Chapter 2

Splitting Off Operation for Graphs and its Applications

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In order to generalize any graph theoretic result to matroids one has to characterize that result in terms of the cycles of the graphs. In this chapter, we characterize cycles and spanning trees of the new graph obtained by applying splitting off operation on the given graph in terms of the cycles and spanning trees of the original graph. We also give some applications of this operation.

1. Introduction

Let $G$ be a connected graph and let $v$ be a vertex of degree at least three in $G$. Given incident edges $x = vv_1$ and $y = vv_2$ in a graph $G$, we can construct a new graph $G_{xy}$ by adding the edge $v_1v_2$ and deleting $x$ and $y$. We say that $G_{xy}$ is obtained from $G$ by splitting off $x$ and $y$. If $v_1 = v_2$ then the resulting loop is deleted. For practical purposes, we denote the new edge $v_1v_2$ in $G_{xy}$ by $a$. The transition from $G$ to $G_{xy}$ is called the splitting off operation. We can retrieve the graph $G$ from $G_{xy}$ by subdividing the edge $a$ by introducing a new vertex say $v'$ and then identifying $v$ and $v'$. The Figure 1 illustrates this construction explicitly.

Splitting off operation was introduced by Lovasz [13] and has important applications in graph theory ([11],[12]). The splitting off operation
is the useful method for solving problems in graph connectivity. This operation may decrease the edge-connectivity of the graph. The essence of the edge-splitting method is to find a pair of edges which can be splitting off maintaining the edge-connectivity properties of the graph. Let $G = (V + s, E)$ be $k$-edge connected in $V$ ($k \geq 2$) and let $d(s)$ be even. Lovasz [13] proved that for every edge $su$ there exists an edge $sv$ for which splitting off the pair $su, sv$ maintains $k$-edge connectivity. If such a good pair exists then one may reduce the problem to a smaller graph which can lead to inductive proofs.

Another typical application is the edge connectivity augmentation problems. Using the edge splitting results and the $g$-polymatroid intersection theorem of Frank [6], Tibor Jordan [11] gave a min-max theorem and a polynomial algorithm for the simultaneous edge-connectivity augmentation problems. In this problem two graphs $G' = (V, E')$, $H' = (V, K')$ and two integers $k, l \geq 2$ are given and the goal is to find a smallest set of new edges whose addition makes $G'$ (and $H'$) $k$-edge connected ($l$-edge connected, respectively) simultaneously. This algorithm finds a feasible solution whose size does not exceed the optimum by more than one. If $k$ and $l$ are both even then the solution is optimal.
2. Cycles in $G_{xy}$

In the following proposition, we characterize the cycles of the graph $G_{xy}$ in terms of the cycles of the graph $G$.

**Proposition 2.1.** Let $G$ be a graph and $x = vv_1, y = vv'_1$ be a pair of adjacent edges in $G$. Let $G_{xy}$ be a graph obtained from $G$ by applying splitting off operation with respect to the pair $\{x, y\}$. Then $C$ is a cycle in $G_{xy}$ if and only if one of the following conditions hold:

1. $C$ is a cycle of $G$ containing neither of the edges $x$ and $y$.
2. $C = (C' - \{x, y\}) \cup \{v_1v'_1\}$, where $C'$ is a cycle of $G$ containing $x$ and $y$.
3. $C = ((C_1 \cup C_2) - \{x, y\}) \cup \{v_1v'_1\}$, where $C_1$ and $C_2$ are disjoint cycles of $G$ each containing precisely one of $x$ and $y$, and $C_1 \cup C_2$ contains no cycle of $G$ containing both $x$ and $y$, or neither.

**Proof. Sufficiency.** Let $C$ be a cycle of $G$ containing neither $x$ nor $y$. Then clearly $C$ is a cycle of $G_{xy}$.

