CHAPTER-3

PERFORMANCE ANALYSIS OF AN $M/M/c/N$ QUEUE WITH BALKING AND RENEGING

3.1 INTRODUCTION

In real life, many queueing situations arise in which there may be a tendency for customers to be discouraged by a long queue. As a result, the customers either decide not to join the queue (i.e. balk) or depart after joining the queue without getting service due to impatience (i.e. renege). Balking and reneging are not only common phenomena in queues arising in daily activities, but also in various machine repair models. In this chapter, we consider an $M/M/c/N$ queueing system with balking and reneging.

Queueing systems with balking, reneging or both have been studied by many researchers. Haight [39] first considered an $M/M/1$ queue with balking. An $M/M/1$ queue with customers reneging was also proposed by Haight [40]. The combined effects of balking and reneging in an $M/M/1/N$ queue have been investigated by Ancker and Gafarian [3], [4]. Abou-EI-Ata and Hariri [1] considered the multiple servers queueing system $M/M/c/N$ with balking and reneging. Reynolds [75] studied the multi-channel Markovian queue with the discouragement. Haghighi et al. [38] discussed the multi-channel Markovian queue $M/M/c$ with both balking and reneging concepts and derived the steady-state probability.

The rest of this chapter is organized as follows. In the next part of chapter, we give a description of the queueing model. In part 3.3 of this chapter, we derive the steady-state equations by the Markov process method. By writing the transition rate matrix as block matrix, we get the matrix form solution of the steady-state probabilities and present a procedure for calculating the steady-state probabilities. In part 3.4 of this chapter, we give some performance measures of the system. Based on the performance analysis, we formulate a cost model to determine the optimal service rate. Some
numerical examples are presented to demonstrate how the various parameters of the model influence the behavior of the system. Conclusions are given in part 3.5 of this chapter.

3.2 SYSTEM MODEL

In this chapter, we consider an $M/M/c/N$ queueing system with balking and reneging. The assumptions of the system model are as follows:

1. Customers arrive at the system one by one according to a Poisson process with rate $\lambda$. On arrival a customer either decides to join the queue with probability $b_n$ or balk with probability $1-b_n$ when $n$ customers are ahead of him ($n = 0, 1, \ldots, N-1$), where $N$ is the maximum numbers of the customers in the system, and

$$0 \leq b_{n-1} \leq b_n < 1, \quad 1 \leq n \leq N-1,$$

$$b_0 = 1, \quad \text{and} \quad b_n = 0, \quad n \geq N.$$

3. After joining the queue each customer will wait a certain length of time $T$ for service to begin. If it has not begun by then, he will get impatient and leave the queue without getting service. This time $T$ is a random variable whose density function is given by

$$d(t) = \alpha e^{-\alpha t}, \quad t \geq 0, \quad \alpha > 0,$$

where $\alpha$ is the rate of time $T$. Let $i$ denote the number of severs being busy and $n$ represent the number of customers in the system. If $n$ is less than or equal to $i$, the customers will get service instantly upon arrival to the server, and the phenomenon of reneging will not occur. If $n$ is greater than $i$, then there are $n-i$ customers who have to wait in the queue. Since the arrival and the departure of the impatien customers without service are independent, the average reneging rate in this state is given by $(n-i) \alpha$.

4. The customers are served on a first-come, first served (FCFS) discipline. Once service commences it always proceeds to completion. The service times are assumed to be distributed according to an exponential distribution with density function as follows:

$$s(t) = \mu e^{-\mu t}, \quad t \geq 0, \quad \mu > 0,$$

where $\mu$ is the service rate.
3.3 STEADY-STATE PROBABILITY

In this part of chapter, we derive the steady-state probabilities by the Markov process method.

