5.74)

i.e., the values of trace and determinant are found to alter for the compositional matrices. Thus the given fuzzy system with relational matrix P in equation (5.70) is stable and periodic in nature. This conclusion is in agreement with the result in Kiszka et al (1985).

**Illustration 5.8**

The proposed algorithm 3 is applied to a fuzzy system (Ibrahim 1999) in Illustration 5.4 of section 5.3.3. Computing trace and determinant for the fuzzy relational matrix P in equation (5.53) as per proposed algorithm 3,
Perform composition on $P$ in equation (5.53) to get $P^2$,

$$P^2 = P \circ P = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.5 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}$$  \hspace{1cm} (5.76)$$

Calculating trace and determinant of $P^2$,

\begin{align*}
T(P^2) &= 1.0 \\
D(P^2) &= 0.001 \hspace{1cm} (5.77)
\end{align*}

Repeating the evaluation of trace and determinant for other compositional matrices and the values are tabulated as in Table 5.14.

<table>
<thead>
<tr>
<th>Compositional Matrices</th>
<th>Trace</th>
<th>Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>0.7</td>
<td>0.012</td>
</tr>
<tr>
<td>$P^2$</td>
<td>1.0</td>
<td>0.001</td>
</tr>
<tr>
<td>$P^3$</td>
<td>1.0</td>
<td>0.0001</td>
</tr>
<tr>
<td>$P^4$</td>
<td>1.0</td>
<td>0.0001</td>
</tr>
<tr>
<td>$P^5$</td>
<td>1.0</td>
<td>0.0001</td>
</tr>
</tbody>
</table>

It is observed from Table 5.14, that from compositional matrix $P^3$ onwards the values of trace and determinant settle at the same particular scalar value, declaring the given fuzzy relational matrix representing the fuzzy system is of stable aperiodic nature. This conclusion is in agreement with Ibrahim (1999).
5.3.6 Proposed Algorithm 4 using the Elements of Compositional Matrices

The stability analysis of fuzzy system represented by its relational matrix is carried out using this algorithm in a simple way. This algorithm uses a simple inspection test using the elemental values of the composition matrices computed from the given fuzzy relational matrix representing the fuzzy system. The intuitive criterion used in this algorithm is derived based on the inferences made on the compositional elemental values computed and compared at each instants for the above three proposed algorithms. Thus compared to three algorithms in section 5.3.3 to 5.3.5, this algorithm is simpler in its approach. The steps involved in the algorithm are as follows:

Step 1: Read the given fuzzy relational matrix P
Step 2: Observe the elements of matrix P
Step 3: Perform composition over the relational matrix P and obtain compositional matrices $P^2$, $P^3$, $P^4$, ... as per the procedure in section 5.3.2.
Step 4: Declare the nature of stability based on the observations over compositional matrices as given below:

Every element in the corresponding rows and columns of $P$, $P^2$, $P^3$, $P^4$, ... is observed and an intuitive criterion for aperiodic and periodic nature of fuzzy system is formed as follows:

“Any row or any column of respective $P^i$ has same elemental value then due to Max-Min rule application, the same value will appear in all $P^i$. This implies that there is no oscillation in the given fuzzy system; otherwise, if there are variations with repetition (in a regular order), then the system has periodic oscillations. If the above mentioned conditions are not satisfied, then the system is unstable.”
The proof for the proposed algorithm 4 is performed based on the energy criterion as given below.

5.3.6.1 Proof for Algorithm 4

The proposed algorithm 4 is substantiated using the energy concept in Kiszka et al (1985).

Let the energy involved in the first to final composition be written as \( E(P^2), E(P^3), \ldots, E(P^n) \).

i) If,
\[
\begin{align*}
E(P^3) - E(P^2) &= 0 \\
E(P^4) - E(P^3) &= 0 \\
&
\end{align*}
\]
\[
\begin{align*}
&
\end{align*}
\]
\[
\begin{align*}
E(P^n) - E(P^{n-1}) &= 0
\end{align*}
\]

Then the system response is aperiodic in nature with no oscillations.

ii) If,
\[
\begin{align*}
E(P^3) - E(P^2) &= \eta_1 \\
E(P^4) - E(P^3) &= \eta_2 \\
E(P^5) - E(P^4) &= \eta_1 \\
E(P^6) - E(P^5) &= \eta_2 \\
E(P^7) - E(P^6) &= \eta_1 \\
&
\end{align*}
\]
\[
\begin{align*}
&
\end{align*}
\]
\[
\begin{align*}
E(P^n) - E(P^{n-1}) &= \eta_2
\end{align*}
\]
Then the system exhibits periodic oscillations with periodic time \( \tau \) (or) with frequency of oscillations \( f = 1/\tau \).

iii) If the conditions given in i) and ii) do not exist, viz, \( \eta \)'s are all increasing in magnitude, then the system is unstable.