Now, suppose $C = (C' - \{x, y\}) \cup \{v_1v'_1\}$, where $C'$ is a cycle of $G$ containing both $x$ and $y$. For convenience, let $C': vxv_1v_2...e_{i-1}v_ie_i...v'_1vy$. By applying the splitting off operation we get the trail $(C' - \{x, y\}) \cup \{v_1v'_1\}$. We prove that $(C' - \{x, y\}) \cup \{v_1v'_1\}$ is a cycle in $G_{xy}$. On the contrary, assume that $(C' - \{x, y\}) \cup \{v_1v'_1\}$ is not a cycle in $G_{xy}$. Then $(C' - \{x, y\}) \cup \{v_1v'_1\}$ is a closed walk in which some vertex is repeated. This vertex must be repeated in $C'$, a contradiction to the fact that $C'$ is a cycle. Therefore, $(C' - \{x, y\}) \cup \{v_1v'_1\}$ is a cycle in $G_{xy}$. 
Next, suppose \( C = ((C_1 \cup C_2) - \{x, y\}) \cup \{v_1v'_1\} \), where \( C_1 \) and \( C_2 \) are disjoint cycles of \( G \), each containing precisely one of \( x \) and \( y \), and \( C_1 \cup C_2 \) contains no cycle of \( G \) containing both \( x \) and \( y \), or neither.

For convenience, let \( C_1 : \ v x v_1 e_1 v_2 \ldots e_{i-1} v_i e_i \ldots e_n v. \)

\( C_2 : \ v y v'_1 e'_1 v'_2 \ldots e'_{j-1} v'_j e'_j \ldots e'_k v. \) By applying the splitting off operation, the set \(( (C_1 \cup C_2) - \{x, y\}) \cup \{v_1v'_1\} \) will result into the sequence:

\[ v e_n v_n e_{n-1} \ldots v_{i+1} e_i v_i e_i \ldots v_2 e_1 v_1 v'_1 e'_1 v'_2 \ldots e'_{j-1} v'_j e'_j \ldots e'_k v. \]

Suppose now \( C_1 \cup C_2 \) does not contain a cycle of type (1) or (2) then we prove that \(( (C_1 \cup C_2) - \{x, y\}) \cup \{v_1v'_1\} \) is a cycle of \( G_{xy} \).

On the contrary, assume that \(( (C_1 \cup C_2) - \{x, y\}) \cup \{v_1v'_1\} \) is not a cycle in \( G_{xy} \). Then it is a closed walk in which some vertex is repeated. Thus some vertex \( v_i \) in \( C_1 \) must be the same as the vertex \( v'_j \) in \( C_2 \) (vertices in \( C_1 \) and also in \( C_2 \) cannot be repeated). But then the sequence \( C''' : \ v e_n v_n e_{n-1} v_{i+1} e_i v_i e_i \ldots v_2 e_1 v_1 v'_1 e'_1 v'_2 \ldots e'_{j-1} v'_j e'_j \ldots e'_k v \) is contained in \( C_1 \cup C_2 \) and contains a cycle of type (1). This is a contradiction to the assumption. Therefore \(( (C_1 \cup C_2) - \{x, y\}) \cup \{v_1v'_1\} \) is a cycle of \( G_{xy} \).

**Necessity.** Suppose \( C \) is a cycle of \( G_{xy} \). By definition of \( G_{xy} \), \( x, y \notin E(G_{xy}) \). Hence \( E(G) \cap \{x, y\} = \phi \).

(i) If \( C \) does not contain the edge \( v_1v'_1 \), then \( C \) is a cycle of \( G \). i.e. a cycle of type (1).

(ii) Suppose that \( C \) contains the edge \( v_1v'_1 \). Then \(( C \cup \{x, y\}) - v_1v'_1 = C' \) is a cycle in \( G \) containing \( x \) and \( y \) which is of type (2).

(iii) \( C \) contains neither \( x \) nor \( y \) but contains an edge \( \{v_1v'_1\} \) and the vertex \( v \). Then we show that \( C \) is a cycle of type (3).

Let \( C : \ v'_1 v_1 e_1 v_2 e_2 \ldots v_i e_i \ldots v_n v'_1. \) Let \( v_i = v \) for some vertex \( v_i \in V(C) \).

