CHAPTER 4

STATE OBSERVER DESIGN ALGORITHMS
FOR MIMO SYSTEM

4.1 INTRODUCTION

In science and engineering problems, state space realizations are often used to model linear and nonlinear dynamical systems. Some of the examples are system monitoring, state feedback control, and fault detection. In all these applications, complete state vector information is necessary to implement the state feedback. However, in practice, all the states are usually not available for feedback because it is often expensive and impractical to have sensors for every state variable, so some form of reconstruction of state variables is required from the measured output. In such scenario, state observer, a computer implemented mathematical model, can be constructed using the mathematical model of the system and the available outputs to obtain an estimate $\hat{x}$ of the true state $x$, and this estimate can then be used as a substitute for $x$. In the classical approach, the reconstruction of state variables comprises the study of unforced system subjected to nonzero initial conditions, and calculation of observer gain to stabilize the error dynamics $\dot{\hat{x}} = x - \hat{x}$, hence achieving asymptotic convergence. Since it is difficult to incorporate the modeling errors in the design of observers for linear systems, much attention has been paid on this area in the recent years (Yang et al 2011) particularly in connection with the problem of residual generation in observer based fault detection applications.
4.1.1 System Monitoring

Many of the control problems require the estimate of certain process variables to effectively monitor the condition of the equipments. For example, a pilot may want to know the pitch angle of the airplane in order to know whether the plane is close to stalling. The observer based system monitoring scheme is shown in Figure 4.1. The observer provides the estimate of the state variables by taking the relevant process inputs and outputs. Dynamical closed loop is not used when the estimate is only used for reference. The main objective in this kind of applications is to simply verify that the estimated variables are very close to the reference value in the presence of disturbances.

![Figure 4.1 Observer based system monitoring](image)

4.1.2 State Feedback Control

![Figure 4.2 Observer based state feedback control](image)
Figure 4.2 illustrates the observer based state feedback controller strategy. The design of state feedback controller starts with the assumption that all the state variables are available for feedback. But such assumption is not always valid because some of the state variables may not be measurable or not available due to faulty sensor. Hence to solve the issue, the state observer algorithm is incorporated into the system to reconstruct the state variables. For example, in case of Linear Quadratic Guassian (LQG) control, an optimal state feedback control can be implemented only when all the state variables are available. Similarly, a key assumption in the eigen value assignment and system stabilization using state feedback control is that the complete state vector information is accessible. So in such cases, the observer is used to estimate the state variables and the estimated variables are used as true variables to implement the state feedback control.

4.1.3 Fault Detection

The conventional approach for fault diagnosis is a hardware based method, in which a particular variable is measured using multiple sensors and a fault is detected by comparing the sensed variables. But this approach has several limitations such as high cost, need for additional equipments and space. Hence analytical redundancy approach is often considered an alternate for the hardware redundancy approach. Fault detection techniques based on analytical redundancy often use an observer to determine the states of the process and from this the conclusion on possible faults is drawn. Analytical redundancy, which uses the redundant analytical relationships among system inputs and measured system outputs, does not require extra hardware to evaluate whether fault has occurred in the system. The principal structure of the system is depicted in Figure 4.3. The size and location of the fault is determined using the fault detection block which makes a decision on
occurrence of the fault based on the observer output. The basic idea is to generate residual signal $e(t)$, which is the difference between real and estimated outputs of the system, to ascertain the possibility of faults. Figure 4.4 illustrates the residual generation scheme using analytical redundancy. The residual signal is zero when there is no fault in the system. In case of faulty system, the value of the residual signal is nonzero. When using an observer for fault detection, there are two main ways in which faults can be represented. First, the fault can be viewed as an external or unmeasured signal into the system. The nonzero residual due to occurrence of fault in such cases can be used as a fault signal. On the other hand, due to unmeasured inputs such as model errors and measurement noise, the fault signal may never satisfy $e(t) = 0$, and must be evaluated with some statistic or deterministic methods. Normally some threshold value is fixed for the residual signal and if the residual signal is greater than the threshold hold value, it is assumed that faults have taken place. The structure of the overall fault detection and isolation technique is shown in Figure 4.5.

\[ \text{Figure 4.3 Observer based fault detection} \]
The design techniques and performance analysis of four different types of state observers are presented for detecting the additive fault in the system. The observers considered for analysis include Luenberger Observer, Kalman Observer, Unknown Input Observer and Sliding Mode Observer. The performance of these observers to a Multiple Input Multiple Output (MIMO) DC servo motor model is assessed. Simulations are carried out to estimate the state variables in the presence of white noise disturbance. The magnitude of residual is taken as a comparison factor to evaluate the performance of the
observers. In case of faulty system, the residual of the observer play an important role in not only identifying the fault but also in estimating the severity of the fault. An additive fault introduced into the system is determined using the residuals of the observers.