Let \( P(n) \) be the probability that there are \( n \) customers in the system. Applying the Markov process theory, we obtain the following set of steady-state equations.

\[
\begin{align*}
\mu P(1) &= \lambda P(0), \\
\lambda P(n-1) + \mu(n+1)P(n+1) &= (\lambda + n\mu)P(n), & n = 0 \\
\lambda P(c-1) + (c\mu + \alpha)P(c+1) &= (\lambda b_c + c\mu)P(2), & 1 \leq n \leq c-1 \\
\lambda b_{n-1}P(n-1) + (c\mu + (n+1-c)\alpha)P(n+1) &= (\lambda b_n + c\mu + (n-c)\alpha)P(n), & n = c \\
\lambda b_{N-1}P(3) &= (c\mu + (N-c)\alpha)P(N), & c < n < N \\
\lambda b_{N-1}P(3) &= (c\mu + (N-c)\alpha)P(N), & n = N.
\end{align*}
\]

For \( N = 4 \) steady state difference equations are

\[
\begin{align*}
\mu P(1) &= \lambda P(0) & n = 0 \\
\lambda P(0) + 2\mu P(2) &= (\lambda + \mu)P(1) & n = 1 \\
\lambda P(1) + (2\mu + \alpha)P(3) &= (\lambda b_2 + 2\mu)P(2) & n = 2 \\
\lambda b_2 P(2) + (2\mu + 2\alpha)P(4) &= (\lambda b_2 + 2\mu + \alpha)P(3) & n = 3 \\
\lambda b_3 P(3) &= (2\mu + 2\alpha)P(4) & n = 4.
\end{align*}
\]

Matrix form of these equations as follows:

\[
\begin{bmatrix}
-\lambda & \mu & 0 & 0 & 0 \\
\lambda & -(\lambda + \mu) & 2\mu & 0 & 0 \\
0 & \lambda & -(\lambda b_2 + 2\mu) & (2\mu + \alpha) & 0 \\
0 & 0 & \lambda b_2 & -(\lambda b_3 + 2\mu + \alpha) & (2\mu + 2\alpha) \\
0 & 0 & 0 & \lambda b_3 & (2\mu + 2\alpha)
\end{bmatrix}
\begin{bmatrix}
P(0) \\
P(1) \\
P(2) \\
P(3) \\
P(4)
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

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On solving these we get,

\[ P(0) = k \]
\[ P(1) = \frac{k\lambda}{\mu} \]
\[ P(2) = \frac{k\lambda^2}{2\mu^2} \]
\[ P(3) = \frac{k\lambda^3b_2}{2\mu^2(2\mu + \alpha)} \]
\[ P(4) = \frac{k\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)} \]

Using

\[ P(0) + P(1) + P(2) + P(3) + P(4) = 1, \]

we get

\[ k = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3b_2}{2\mu^2} + \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)}\right)^{-1} \]

Therefore,

\[ P(0) = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3b_2}{2\mu^2} + \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)}\right)^{-1} \]
\[ P(1) = \frac{\lambda}{\mu} \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3b_2}{2\mu^2} + \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)}\right)^{-1} \]
\[ P(2) = \frac{\lambda^2}{2\mu^2} \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3b_2}{2\mu^2} + \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)}\right)^{-1} \]
\[ P(3) = \frac{\lambda^3b_2}{2\mu^2(2\mu + \alpha)} \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3b_2}{2\mu^2} + \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)}\right)^{-1} \]
\[ P(4) = \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)} \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3b_2}{2\mu^2} + \frac{\lambda^4b_2b_3}{2\mu^2(2\mu + \alpha)(2\mu + 2\alpha)}\right)^{-1} \]
For $N = n$

$$P(n) = \frac{(\lambda^n b_2 b_3 \ldots b_{n-1})k}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (n-2)\alpha)}$$

Using

$$P(0) + P(1) + P(2) + P(3) + P(4) + \ldots + P(N) = 1,$$

we get

$$k = \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2 b_2}{2\mu^2} + \frac{\lambda^3 b_3}{2\mu^2} + \frac{\lambda^4 b_2 b_3}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha)} + \ldots + \frac{\lambda^n b_2 b_3 \ldots b_{n-1}}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (N-2)\alpha)}\right)^{-1}.$$ 

Therefore,

$$P(n) = \frac{(\lambda^n b_2 b_3 \ldots b_{n-1})}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (n-2)\alpha)} \left(1 + \frac{\lambda}{\mu} + \frac{\lambda^2 b_2}{2\mu^2} + \frac{\lambda^3 b_3}{2\mu^2} + \frac{\lambda^4 b_2 b_3}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha)} + \ldots + \frac{\lambda^n b_2 b_3 \ldots b_{n-1}}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (N-2)\alpha)}\right)^{-1}.$$ 

### 3.4 PERFORMANCE MEASURES AND COST MODEL

In this part of this chapter, we give some performance measures of the system. Based on these performance measures, we develop a cost model to determine the optimal service rate.