The proposed algorithm 4 is applied to the following illustrative examples.

**Illustration 5.9**

The proposed algorithm 4 is applied to a fuzzy system (Kiszka et al 1985) in Illustration 5.3 of section 5.3.3.

Performing the first composition to the fuzzy relational matrix \( P \) in equation (5.43) to get \( P^2 \),

\[
P^2 = P \circ P = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}
\]  

(5.80)

Computing the next composition,

\[
P^3 = P^2 \circ P = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.2 \end{bmatrix}
\]  

(5.81)

Further composition over \( P^3 \),

\[
P^4 = P^3 \circ P = \begin{bmatrix} 0.3 & 0.2 \\ 0.2 & 0.3 \end{bmatrix}
\]  

(5.82)
Repeating composition over $P^4$ to obtain $P^5$,

$$P^5 = P^4 \circ P = \begin{bmatrix} 0.2 & 0.3 \\ 0.3 & 0.2 \end{bmatrix}$$  \hspace{1cm} (5.83)

Applying the step 4 of proposed algorithm 4 to the matrices given in equation (5.80) through equation (5.83), it can be observed that the elemental membership values of compositional matrices repeats in a regular order i.e., $P^2$, $P^4$, $P^6$, etc. are found to contain same elemental values while $P^3$, $P^5$, $P^7$, etc. are found to contain same elemental values. This infers that the given fuzzy system is stable periodic (with oscillations) in nature. This conclusion is in agreement with Kiszka et al (1985).

**Illustration 5.10**

Applying the proposed algorithm 4 to a fuzzy system (Ibrahim 1999) in Illustration 5.4 of section 5.3.3, the following observations are made.

As per the proposed algorithm performing the composition to relational matrix $P$ in equation (5.53),

$$P^2 = P \circ P = \begin{bmatrix} 0.3 & 0.2 & 0.3 \\ 0.5 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}$$  \hspace{1cm} (5.84)

Performing further composition,

$$P^3 = P^2 \circ P = \begin{bmatrix} 0.3 & 0.3 & 0.3 \\ 0.4 & 0.3 & 0.4 \\ 0.4 & 0.3 & 0.4 \end{bmatrix}$$  \hspace{1cm} (5.85)
Further composition over $P^3$, to get $P^4$,

$$P^4 = P^3 \circ P = \begin{bmatrix}
0.3 & 0.3 & 0.3 \\
0.4 & 0.3 & 0.4 \\
0.4 & 0.3 & 0.4
\end{bmatrix}$$ \hspace{1cm} (5.86)

Also, further compositions over $P^4$ and so on will yield the same compositional matrix as in equation (5.86).

Applying step 4 of proposed algorithm 4 to the compositional matrices in equations (5.85), (5.86), it can be observed that elemental membership values remain same in all entries from compositional matrix $P^3$ onwards without any variations. This shows the stable aperiodic nature of the given fuzzy system. The conclusion obtained using the proposed algorithm 4 is in agreement with Ibrahim (1999).

5.4 STABILIZATION AND STABILITY ANALYSIS OF AN INVERTED PENDULUM MOTION USING FUZZY LOGIC CONTROLLER

Most real life physical systems are actually non-linear systems. Conventional design approaches use different approximation methods to handle non-linearity. A linear approximation technique is simple, but it tends to limit control performance. A piecewise linear technique is better but it is tedious to implement because it often requires the design of several linear controllers. A look up table process improves control performance, but it is difficult to debug and tune. Further, in complex systems where multiple inputs exist, a look up table is impractical and costlier due to its large memory requirements. As a result, fuzzy logic controller (FLC) provides an alternative solution to non-linear control because it is closer to the real world. Non-linearity is handled by rules, membership functions, and the inference
process which results in improved performance, simpler implementation and reduced design costs. In this section, a fuzzy logic controller is designed with a new set of rules for making the given non-linear inverted pendulum model completely stable. Also, the output membership values obtained from the designed FLC based inverted pendulum is utilized for inferring its stability nature using the algorithms discussed in section 5.3.