Now, subdivide the edge \( v_1v'_1 \) so that \( v''v_1 = x \) and \( v''v'_1 = y \), where \( v'' \)
is a new vertex of degree 2. Then decompose the cycle $C$ into two parts as $C_1: v''v_1e_1...e_{i-1}v_i = v$ and $C_2: v = v_ie_i+v_{i+1}e_{i+1}...v_{n-1}v_nv_1'v''$. Hence $C_1: x = v''v_1e_1...e_{i-1}v_i = v$ and $C_2: v = v_ie_i+v_{i+1}e_{i+1}...v_{n-1}v_nv_1'v'' = y$.

Now, identify the vertices $v''$ and $v$ so that $C_1$ and $C_2$ are edge disjoint cycles in $G$ such that $x \in C_1$, $y \in C_2$.

We show that $C_1 \cup C_2$ does not contain any cycle of type (1) or (2). On the contrary, suppose that $C_1 \cup C_2$ contains a cycle say $C^{iv}$ of type (2) say. Then $x, y \in C^{iv} \subseteq C_1 \cup C_2$. Let $C^{iv}: v_nv_1v_1e_1v_2...v_{k-1}e_{k-1}v_k(=v_m)e_{m+1}v_{m+1}...v_{n-1}e_{n-1}v_n$. But then the cycle $C^{vi}: v_nv_1v_1e_1v_2...v_{k-1}e_{k-1}v_ke_{m+1}...v_{n-1}e_{n-1}v_n$ of $G_{xy}$ will be properly contained in $C$ which is a contradiction to the fact that $C$ is a cycle in $G_{xy}$. Suppose that $C_1 \cup C_2$ contains a cycle say $C^{vi}$ of type (1). Then $x, y \notin C^{vi} \subseteq C_1 \cup C_2$. Let $C^{vi}: v_nv_1v_1e_1v_2...v_{k-1}e_{k-1}v_k(=v_m)e_{m+1}v_{m+1}...v_{n-1}e_{n-1}v_n$. But then the cycle $C^{vi}$ will be properly contained in $C$ which is a contradiction to the fact that $C$ is a cycle of $G_{xy}$. This completes the proof of the proposition.

If $C$ denotes the set of all cycles of $G_{xy}$, and $C_0$, $C_1$, $C_2$ denote the set of cycles of $G_{xy}$ of type (1),(2) and (3) respectively, then $C = C_0 \cup C_1 \cup C_2$.

### 3. Spanning Trees in $G_{xy}$

In this section, we characterize the spanning trees of the graph $G_{xy}$ in terms of the spanning trees of the graph $G$. We consider two cases depending on whether $\{x, y\}$ forms a cutset in $G$ or not. Firstly, we prove the following Lemma.
Lemma 3.1. Let $T$ be a spanning tree of a graph $G$ and let $x, y \in E(G)$. If $\{x, y\}$ forms a cut set in $G$, then every spanning tree of $G$ contains at least one of $x$ and $y$.

Proof. On the contrary, suppose that $T$ is a spanning tree of $G$ such that $x, y \notin T$. Let $C$ be a fundamental cycle contained in $T \cup \{x\}$. Then $x \in C$ while $y \notin C$, a contradiction to the fact that $\{x, y\}$ form a cut set in $G$.

Theorem 3.2. Let $T$ be a spanning tree of a graph $G$ and let $x, y \in E(G)$. Suppose $\{x, y\}$ forms a cut set in $G$, $a \in E(G_{xy}) - E(G)$. Then $T'$ is a spanning forest of $G_{xy}$ if and only if one of the following conditions hold:

(i) $T' = T - \{x\}$, where $T$ is a spanning tree of $G$ containing $x$ but not $y$.

(ii) $T' = T - \{y\}$, where $T$ is a spanning tree of $G$ containing $y$ but not $x$.

(iii) $T' = (T - \{x, y\}) \cup \{a\}$, where $T$ is a spanning tree of $G$ containing both $x$ and $y$.

Proof. Let $T$ be a spanning tree of $G$, $x \in E(T)$ and $y \notin E(T)$. We prove that $T' = T - \{x\}$ is a spanning forest of $G_{xy}$ by showing that it is maximal acyclic subgraph of $G_{xy}$.