4.2 MODELING OF DC SERVO MOTOR

A second order DC motor with multiple inputs and multiple outputs is chosen as a plant for the observer implementation. From the first principles the mathematical model of the plant in terms of state space model is derived. The parameters of the motor model are given in Table 4.1. The inputs to the plant are armature voltage \( V_a(t) \) and the load torque \( M_L(t) \). The measured output signals are armature current \( I_a(t) \) and speed of the motor \( \omega(t) \). In simulation the armature voltage is given as a step function while the load torque is chosen to be a fixed value of 0.1. Figure 4.6 shows the signal flow diagram of the DC motor.

![Figure 4.6 Signal flow diagram of the DC Motor](image-url)
Table 4.1 List of parameters of DC motor model

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ra</td>
<td>Armature Resistance</td>
<td>1.52 Ω</td>
</tr>
<tr>
<td>La</td>
<td>Armature Inductance</td>
<td>6.82*10^{-3} H</td>
</tr>
<tr>
<td>ψ</td>
<td>Magnetic Flux</td>
<td>0.33 Wb</td>
</tr>
<tr>
<td>J</td>
<td>Inertia constant</td>
<td>0.0192 kg m^2</td>
</tr>
<tr>
<td>Mfi</td>
<td>Viscous Friction</td>
<td>0.36*10^{-3} N ms</td>
</tr>
</tbody>
</table>

The first order differential equation of armature current \( I_A(t) \) and speed \( \omega(t) \) are represented as

\[
L_a \dot{I}_A = -R_a I_A(t) - \psi \omega(t) - V_A(t) \quad (4.1)
\]

\[
J \dot{\omega}(t) = \psi I_A(t) - M_f(t) \omega(t) - M_f(t) \quad (4.2)
\]

The general continuous state space form with faults or disturbance is represented as

\[
\dot{x}(t) = Ax(t) + Bu(t) + Lf(t) \quad (4.3)
\]

\[
y(t) = Cx(t) + Du(t) + M_f(t) \quad (4.4)
\]

where \( x(t) \in \mathbb{R}^n \) is a state vector, \( u(t) \in \mathbb{R}^m \) is a control input vector, \( y(t) \in \mathbb{R}^p \) is a measurement output vector, \( A, B, C \) and \( D \) are known constant system matrices. The continuous time system in Equations (4.3) and (4.4) can be discretised using sampling time to obtain the discrete time model as represented in Equations (4.5) and (4.6)

\[
x(k+1) = Ax(k) + Bu(k) + Lf(k) \quad (4.5)
\]
\[ y(k) = Cx(k) + Du(k) + Mf_o(k) \]  \hspace{1cm} (4.6)

where \( T \) is the sampling time, \( L = T^*A^*L \) and \( M = M_m \).

The continuous time model of a DC motor in state space form can be represented as,

\[
\begin{bmatrix}
  \dot{I}_d(t) \\
  \dot{\omega}(t)
\end{bmatrix} = \begin{bmatrix}
  -R_i/L_c & -\psi/L_c \\
  \psi/J & 0
\end{bmatrix} \begin{bmatrix}
  I_d(t) \\
  \omega(t)
\end{bmatrix} + \begin{bmatrix}
  1/L_c & 0 \\
  0 & -1/J
\end{bmatrix} \begin{bmatrix}
  V_d(t) \\
  V_{\omega}(t)
\end{bmatrix}
\] \hspace{1cm} (4.7)

\[
\begin{bmatrix}
  y_1(t) \\
  y_2(t)
\end{bmatrix} = \begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix} \begin{bmatrix}
  I_d(t) \\
  \omega(t)
\end{bmatrix} + \begin{bmatrix}
  0 & V_d(t) \\
  0 & V_{\omega}(t)
\end{bmatrix}
\] \hspace{1cm} (4.8)

### 4.3 MODELS OF OBSERVERS

An observer is a subsystem which estimates the unmeasured state variables from the available output and control input along with the plant model. If the system satisfies the observability condition, it is possible to estimate the state variables. In the following section, four observer models are considered for estimating the variables of the DC servo system.