5.4.1 Overview of Inverted Pendulum (IP) Model

The non-linear system to be stabilized consists of the cart and a rigid pendulum hinged to the top of the cart. The cart is free to move left or right on a straight bounded track and the pendulum can swing in the vertical plane determined by the track. Figure 5.13 shows the cart with an inverted pendulum (Passino and Yurkovich 1998). The dynamical equations of the linearized model (Ross 1997) are,

\[
(I_p + (m_p)l^2)\ddot{\theta} - (m_p)gl\theta = (m_p)l\ddot{x}
\]

\[
(M_p + (m_p))\ddot{x} + b\dot{x} - (m_p)l\dot{\theta} = u
\]

(5.87)

where, $M_p$ - mass of the cart
$m_p$ - mass of the pendulum
$b$ - friction of the cart
$l$ - length to pendulum centre of mass
$I_p$ - inertia of the pendulum
$x$ - cart position coordinate
$\theta$ - pendulum angle from vertical position
$\dot{x}$ - velocity of the cart
$\dot{\theta}$ - angular velocity of the pendulum
$g$ - gravitational acceleration
$u$ - input to the pendulum

and $F$ - force applied to the cart
The control force $F$ is applied to the cart to prevent the pendulum from falling while keeping the cart within the specified bounds on the track. It is assumed that $M_p = 0.5$ kg, $m_p = 0.2$ kg, $b = 0.1$ M/m/sec, $l = 0.3$ m, $I_p = 0.006$ kgm$^2$ and $g = 9.81$ m/s$^2$. From equation (5.87), the transfer function of the pendulum model can be deduced and is given below:

$$\frac{\theta(s)}{u(s)} = \frac{\frac{(m_p)l}{q_p}}{\frac{s^2 + b(I_p + m_pl^2)}{q_p} - \left(\frac{(M_p + m_p)(m_pl)}{q_p}\right)s - \frac{bm_pg}{q_p}}$$

(5.88)

where, $q_p = \left\{\left(M_p + m_p\right)(I_p + m_pl^2) - (m_pl)^2\right\}$.

On substituting the values of parameters in equation (5.88), the transfer function of the linearized inverted pendulum model becomes,

$$\frac{\theta(s)}{u(s)} = \frac{4.5455s}{s^2 + 0.1818s^2 - 31.1818s - 4.4545}$$

(5.89)

Equation (5.89) represents an inverted pendulum model and is further used along with the fuzzy logic controller for stabilization process.

![Figure 5.13 Model of an inverted pendulum](image-url)
Inverted pendulum with non-linearity is a very good example for control engineers to verify modern control theory. In fact, stabilization of the inverted pendulum is also a model for the attitude control of a space booster rocket and a satellite, an automatic aircraft landing system, aircraft stabilization in the turbulent air-flow, stabilization of a cabin in a ship, a biped locomotion system, stabilization of nuclear fuel rods in a reactor and so on. For the inverted pendulum represented by equation (5.89), unit step response was applied to its open loop system as shown in Figure 5.14, and the plot shown in Figure 5.15 is obtained. The response curve in Fig 5.15 shows the system is unstable. Thus a FLC is designed with a set of rules to completely stabilize the pendulum model as well as to analyze its nature of stability using the output membership values of the designed FLC.

\[
G(s) = \frac{4.5455s}{s^3 + 0.1818s^2 - 31.1818s - 4.4545}
\]

**Figure 5.14** Open loop inverted pendulum model

**Figure 5.15** Step response of inverted pendulum represented by equation (5.89)
5.4.2 Design of Fuzzy Logic Controller for the inverted pendulum

The fuzzy logic controller is to be designed for stabilizing the inverted pendulum model. Fuzzy Logic Control describes the algorithm for plant control as a fuzzy relation between information on the condition of the plant to be controlled and the control action. It is thus distinguished from conventional control algorithms, since information (linguistic or fuzzy model) about the system is needed rather than its mathematical model. A simple fuzzy logic control system is shown in Figure 5.16.

![Figure 5.16 A simple fuzzy logic control system](image)

The basic steps involved in designing a simple fuzzy logic controller (Ross 1997) is,

i) Identify the inputs and outputs using linguistic variables

ii) Assign membership functions to the variables

iii) Build a rule base

iv) Generate a crisp control action (defuzzification)
Using the basic concept of fuzzy control system, the block diagram of fuzzy logic controller based inverted pendulum is as shown in Figure 5.17.