#### 3.4.1 PERFORMANCE MEASURES

Using the steady-state probability presented in the part 3.3 of this chapter, we can obtain some performance measures of the system, such as the busy probability of the server $P_B$, the expected number of the waiting customers $E(N_q)$ and the expected number of the customers in the system $E(N)$ as follows:
\[ E(N_q) = \sum_{n=1}^{N} (n-1)P(n) \]

\[ = \sum_{n=1}^{N} (n-1) \left( \frac{(\lambda^n b_2 b_3 \cdots b_{n-1}) \left( 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3 b_2}{2\mu^2} + \frac{\lambda^4 b_2 b_3}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha)} + \ldots \right)}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (N-2)\alpha)} \right)^{-1} \]

\[ E(N) = \sum_{n=1}^{N} nP(n) \]

\[ = \sum_{n=1}^{N} n \left( \frac{(\lambda^n b_2 b_3 \cdots b_{n-1}) \left( 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3 b_2}{2\mu^2} + \frac{\lambda^4 b_2 b_3}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha)} + \ldots \right)}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (N-2)\alpha)} \right)^{-1} \]

\[ B.R. = \sum_{n=1}^{N} \lambda(1 - b_n)P(n) \]

\[ = \sum_{n=1}^{N} \lambda(1 - b_n) \left( \frac{(\lambda^n b_2 b_3 \cdots b_{n-1}) \left( 1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3 b_2}{2\mu^2} + \frac{\lambda^4 b_2 b_3}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha)} + \ldots \right)}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) \ldots (2\mu + (N-2)\alpha)} \right)^{-1} \]
\[ R.R = \sum_{n=1}^{N} (n-1)\alpha P(n) \]

\[ = \sum_{n=1}^{N} (n-1)\alpha \left( \frac{1}{\mu} \frac{1}{\mu} \frac{1}{\mu} \ldots \frac{1}{\mu} \right) \left( \frac{\lambda + \lambda^2 b_2 + \lambda^3 b_3 + \lambda^4 b_2 b_3 + \ldots}{2\mu^2 (2\mu + \alpha)(2\mu + 2\alpha) + \ldots} \right)^{-1} \]

\[ L.R. = B.R. + R.R \]  
(3.5)

### 3.4.2 COST MODEL

In this subpart of this chapter, we develop an expected cost model, in which service rate \( \mu \) is the control variable. Our objective is to control the service rate to minimize the system’s total average cost per unit. Let

- \( C_1 \equiv \text{cost per unit time when the server is busy}, \)
- \( C_2 \equiv \text{cost per unit time when a customer joins in the queue and waits for service}, \)
- \( C_3 \equiv \text{cost per unit time when a customer balks or reneges}. \)

Using the definitions of each cost element listed above, the total expected cost function per unit time is given by

\[ F(\mu) = C_1 P_B + C_2 E(N_q) + C_3 L.R. \]

\[ F(\mu) = C_1 + C_2 \sum_{n=1}^{N} (n-1)P(n) + C_3 \left[ \sum_{n=1}^{N} \lambda (1-b_n)P(n) + \sum_{n=1}^{N} \alpha (n-1)P(n) \right] \]
\[
= C_1 + C_2 \sum_{n=1}^{N} (n-1) \left( \frac{1 + \frac{\lambda}{\mu} + \frac{\lambda^2 b_2}{2 \mu^2} + \frac{\lambda^3 b_2}{2 \mu^2} + \frac{\lambda^4 b_2}{2 \mu^2} + \frac{\lambda^n b_2 b_3}{2 \mu^2 (2 \mu + \alpha)(2 \mu + 2 \alpha)\cdots(2 \mu + (N-2)\alpha)} + \cdots}{2 \mu^2 (2 \mu + \alpha)(2 \mu + 2 \alpha)\cdots(2 \mu + (n-2)\alpha)} \right)^{-1} \]