On the contrary, let $X$ be a cycle of $G_{xy}$ contained in $T'$. Since $a \notin X$, $X$ is a cycle of $G$ containing none of $x$ and $y$, and contained in $T - \{x\}$, a contradiction to the fact that $T$ is a spanning tree of $G$. Thus $T'$ is an acyclic subgraph of $G_{xy}$.

We show that it is maximal acyclic. Let $\alpha \in E(G_{xy}) - T'$. Suppose $\alpha \neq a$. We show that $T' \cup \{\alpha\}$ contains a cycle of $G_{xy}$. Let $C_1$ be the fundamental cycle of $G$ contained in $T \cup \{\alpha\}$. Since $y \notin T$, and $\{x, y\}$ form a cut set, $C_1$ contains none of $x$ and $y$. Therefore, it is a cycle of $G_{xy}$ and $C_1 \subseteq (T - \{x\}) \cup \{\alpha\} = T' \cup \{\alpha\}$. Thus $(T - \{x\}) \cup \{\alpha\}$ contains a cycle...
of $G_{xy}$.

We complete the proof by showing that $(T - \{x\}) \cup \{a\}$ contains a cycle of $G_{xy}$. Let $C$ be a cycle of $G$ contained in $T \cup \{y\}$. Since $y \in C$ and there is no cycle of $G$ containing exactly one of $x$ and $y$; $x, y \in C$. Now $(C - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ that is contained in $(T - \{x\}) \cup \{a\}$. Therefore $T'$ is a spanning forest of $G_{xy}$.

By the similar argument, one can prove that if $T' = T - \{y\}$, where $T$ is a spanning tree of $G$ containing $y$ but not $x$, then $T'$ is a spanning forest of $G_{xy}$.

Next, let $T$ be a spanning tree of $G$ such that $x, y \in T$. We show that $T' = (T - \{x, y\}) \cup \{a\}$ is an acyclic subgraph of $G_{xy}$. On the contrary, let $X$ be a cycle of $G_{xy}$ which is contained in $T'$. Since $a \in X$, and there is no cycle of $G$ containing exactly one of $x$ and $y$, there exists a cycle of $G$, say $C$, such that $x, y \in C$ and $X = (C - \{x, y\}) \cup \{a\}$. But then $C \subseteq T$, and this is a contradiction to the fact that $T$ is a spanning tree of $G$. Hence $T'$ must be acyclic subgraph of $G_{xy}$. Since $|E(T')| = |E(T)| - 1$, $T'$ is a spanning forest of $G_{xy}$.

Conversely, suppose that $T'$ is a spanning forest of $G_{xy}$. We consider the following cases:

Case (i) $a \notin T'$. We show that $T = T' \cup \{x\}$ is a spanning tree of $G$. Since $|E(T)| = |E(T')| + 1$, it is enough to prove that $T$ is acyclic subgraph of $G$. On the contrary, assume that $C$ is a cycle of $G$ contained in $T$. Since $C$ does not contain $x$ and $y$, it is a cycle of $G_{xy}$ that is contained in $T'$, a contradiction. Thus $T$ is a spanning tree of $G$. By the similar argument, one can prove that $T' \cup \{y\}$ is also a spanning tree of $G$.

Case (ii) $a \in T'$. We show that $T = (T' - \{a\}) \cup \{x, y\}$ is acyclic subgraph
of $G$. If not, then it contains a cycle of $G$. Let $C$ be a cycle of $G$ contained in $T$. Then $C$ contains neither or both $x$ and $y$. If $C$ contains none of $x$ and $y$, then it will be a cycle of $G_{xy}$ that is contained in $T'$, a contradiction. If $x, y \in C$, then $(C - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ which is contained in $T'$, again a contradiction. Thus $T$ is a spanning tree of $G$. \hfill $\square$

**Theorem 3.3.** Let $T$ be a spanning tree of a graph $G$, $x, y \in E(G)$. Suppose $\{x, y\}$ does not form a cut set in $G$, $a \in E(G_{xy}) - E(G)$. Then $T'$ is a spanning tree of $G_{xy}$ if and only if one of the following conditions hold:

(i) $T' = T$, $T$ is a spanning tree of $G$ containing none of $x$ and $y$.