![Diagram of Fuzzy Logic Controller for Inverted Pendulum]

**Figure 5.17 Fuzzy logic controller for an inverted pendulum on a cart**

The fuzzy logic controller is constructed by considering the angular position and angular velocity of the pendulum as conditional variables and the force as reaction variable i.e., the FLC is a two input, single output controller described by,

\[ Q = \Phi(e, \dot{e}) \]  \hspace{1cm} (5.90)

where \( e \) and \( \dot{e} \) denote error and change of error respectively and \( Q \) is the output of the controller.

The membership functions are defined for the input \((e, \dot{e})\) and output \((Q)\) variables using the linguistic variables. For input and output variables, seven linguistics - negative large (NL), negative medium (NM), negative small (NS), zero (ZE), positive small (PS), positive medium (PM) and positive large (PL) are assigned. The membership plot of \( e, \dot{e} \) and \( Q \) are as shown in Figure 5.18.
On defining the membership functions, fuzzy rule base is formed in a Fuzzy Associative Memory (FAM) table as shown in Table 5.15. The rule base is formulated in the FAM table shown in Table 5.15 using IF-THEN statements with two input variables and one output variable as follows:

- IF $e = NL$ and $\dot{e} = NL$ THEN $Q = NL$
- IF $e = NL$ and $\dot{e} = NM$ THEN $Q = NL$
- IF $e = NL$ and $\dot{e} = ZE$ THEN $Q = NM$

**Figure 5.18 Membership functions of $e$, $\dot{e}$ and $Q$**
and so on. In this manner, the rule base is created and FAM table is formulated. The inference from Table 5.15 gives the output membership values $Q$ of the designed FLC based inverted pendulum system.

**Table 5.15 Control rules (FAM Table)**

<table>
<thead>
<tr>
<th>$\dot{e}$</th>
<th>NL</th>
<th>NM</th>
<th>NS</th>
<th>ZE</th>
<th>PS</th>
<th>PM</th>
<th>PL</th>
</tr>
</thead>
<tbody>
<tr>
<td>NL</td>
<td>NL</td>
<td>NL</td>
<td>NL</td>
<td>NM</td>
<td>NS</td>
<td>NS</td>
<td>ZE</td>
</tr>
<tr>
<td>NM</td>
<td>NL</td>
<td>NM</td>
<td>NM</td>
<td>NM</td>
<td>NS</td>
<td>ZE</td>
<td>ZE</td>
</tr>
<tr>
<td>NS</td>
<td>NM</td>
<td>NM</td>
<td>NS</td>
<td>NS</td>
<td>ZE</td>
<td>ZE</td>
<td>PS</td>
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<tr>
<td>ZE</td>
<td>NS</td>
<td>NS</td>
<td>ZE</td>
<td>ZE</td>
<td>PS</td>
<td>PS</td>
<td></td>
</tr>
<tr>
<td>PS</td>
<td>NS</td>
<td>ZE</td>
<td>ZE</td>
<td>PS</td>
<td>PS</td>
<td>PM</td>
<td>PM</td>
</tr>
<tr>
<td>PM</td>
<td>ZE</td>
<td>ZE</td>
<td>PS</td>
<td>PM</td>
<td>PM</td>
<td>PM</td>
<td>PL</td>
</tr>
<tr>
<td>PL</td>
<td>ZE</td>
<td>PS</td>
<td>PM</td>
<td>PM</td>
<td>PL</td>
<td>PL</td>
<td>PL</td>
</tr>
</tbody>
</table>

Table 5.15 indicates the set of 49 rules which makes the given inverted pendulum model to be stable. From Table 5.15, on evaluating the rules, the relational matrix corresponding to the output $Q$, representing membership values is obtained as,

$$Q = \begin{bmatrix}
1.0 & 0.4 & 0.4 & 0.4 & 0.3 & 0.1 & 0 \\
0.1 & 1.0 & 0.5 & 0.8 & 0.3 & 0.4 & 0.2 \\
0.4 & 0.4 & 1.0 & 0.3 & 0.3 & 0.2 & 0.1 \\
0.3 & 0.3 & 0.3 & 1.0 & 0.3 & 0.3 & 0.3 \\
0.2 & 0.3 & 0.3 & 0.4 & 1.0 & 0.4 & 0.1 \\
0.1 & 0.2 & 0.4 & 0.8 & 1.0 & 0.3 & 0.1 \\
0 & 0.1 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0 \\
\end{bmatrix}$$