\[
+ C_3 \sum_{n=1}^{N} (\lambda(1-b_n) + \alpha(n-1)) \left( \frac{1 + \frac{\lambda}{\mu} + \frac{\lambda^2 b_2}{2 \mu^2} + \frac{\lambda^3 b_2}{2 \mu^2} + \frac{\lambda^4 b_2 b_3}{2 \mu^2 (2 \mu + \alpha)(2 \mu + 2 \alpha)\cdots(2 \mu + (N-2)\alpha)} + \cdots}{2 \mu^2 (2 \mu + \alpha)(2 \mu + 2 \alpha)\cdots(2 \mu + (n-2)\alpha)} \right)^{-1} \]

where \( E(N_q), E(N), B.R., R.R. \) and \( L.R. \) are given in Eqs. (3.1) – (3.5). The first item is the cost incurred by the server. The second item \( C_2 E(N_q) \) is the cost incurred by the customer’s waiting. The last item \( C_3 L.R. \) is the cost incurred by the customer loss.

### 3.4.3 NUMERICAL RESULTS

In this subpart of this chapter, we present some numerical examples to demonstrate how the various parameters of the model influence the optimal service rate \( \mu^* \), the optimal expected cost of the system \( F(\mu^*) \) and other performance measures of the system. We fix the maximum number of customers in the system \( N = 3 \), the probability \( b_n = 1/(n+1) \) and the cost elements \( C_1 = 15, \ C_2 = 12, \ C_3 = 18. \)
First, we select the rate of the waiting time $\alpha = 0.1$ and change values of arrival rate of customer $\lambda$. The numerical results are summarized in Table 3.1. This shows that: (i) the optimal service rate $\mu$ increases with the increasing $\lambda$ and its minimum expected cost $F(\mu^*)$ increases with the increasing $\lambda$; (ii) the expected number of the waiting customers $E(N_q)$, the expected number of customers in the system $E(N)$ first increases with the increase of $\lambda$ and then decreases with the increase of $\lambda$; (iii) average rate of customer loss $L.R.$ increases with the increasing $\lambda$ this is because the number of the customers in the system increases with the increasing $\lambda$. Thus, $F(\mu^*)$ increases with the increasing $\lambda$.

Finally, we select $\lambda = 0.4$, and change values of $\alpha$. The numerical results are summarized in Table 3.2. This shows that: (i) the optimal service rate $\mu^*$ decreases with the increasing $\alpha$, and its minimum expected cost $F(\mu^*)$ increases with the increasing $\alpha$; (ii) $E(N_q)$, the expected number of
customers in the system $E(N)$ first increases with the increase of $\alpha$ and then decreases with the increase of $\alpha$. Thus $F(\mu^*)$ first decreases with the increase of $\alpha$ and then increases with the increase of $\alpha$. The following graph shows the performance of the model at $\alpha = 0.1$ and $\lambda = 0.4$. Figure 3.1 and 3.5 are for $\alpha = 0.1$ and they depict the performance of the model. Similarly, Figure 3.6 and 3.10 are for $\lambda = 0.4$.

![Figure 3.1: Graph between arrival rate and optimal service rate](image1)

![Figure 3.2: Graph between arrival rate and expected optimal cost function](image2)
Figure 3.3: Graph between arrival rate and expected queue length

Figure 3.4: Graph between arrival rate and expected length of system
Figure 3.5: Graph between arrival rate and average rate loss of customers

Figure 3.6: Graph between reneging rate and optimal service rate
Figure 3.7: Graph between reneging rate and expected optimal cost function

Figure 3.8: Graph between reneging rate and expected queue length
**Figure 3.9:** Graph between reneging rate and expected length of system

**Figure 3.10:** Graph between reneging rate and average rate loss of customers
3.5 CONCLUSION

In this chapter, we considered an $M/M/c/N$ queueing system with balking and reneging. We developed the equations of the steady state probabilities and derived the matrix form solution of the steady-state probabilities. We also gave some performance measures of the system, and formulated a cost model to determine the optimal service rate. Although the function of the cost is too complicated to derive the explicit expression of the optimal service rate, the performance measures and the optimal service rate can be numerically evaluated by the formula in the part 3.4 of this chapter. Some numerical examples were presented to demonstrate how the various parameters of the model influence the behavior of the system.