#### 4.3.1 Luenberger Observer

![Figure 4.7 Block diagram of Luenberger observer](image)
Figure 4.7 shows the block diagram of Luenberger Observer. Luenberger observer is the fundamental observer for most of the state estimation techniques available today. The structure of the observer is same as the plant model except the error correction term. By tuning the observer gain, the error is made to zero when time t approaches infinity. The gain of the observer is normally adjusted in such a way that the observer responds faster than the system, so that the estimated variables can be used for implementing the state feedback control. Consider a continuous linear time invariant system

\[
\dot{x} = Ax + Bu \tag{4.9}
\]

\[
y = Cx + Du \tag{4.10}
\]

where, \(x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^r\).

For such linear system observer equation is given by,

\[
\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}) \tag{4.11}
\]

\[
\hat{y} = C\hat{x} \tag{4.12}
\]

Observer error is defined as

\[
e = x - \hat{x} \tag{4.13}
\]

Differentiating Equation (4.13)

\[
\dot{e} = \dot{x} - \dot{\hat{x}} \tag{4.14}
\]

Substituting Equations (4.9), (4.10) and (4.11) into (4.14),

\[
\dot{e} = Ax + Bu - [A\hat{x} + Bu + L(y - \hat{y})] \tag{4.15}
\]
Further simplifying the Equation (4.15),

\[ \dot{e} = A(x - \hat{x}) - L(y - \hat{y}) \quad (4.16) \]

Using Equations (4.9) and (4.10) into (4.14),

\[ \dot{e} = A(x - \hat{x}) - LC(x - \hat{x}) \quad (4.17) \]
\[ \dot{e} = (A - LC)e \quad (4.18) \]

Solution of Equation (4.18) is given by,

\[ e(t) = e^{\exp ((A - LC)t)}e(0) \quad (4.19) \]

The eigenvalues of the matrix \((A - LC)\) can be made arbitrarily by appropriate choice of the observer gain \(L\), when the pair \([A, C]\) is observable (Observability condition holds). So the observer error \(e \rightarrow 0\) when \(t \rightarrow \infty\). The given system is observable if and only if the observability matrix \([C^T : A^T C^T : \ldots : A^{T(n-1)} C^T]\) has a rank of \(n\). The gains of the state observer matrix can be determined by equating the desired characteristic equation of the system to its desired characteristic equation.

\[ |sI - A + LC| = (s - \mu_1)(s - \mu_2)\ldots(s - \mu_n) \quad (4.20) \]

where, \(\mu_1, \mu_2, \ldots, \mu_n\) are the desired closed loop poles of plant and \(L\) is state observer gain matrix. By equating the actual characteristic equation with the desired characteristic equation, as given in Equation (4.21), the coefficients of \(L\) matrix can be obtained.

\[ |sI - A + LC| = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_{n-1} s + \alpha_n \quad (4.21) \]
The observer gain is chosen in such a way that observer responds 5 to 10 times faster than plant response. Normally the system which responds faster requires more energy to control. Hence to make the system to respond faster, the desired dominant poles should be located far away from the \( j\omega \) axis.

**4.3.2 Kalman Observer**

Kalman observer is a recursive predictive filter that is based on the use of state space techniques and recursive algorithms, in which only the estimated state from the previous time step and the current measurement are needed to compute the estimate of the current state. The Kalman filter operates by propagating the mean and covariance of the state through time. The notation \( \hat{X}_{n|m} \) represents the estimate of the state vector \( X \) at time \( n \) given observations up to \( m \).

The state of the filter is represented by two variables:

- \( \hat{X}_{k|k} \), the *a posteriori* state estimate at time \( k \) given observations up to and including at time \( k \)
- \( P_{k|k} \), the *a posteriori* error covariance matrix (a measure of the estimated accuracy of the state estimate).

The block diagram of Kalman observer is shown in Figure 4.8. The Kalman filter has two distinct phases namely, prediction and correction. The prediction phase uses the state estimate from the previous time step to produce an estimate of the state at the current time step. This predicted state estimate is also known as the a priori state estimate. Although it is an estimate of the state at the current time step, it does not include observation information from the current time step. In the correction phase, the current a
priori prediction is combined with current observation information to refine the state estimate. This improved estimate is termed the a posteriori state estimate.