(ii) $T' = (T - \{x\}) \cup \{a\}$ where $T$ is a spanning tree of $G$ containing $x$ but not $y$ and $(T - \{x\}) \cup \{y\}$ is not a spanning tree of $G$.

(iii) $T' = (T - \{y\}) \cup \{a\}$ where $T$ is a spanning tree of $G$ containing $y$ but not $x$ and $(T - \{y\}) \cup \{x\}$ is not a spanning tree of $G$.

**Proof.** Let $T$ be a spanning tree of $G$ such that $x, y \notin T$. We prove that $T$ is a spanning tree of $G_{xy}$ by showing that it is a maximal acyclic subgraph of $G_{xy}$.

On the contrary, suppose that $C$ is a cycle of $G_{xy}$ contained in $T$. Since $a \notin C$, $C$ is a cycle of $G$ such that $x, y \notin C$, a contradiction to the fact that $T$ is a spanning tree of $G$. We conclude that $T$ is acyclic subgraph of $G_{xy}$.

We show that for any $\alpha \in E(G_{xy}) - T$, $T \cup \{\alpha\}$ contains a cycle of $G_{xy}$. If $\alpha \neq a$, then $T \cup \{\alpha\}$ contains a cycle of $G$. Let $C$ be a cycle of $G$ contained in $T \cup \{\alpha\}$. Since $x, y \notin C$, $C$ is a cycle of $G_{xy}$ contained in $T \cup \{\alpha\}$. \hfill $\square$
Next, assume that $\alpha = a$, we show that $T \cup \{a\}$ contains a cycle of $G_{xy}$. $T \cup \{x\}$ as well as $T \cup \{y\}$ contains a cycle of $G$. Let $C_1$ and $C_2$ be the fundamental cycles of $G$, contained in $T \cup \{x\}$ and $T \cup \{y\}$, respectively. If $C_1 \cap C_2 \neq \phi$, then there exists a cycle of $G$, say $C$, such that $x, y \in C \subseteq C_1 \cup C_2$. Then $X = (C - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ which is contained in $T \cup \{a\}$. Next, let $C_1 \cap C_2 = \phi$. If there is a cycle of $G$, say $C$, containing either none, or both of $x$ and $y$, such that $C \subseteq C_1 \cup C_2$, then $C$ or $(C - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$, which is contained in $T \cup \{a\}$. If there is no such $C$, then $(C_1 \cup C_2 - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ contained in $T \cup \{a\}$. It means that $T \cup \{a\}$ contains a cycle of $G_{xy}$. Thus $T$ is a spanning tree of $G_{xy}$.

Let $T$ be a spanning tree of $G$ containing exactly one of $x$ and $y$, say $x$, such that $(T - \{x\}) \cup \{y\}$ is not a spanning tree of $G$. We show that $T' = (T - \{x\}) \cup \{a\}$ contains a cycle of $G_{xy}$.

Suppose that $X$ is a cycle of $G_{xy}$ contained in $T'$. Obviously $a \in X$. We consider the following cases:

Case (i) $X = (C - \{x, y\}) \cup \{a\}$ where $C$ is a cycle of $G$ containing $x$ and $y$. Therefore, $C \subseteq T \cup \{y\}$. Since $y \in C$, we have $T \cup \{y\} = T_1 \cup \{x\}$, where $T_1 = (T - \{x\}) \cup \{y\}$ is a spanning tree of $G$, a contradiction.

Case (ii) $X = (C_1 \cup C_2 - \{x, y\}) \cup \{a\}$ where $C_1$ and $C_2$ are disjoint cycles of $G$ such that $x \in C_1$, $y \in C_2$ and $C_1 \cup C_2$ does not contain any cycle of $G$ containing either none, or both $x$ and $y$. Now $(C_1 \cup C_2 - \{x, y\}) \cup \{a\} \subseteq (T - \{x\}) \cup \{a\}$ implies that $C_1 \subseteq T$, a contradiction. Therefore, $T' = (T - \{x\}) \cup \{a\}$ is an acyclic subgraph of $G_{xy}$. Since $|E(T')| = |E(T)|$, $T'$ is a spanning tree of $G_{xy}$.