(5.91)
The elements in the Q matrix are evaluated as follows:

For example, the first row, second column element, is based on the rule,

IF $e = \text{NL}$ and $\dot{e} = \text{NM}$ THEN $Q = \text{NL}$

On evaluating this rule, the membership values are,

At $e = \text{NL}$, $\mu_e = 0.4$ and
At $\dot{e} = \text{NM}$, $\mu_{\dot{e}} = 1$

Here, min rule is used for evaluation of the rules, thus membership value of the output $Q$ at this instant becomes,

$$\mu_Q = \min(\mu_e, \mu_{\dot{e}}) = \min(0.4, 1) = 0.4$$

This process is performed for evaluating all the rules and the entire elements of output matrix Q are computed.

This output controlled force Q matrix obtained from fuzzy logic controller is further used to analyze the nature of stability of FLC based inverted pendulum in the next section. Correspondingly, the defuzzified output from FLC is given to the plant inverted pendulum as shown in Figure 5.17. On applying unit step input to the FLC based inverted pendulum, the plot shown in Figure 5.19 is obtained, which shows that the plant represented by equation (5.89) is stable for the given input signal.
The Fuzzy logic controller was designed in MATLAB 7.0 (Alberto Cavallo et al 1996) environment using Mamdani Fuzzy Inference system Editor and the control surface of the fuzzy controller as shown in Figure 5.20 is obtained for the response shown in Figure 5.19. The surface represents the information about the output $Q$ based on $e$ and $\dot{e}$ in the fuzzy controller. Appendix 6 provides the program of Mamdani Fuzzy Inference System developed for inverted pendulum model.
The rules formulated for this fuzzy inference system is as shown in Figure 5.21.

![Figure 5.21 Fuzzy inference rules designed for inverted pendulum model](image)

The MATLAB – SIMULINK model for stabilization of inverted pendulum using FLC is as shown in Figure 5.22. Thus, fuzzy logic controller using its membership functions and rules has made the inverted pendulum model stable.

![Figure 5.22 MATLAB-SIMULINK model for Fuzzy Logic Controller based inverted pendulum](image)
5.4.3 Analyzing Stability Nature of Inverted Pendulum Motion

The various algorithms proposed in section 5.3 are utilized for analyzing the stability nature of FLC based inverted pendulum defined by its output relational matrix in equation (5.91).

5.4.3.1 Stability analysis using step response of compositional matrices

The algorithm proposed in section 5.3.3 is applied over the output relational matrix given in equation (5.91) representing the FLC based inverted pendulum. The steps involved in analyzing nature of stability for inverted pendulum is as follows:

Step 1: Read the output relational matrix $Q$ in equation (5.91) and normalize it using the product of number of rows and number of columns. On scaling $Q$ is equation (5.91), $Q_0$ is obtained as,

$$Q_0 = \begin{bmatrix}
0.020408 & 0.0081633 & 0.0081633 & 0.0081633 & 0.0061244 & 0.0020408 & 0 \\
0.0020408 & 0.020408 & 0.010204 & 0.016327 & 0.0061244 & 0.0081633 & 0.0040816 \\
0.0081633 & 0.0081633 & 0.020408 & 0.0061224 & 0.0061224 & 0.0061224 & 0.0061224 \\
0.0020408 & 0.0040816 & 0.0040816 & 0.0081633 & 0.020408 & 0.0081633 & 0.0020408 \\
0.0061224 & 0.0061224 & 0.0061244 & 0.0020408 & 0.0061224 & 0.0061224 & 0.0061224 \\
0.0040816 & 0.0061224 & 0.0061244 & 0.0081633 & 0.020408 & 0.0081633 & 0.0020408 \\
0.0020408 & 0.0040816 & 0.0040816 & 0.0081633 & 0.016327 & 0.020408 & 0.0061244 \\
0 & 0.0020408 & 0.0061224 & 0.0081633 & 0.0081633 & 0.0081633 & 0.0020408
\end{bmatrix}$$

(5.92)