Conversely, suppose that $T'$ is a spanning tree of $G_{xy}$. We consider the
following cases:

(i) $a \notin T'$. We prove that $T'$ is a spanning tree of $G$ containing none of $x$ and $y$, by showing that it is acyclic in $G$. On the contrary, let $C$ be a cycle of $G$, contained in $T'$. Since $x, y \notin C$, $C$ is a cycle of $G_{xy}$ contained in $T'$, a contradiction to the fact that $T'$ is a spanning tree of $G_{xy}$.

(ii) $a \in T'$. We define $D = T' - \{a\}$ and show that $D$ is acyclic in $G$. If $C$ is a cycle of $G$ contained in $D$, then $x, y \notin C$ shows that $C$ is a cycle of $G_{xy}$ contained in $T'$, a contradiction. Therefore, $D$ is acyclic in $G$.

We show that exactly one of $D \cup \{x\}$ and $D \cup \{y\}$ is acyclic in $G$. We Consider the following two cases:

**Case (i).** Suppose that both contains a cycle of $G$, and let $C_1, C_2$ be the cycles of $G$ such that $C_1 \subseteq D \cup \{x\}$ and $C_2 \subseteq D \cup \{y\}$. Obviously $x \in C_1$ and $y \in C_2$.

If $C_1 \cap C_2 \neq \emptyset$, then there exists a cycle of $G$, say $C$, such that $x, y \in C \subseteq C_1 \cup C_2$. Consequently, $(C - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ contained in $T'$, a contradiction.

Next, assume that $C_1 \cap C_2 = \emptyset$. If there is a cycle of $G$, say $C$, containing either none, or both $x$ and $y$, and contained in $C_1 \cup C_2$, then $C$ or $(C - \{x, y\}) \cup \{a\}$ will be a cycle of $G_{xy}$ contained in $T'$. Otherwise $(C_1 \cup C_2 - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ contained in $T'$, again a contradiction. Therefore, one of $D \cup \{x\}$ and $D \cup \{y\}$ is acyclic in $G$.

**Case (ii).** Suppose that $D \cup \{x\}$ and $D \cup \{y\}$ are both acyclic in $G$. We show that $D \cup \{x, y\}$ is also acyclic. If not, assume that $C$ is a cycle of $G$ contained in $D \cup \{x, y\}$. Since $D \cup \{x\}$ and $D \cup \{y\}$ are acyclic, $x, y \in C$. Then $(C - \{x, y\}) \cup \{a\}$ is a cycle of $G_{xy}$ and contained in $T'$, a
contradiction. Thus exactly one of $D \cup \{x\}$ and $D \cup \{y\}$ is acyclic.

Let $D \cup \{x\}$ be acyclic while $D \cup \{y\}$ contains a cycle of $G$. Then by the first part of the proof, $T = D \cup \{x\} = (T' - \{a\}) \cup \{x\}$ is a spanning tree of $G$. In fact, $T' = (T - \{x\}) \cup \{a\}$ is a spanning tree of $G$ such that $(T' - \{x\}) \cup \{y\}$ is not spanning tree. This completes the proof of the theorem. \hfill \Box

4. Bipartiteness of $G_{xy}$

If the graph $G$ is bipartite then $G_{xy}$ may not be bipartite for some edges $x, y$ of $G$. In the following theorem, we provide the condition under which $G_{xy}$ remains bipartite if $G$ is bipartite.

**Theorem 4.1.** Let $G$ be a graph and $x, y \in E(G)$, $a \in E(G_{xy}) - E(G)$ and suppose $G$ is bipartite. Then $G_{xy}$ is bipartite if and only if $x$ or $y$ is a bridge in $G$.

**Proof.** Suppose $G_{xy}$ is bipartite. Assume that $x$ and $y$ are non-bridges. Let $C_x, C_y$ be cycles of $G$ containing $x, y$ respectively. If $C_x \cap C_y = \phi$, then $C = (C_x \cup C_y) - \{x, y\} \cup \{a\}$ is a cycle of $G_{xy}$ containing $a$. If $C_x \cap C_y \neq \phi$, then by Lemma 2.1, there is a cycle of $G$ containing both $x$ and $y$. Hence $C = C_1 - \{x, y\} \cup \{a\}$ is a cycle in $G_{xy}$. Since $G$ is bipartite, each cycle of $G$ is of even size. Hence each cycle of $G_{xy}$ containing $a$ is of odd size. Hence $G_{xy}$ is not bipartite, a contradiction.

Conversely, suppose that $x$ or $y$ is a bridge. Then no cycle of $G_{xy}$ contains $a$. Hence each cycle of $G_{xy}$ is a cycle of $G$. This implies that $G_{xy}$ is bipartite. \hfill \Box
5. Applications

Splitting off operation and the connectedness

The Splitting off operation on a connected graph may not yield a connected graph. The graph $G$ and the corresponding graph $G_{xy}$ of the Figure 2 exhibit this fact.

![Figure 2](image)

We explore the conditions for a graph $G$ for which $G_{xy}$ is connected. The connectedness of $G_{xy}$ depends on the choice of the pairs $\{x, y\}$ of the edges in $G$.

In the following Lemma we provide conditions for splitting off operation to preserve the connectedness of the graph $G$.

**The Splitting Lemma 5.1.** Let $G$ be a connected bridgeless graph. Suppose $v \in V(G)$ with $d(v) > 3$ and $e_1, e_2, e_3$ are the edges incident at $v$. Form the graphs $G_{12}$ and $G_{13}$ by applying the splitting off operation using the pairs $e_1, e_2$ and $e_1, e_3$, respectively and assume that $e_1$ and $e_3$ belong to different blocks if $v$ is a cut vertex of $G$. Then either $G_{12}$ or $G_{13}$ is connected and bridgeless. In particular, if $v$ is a cut vertex, then $G_{13}$ has this property. Finally, if $B \subseteq G$ is a block and $e_1, e_2, e_3$ are edges in $B$ then both $G_{12}$ and $G_{13}$ are connected.
Figure 3 illustrates the Splitting Lemma explicitly. The result is important and can be used as a useful tool in graph theory.

The splitting off operation on a 2-connected graph, in general, may not yield a 2-connected graph.

![Image](image1.png)  
**Figure 3(a)** $v$ is a cut vertex.

![Image](image2.png)  
**Figure 3(b)** $G$ is a block.

**Figure 3**: Illustration of the Splitting Lemma.

Example 5.2 exhibits the fact that even if the given graph is 3-connected, the splitting off operation on it may not produce a 2-connected graph.

**Example 5.2.** Consider the complete graph $K_4$ on 4 vertices. The graph

![Image](image3.png)  
**Figure 4**

$(K_4)_{12}$ is not 2-connected. In fact, the element 6 is a bridge in it. Further,
the set $C'$ of cycles in the definition of $(K_4)_{12}$ consists of just members of $C_0$ and $C_1$. We provide a sufficient condition for $G_{xy}$ to be 2-connected.

**Theorem 5.3.** Let $G$ be a 2-connected graph and $\{x, y\}$ be a pair of edges of $G$. If $C_2 \neq \phi$ then $G_{xy}$ is 2-connected.

**Proof.** Assume that $C_2 \neq \phi$ and let $Z = ((C_1 \cup C_2) - \{x, y\}) \cup \{a\}$ be a member of $C_2$. Suppose $e$ and $f$ are two arbitrary edges of $G_{xy}$. We show that there is a cycle of $G_{xy}$ containing both $e$ and $f$. We consider the following two cases.

**Case (i).** Neither of $e$ and $f$ is $a$. Then $e, f \in E(G)$. Since $G$ is 2-connected there is a cycle $C$ of $G$ containing both the edges $e$ and $f$. If $C$ contains neither of $x$ and $y$ then it is also a cycle of $G_{xy}$ and we are through. Further, if $C$ contains $x$ and $y$ both then $(C - \{x, y\}) \cup \{a\}$ is the cycle of $G_{xy}$ containing $e$ and $f$.