Step 2: Formulating the characteristic polynomial for equation (5.92),

$$C(z) = z^7 - 0.14286 z^6 + 0.0078676 z^5 - 0.00022387 z^4 + 3.6147 e^{-06} z^3 - 3.3429 e^{-08} z^2 + 1.6489 e^{-10} z - 3.3616 e^{-13}$$

(5.93)

Step 3: Assuming an open loop all pole system of $C(z)$ in equation (5.93),

$$Y(z) = \frac{1}{C(z)} = \frac{1}{z^7 - 0.14286 z^6 + 0.0078676 z^5 - 0.00022387 z^4 + 3.6147 e^{-06} z^3 - 3.3429 e^{-08} z^2 + 1.6489 e^{-10} z - 3.3616 e^{-13}}$$

(5.94)
Step 4: Obtaining the unit step response of \( Y(z) \) in equation (5.94) as shown in Figure 5.23 and observing its amplitude,

\[
A(Q_s) = 1.1563
\]  

(5.95)

![Figure 5.23 Unit step response of \( Y(z) \) in equation (5.94)](image)

Step 5: Repeating steps 2 to 4 for compositions obtained using \( Q_s \).

The first composition obtained using \( Q_s \) is,

\[
Q_s^2 = Q_s \cdot Q_s
\]

\[
= \begin{bmatrix}
0.020408 & 0.0081633 & 0.0081633 & 0.0081633 & 0.0061224 & 0.0081633 & 0.0061244 \\
0.0081633 & 0.020408 & 0.010204 & 0.016327 & 0.0081633 & 0.0081633 & 0.0061244 \\
0.0081633 & 0.0081633 & 0.020408 & 0.0081633 & 0.0061244 & 0.0081633 & 0.0061244 \\
0.0061244 & 0.0061244 & 0.0061244 & 0.020408 & 0.0061244 & 0.0061244 & 0.0061244 \\
0.0061244 & 0.0061244 & 0.0061244 & 0.0081633 & 0.020408 & 0.0081633 & 0.0061244 \\
0.0061244 & 0.0061244 & 0.0061244 & 0.0081633 & 0.016327 & 0.020408 & 0.0061244 \\
0.0061244 & 0.0061244 & 0.0061244 & 0.0081633 & 0.0081633 & 0.0081633 & 0.020408 \\
\end{bmatrix}
\]

(5.96)

Forming the characteristic polynomial of \( Q_s^2 \) in equation (5.96) and assuming an open loop all pole system, \( Y(z) \) in equation (5.97) is obtained.
\[
Y(z) = \frac{I}{C(z)} = \frac{I}{z^2 - 0.14286z^6 + 0.0075594z^5 - 0.00020524z^4 + 3.1757e-06z^3 - 2.8417e-08z^2 + 1.3733e-10z - 2.7801e-13}
\]

(5.97)

Plotting unit step response for equation (5.97) as shown in Figure 5.24 and observing its peak amplitude,

\[A(Q^2) = 1.1567\]

(5.98)

![Figure 5.24 Unit step response of Y(z) in equation (5.97)](image)

The process is repeated for other compositional matrices and peak amplitude observed in all cases is tabulated as shown in Table 5.16.
Table 5.16  Peak amplitudes for compositional matrices of Fuzzy Logic Controller based inverted pendulum model

<table>
<thead>
<tr>
<th>Peak Amplitude of output response for unit step input</th>
<th>( A(Q_s) )</th>
<th>( A(Q_s^2) )</th>
<th>( A(Q_s^3) )</th>
<th>( A(Q_s^4) )</th>
<th>( A(Q_s^5) )</th>
<th>( A(Q_s^6) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1563</td>
<td>1.1567</td>
<td>1.1568</td>
<td>1.1568</td>
<td>1.1568</td>
<td>1.1568</td>
<td>( \ldots )</td>
</tr>
</tbody>
</table>

From Table 5.16, it can be noted that the peak value of response curves for compositional matrices from \( Q_s^3 \) onwards remains the same showing stable aperiodic nature of the designed FLC based inverted pendulum model. The process can be stopped if the required observation is met. The proposed approach is simple and straight forward in application as well as the stability information is depicted easily compared to the energy stability criterion in Kiszka et al (1985).