Now suppose that $x \in C$ but $y \notin C$. We show that for every edge $z$ in the set $E(C \cup C_1)$ and the edge $x$ there is a cycle of $G_{xy}$ containing $a$ and $z$. If $z$ is an edge of $C_1$ then $((C_1 \cup C_2) - \{x, y\}) \cup \{a\}$ is the desired cycle of $G_{xy}$. Assume that $z \in E(C) - E(C_1)$. Since $x \in E(C_1) \cap E(C)$, there is a cycle $C_3$ of $G$ such that $z \in C_3 \subseteq (C \cup C_1)$ and $x \notin C_3$. The cycle $C_3$ contains neither of the edges $x$ and $y$ therefore it is a cycle of $G_{xy}$. Further, it contains at least one element of $C_1$ say $w$. Then $G_{xy}$ has the cycle $Z$ containing the edges $w$ and $a$ and also the cycle $C_3$ containing the edges $w$ and $z$. Consequently $G_{xy}$ has a cycle containing $z$ and $a$ as desired. Now $e$ and $f$ are the edges of $E(C \cup C_1)$, therefore $G_{xy}$ has cycles containing the pairs $\{e, a\}$ and $\{f, a\}$. This implies that $G_{xy}$ has a cycle containing the edges $e$ and $f$. 

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Case (ii). Suppose one of $e$ and $f$ is $a$, say $e = a$. Then we want to show that there is a cycle of $G_{xy}$ containing $a$ and $f$. But this follows from the proof of Case (i).

In the following Theorem, we characterize Eulerian graphs in terms of splitting off operation. First we prove the following lemma.

**Lemma 5.4.** Let $G$ be a graph with an edge set $E(G)$ and $x, y \in E(G)$. Suppose that $G_{xy}$ is connected. Then $G$ is Eulerian if and only if $G_{xy}$ is Eulerian.

**Proof.** Suppose $G$ is an Eulerian graph. Then each vertex of $G$ has even degree. Let the edges $x$ and $y$ be incident at the vertex $v$ of $G$. For convenience, let $vv_1 = x$, $vv'_1 = y$. After splitting off operation using the pair $\{x, y\}$, the degree of vertex $v$ in $G_{xy}$ decreases by two and remains even. Degrees of $v_1$ and $v'_1$ remain same in $G_{xy}$. The degrees of the other vertices in $G$ and also in $G_{xy}$ are unaffected by the application of the operation. Also it is given that $G_{xy}$ is connected. Hence $G_{xy}$ is Eulerian.

Conversely, suppose $G_{xy}$ is an Eulerian graph. So, each vertex of $G_{xy}$ is of even degree. Now, subdivide the edge $v_1v'_1$ by introducing the new vertex $v''$ on it. Take the newly created edges $v_1v''$ and $v''v'_1$ as $x$ and $y$, respectively. The degree of $v''$ will be two, an even number. Then, identify $v''$ and $v$ to get a graph $G$. The degrees of all the vertices in $G$ are even. Then $G$ is eulerian and the splitting off operation on it with respect to the pair $\{x, y\}$ gives the graph $G_{xy}$.

**Theorem 5.5.** A graph $G$ is Eulerian if and only if $G$ can be transformed through repeated applications of splitting off procedure into a graph which is a cycle.
Proof. Suppose $G$ is an Eulerian graph with an edge set $E$. Then, through a sequence of splitting off operation performed on vertices of degree exceeding 2 in such a way that at each step the resulting graph $G_{xy}$ is still connected by Splitting Lemma [4], we arrive to a graph which is a cycle.

Conversely, suppose that $G$ can be transformed into a cycle $C$ through repeated applications of the splitting off procedure. Let the sequence of graphs involved in this process be $G = G_0, G_1, G_2, ..., G_i, G_{i+1}, ... , G_n = C$ where $G_{i+1}$ is obtained from $G_i$ by performing a splitting off operation on a vertex of $G_i$ of degree greater than 2. It is now enough to show that if $G_i$ is Eulerian then $G_{i-1}$ is Eulerian but this follows from Lemma 5.4. Now, $G_n = C$ is Eulerian, therefore, $G_{n-1}$ is Eulerian. Continuing in this way, we conclude that $G_{n-2}, G_{n-3}, ... , G_0 = G$ is Eulerian. This completes the proof of the theorem. $\square$

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