5.4.3.2 Stability analysis using necessary conditions of compositional matrices

The proposed algorithm in section 5.3.4 is applied to the output relational matrix given in equation (5.91) of FLC based inverted pendulum. The steps involved in analyzing its stability nature are as follows:

The output relation matrix in equation (5.91) is scaled and the characteristic polynomial in equation (5.93) is obtained. Evaluating \( C(z) \) in equation (5.93) as per proposed algorithm in section 5.3.4, the values are,

\[
C(0) = -3.3616 \times 10^{-13}
\]

\[
C(1) = 0.86479
\]

\[
C(-1) = -1.1510
\]  

(5.99)
Similarly, evaluating \( C(z) \) obtained for the first compositional matrix \( Q_s^2 \) given in equation (5.96) at \( z = 0, 1 \) and -1, the values are,

\[
\begin{align*}
C(0) &= -2.7801 \times 10^{-13} \\
C(1) &= 0.8645 \\
C(-1) &= -1.1506
\end{align*}
\]

(5.100)

The process is repeated for other compositional matrices and the evaluation of respective characteristic equations are carried out at \( z = 0, 1 \) and -1. The values corresponding to each compositional matrix are tabulated as shown in Table 5.17.

**Table 5.17 Evaluation of \( C(z) \) for each compositional matrices at \( z = 0, 1 \) and -1 of Fuzzy Logic Controller based inverted pendulum**

<table>
<thead>
<tr>
<th>Compositional Matrices</th>
<th>( C(0) )</th>
<th>( C(1) )</th>
<th>( C(-1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( Q_s )</td>
<td>-3.3616e-13</td>
<td>0.86479</td>
<td>-1.1510</td>
</tr>
<tr>
<td>( Q_s^2 )</td>
<td>-2.7801e-13</td>
<td>0.8645</td>
<td>-1.1506</td>
</tr>
<tr>
<td>( Q_s^3 )</td>
<td>-2.7066e-13</td>
<td>0.86448</td>
<td>-1.1506</td>
</tr>
<tr>
<td>( Q_s^4 )</td>
<td>-2.7066e-13</td>
<td>0.86448</td>
<td>-1.1506</td>
</tr>
<tr>
<td>( Q_s^5 )</td>
<td>-2.7066e-13</td>
<td>0.86448</td>
<td>-1.1506</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
<td>.</td>
<td>.</td>
</tr>
</tbody>
</table>

From Table 5.17, it can be observed that the values of \( C(0), C(1) \) and \( C(-1) \) for compositional matrices \( Q_s^3 \) onwards settles at the same value showing stable aperiodic nature of the designed FLC based inverted pendulum model. On comparing this inference with that of the unit step response of the closed loop system having the inverted pendulum along with
the FLC shown in Figure 5.19, the aperiodic stability of the system is validated.

### 5.4.3.3 Stability analysis using trace and determinant

The algorithm proposed in section 5.3.5 is applied over the output relational matrix of the FLC based inverted pendulum given in equation (5.91).

As per the proposed algorithm, computing trace and determinant of the output relational matrix in equation (5.91), the values are,

\[ \text{Trace of } Q = T(Q) = 7 \]
\[ \text{Determinant of } Q = D(Q) = 0.22799 \quad (5.101) \]

Performing the first composition for \( Q \) in equation (5.91), \( Q^2 \) is computed as,

\[
Q^2 = \begin{bmatrix}
1.0 & 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 \\
0.4 & 1.0 & 0.5 & 0.8 & 0.4 & 0.4 & 0.3 \\
0.4 & 0.4 & 1.0 & 0.4 & 0.3 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 1.0 & 0.3 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 1.0 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.8 & 1.0 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0
\end{bmatrix}
\quad (5.102)
\]

Computing trace and determinant of \( Q^2 \) in equation (5.102),

\[ T(Q^2) = 7 \]
\[ D(Q^2) = 0.18856 \quad (5.103) \]
The process is repeated for other compositional matrices \((Q^3, Q^4, \ldots)\) and the respective trace and determinant for each compositional matrices are evaluated and tabulated as shown in Table 5.18.

### Table 5.18 Trace and determinant for compositional matrices of Fuzzy Logic Controller based inverted pendulum

<table>
<thead>
<tr>
<th>Compositional Matrices</th>
<th>Trace</th>
<th>Determinant</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Q)</td>
<td>7</td>
<td>0.22799</td>
</tr>
<tr>
<td>(Q^2)</td>
<td>7</td>
<td>0.18856</td>
</tr>
<tr>
<td>(Q^3)</td>
<td>7</td>
<td>0.18357</td>
</tr>
<tr>
<td>(Q^4)</td>
<td>7</td>
<td>0.18357</td>
</tr>
<tr>
<td>(Q^5)</td>
<td>7</td>
<td>0.18357</td>
</tr>
<tr>
<td>(\cdot)</td>
<td>(\cdot)</td>
<td>(\cdot)</td>
</tr>
<tr>
<td>(\cdot)</td>
<td>(\cdot)</td>
<td>(\cdot)</td>
</tr>
<tr>
<td>(\cdot)</td>
<td>(\cdot)</td>
<td>(\cdot)</td>
</tr>
</tbody>
</table>

From Table 5.18, it can be observed that the trace and determinant values from compositional matrices \(Q^3\) onwards remains the same, declaring the designed FLC based inverted pendulum model is aperiodically stable. The proposed algorithmic approach is simple and the stability information is obtained easily compared to the energy stability criterion in Kiszka et al (1985).

### 5.4.3.4 Stability analysis using the elements of compositional matrices

The proposed algorithm in section 5.3.6 is applied over the relational matrix of the FLC based inverted pendulum given in equation (5.91).
As per the proposed algorithm performing the composition to relational matrix $Q$ in equation (5.91),

$$Q^2 = Q \circ Q$$

$$= \begin{bmatrix}
1.0 & 0.4 & 0.4 & 0.4 & 0.3 & 0.4 & 0.3 \\
0.4 & 1.0 & 0.5 & 0.8 & 0.4 & 0.4 & 0.3 \\
0.4 & 0.4 & 1.0 & 0.4 & 0.3 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 1.0 & 0.3 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 1.0 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.8 & 1.0 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0
\end{bmatrix} \quad (5.104)$$

On performing next composition over $Q^2$,

$$Q^3 = Q^2 \circ Q$$

$$= \begin{bmatrix}
1.0 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.3 \\
0.4 & 1.0 & 0.5 & 0.8 & 0.4 & 0.4 & 0.4 & 0.3 \\
0.4 & 0.4 & 1.0 & 0.4 & 0.4 & 0.4 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 1.0 & 0.3 & 0.3 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 1.0 & 0.4 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.8 & 1.0 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0 & 0.3
\end{bmatrix} \quad (5.105)$$

Further composition over $Q^3$,

$$Q^4 = Q^3 \circ Q$$

$$= \begin{bmatrix}
1.0 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.4 & 0.3 \\
0.4 & 1.0 & 0.5 & 0.8 & 0.4 & 0.4 & 0.4 & 0.3 \\
0.4 & 0.4 & 1.0 & 0.4 & 0.4 & 0.4 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 1.0 & 0.3 & 0.3 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 1.0 & 0.4 & 0.4 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.8 & 1.0 & 0.3 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0 & 0.3 \\
0.3 & 0.3 & 0.3 & 0.4 & 0.4 & 0.4 & 1.0 & 0.3
\end{bmatrix} \quad (5.106)$$
Also, further compositions over \( Q^4 \) and so on will yield the same compositional matrix as in equation (5.106). Applying step 4 of proposed algorithm in section 5.3.6 to the compositional matrices from \( Q^3 \) onwards, it can be observed that elemental membership values remain the same in all entries without any variations. This shows that the designed FLC based inverted pendulum model is aperiodically stable.

**Note:** It should be noted that the inferences made by applying the four proposed algorithms over the relational matrix of FLC based inverted pendulum remains the same and declares the system is aperiodically stable. Amidst, all the four algorithms, the fourth algorithm which is an intuitive criterion and inspection test is the simplest.

### 5.5 SUMMARY

In this Chapter, simple algebraic procedures are proposed for analyzing the stability nature of the given fuzzy system represented by its system matrices. Also, certain sufficient conditions are derived for inferring instability situations of the given fuzzy system. Four algorithms are proposed using compositional matrices for performing stability investigations over the fuzzy systems represented by its relational matrices. The advantage of these algorithms are, they are simple and straight forward in application for studying the stability nature of fuzzy systems involving minimum number of computations. The sample program for the proposed algorithms using compositional matrices is given in Appendix 5. Further, a fuzzy logic controller is designed for stabilizing the non-linear motion of the inverted pendulum and stability analysis is carried out for the FLC based inverted pendulum using the proposed algorithms. The forthcoming Chapter 6 discusses on the lower order modeling of a system given its higher order model employing Particle Swarm Optimization and auxiliary polynomial approach discussed in Chapter 2.