Consider a linear time invariant discrete system given by the following equation

\[
X_{k+1} = FX_k + BU_k \\
Z_{k+1} = HX_{k+1} + V_{k+1}
\]

where, \(F\) is the state transition matrix, \(B\) is the control input matrix, \(W_k\) is the process noise with zero mean multivariate normal distribution having covariance \(Q_k\), \(U_k\) is the control input, \(H\) is the observation matrix and \(V_{k+1}\) is the observation noise, which is zero mean Gaussian white noise having covariance \(R_k\).
a) Prediction (Time update) Equations

Predicted state estimate

\[ \hat{x}_{k|k-1} = F_k \hat{x}_{k-1|k-1} + B u_k \]  
(4.25)

Predicted estimate covariance

\[ P_{k|k-1} = F_k P_{k-1|k-1} F_k^T + Q_k \]  
(4.26)

b) Correction (Measurement update) Equations

Innovation or measurement residual

\[ y_k = z_k - H \hat{x}_{k|k-1} \]  
(4.27)

Innovation (or residual) covariance

\[ S_k = H P_{k|k-1} H^T + R_k \]  
(4.28)

Optimal Kalman gain

\[ K_k = P_{k|k-1} H^T S_k^{-1} \]  
(4.29)

Updated (a posteriori) state estimate

\[ \hat{x}_{k|k} = \hat{x}_{k|k-1} + K_k y_k \]  
(4.30)

Updated (a posteriori) estimate covariance

\[ P_{k|k} = (I - K_k H) P_{k|k-1} \]  
(4.31)
4.3.3 Unknown Input Observer

Consider a continuous linear time invariant space model of the system

\[ \dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \]  
\[ y(t) = Cx(t) \]  
\[ x \in \mathbb{R}^{n_x} \) represents the state vector, \( u \) represents input vector, \( y \) represents sensor output, \( A \) represents system coefficient matrix, \( B \) represents input coefficient matrix, \( C \) represents output coefficient matrix, \( d \in \mathbb{R}^{n_u} \) represents the unknown input vector, and \( E \in \mathbb{R}^{n_x} \) represents the unknown input distribution matrix. The structure of the UIO can be described as,

\[ \hat{z}(t) = Fx(t) + TBu(t) + Ky(t) \]  
\[ \hat{x}(t) = z(t) + Hy(t) \]
\( \hat{x} \in \mathbb{R}^{n\times n} \) represents the estimated state vector and \( T \in \mathbb{R}^{n\times n} \), \( K \in \mathbb{R}^{n\times n} \) and \( H \in \mathbb{R}^{n\times n} \) are matrices satisfying UIO requirements. The block diagram of Unknown Input Observer is shown in Figure 4.9. The error vector is given by

\[
e(t) = x(t) - \hat{x}(t)
\]

The state estimation error is governed by the following equation.

\[
e(t) = x(t) - \hat{x}(t) = x(t) - z(t) - Hy(t) = (I - HC)x(t) - z(t) \tag{4.37}
\]

Using Equation (4.37), derivative of the error vector is obtained as

\[
\dot{e}(t) = (A - HCA - K_1 C)\dot{e}(t) + (A - HCA - K_1 C)x(t) + (A - HCA - K_2 C)Hy(t) + (I - HC)\dot{u}(t) + (I - HC)Ed(t) - Fx(t) - TB\dot{u}(t) - K_2 y(t)
\]

\[
= (A - HCA - K_1 C)e(t) - [F - (A - HCA - K_1 C)]e(t) + [K_2 - (A - HCA - K_1 C)]Hy(t) - [T - (I - HC)]\dot{u}(t) - (I - HC)Ed(t) \tag{4.38}
\]

The following relations hold true

\[
(HC - I)E = 0 \tag{4.39}
\]

\[
T = (I - HC) \tag{4.40}
\]

\[
F = A - HCA - K_1 C K_2 \tag{4.41}
\]

\[
K_2 = FH \tag{4.42}
\]

\[
K = K_1 + K_2 \tag{4.43}
\]

Derivative of the error vector is \( \dot{e}(t) = Fe(t) \). Hence, the solution of the error vector will be \( e(t) = e^t e(0) \). If \( F \) is chosen as a Hurwitz matrix, the solution of the error equation goes to zero asymptotically. So, \( \hat{x} \) converges to \( x \). Necessary and sufficient conditions for the observer to be a UIO for the system defined in Equations (4.32) and (4.33) are
(i) \[ \text{rank}(CE) = \text{rank}(E) \]

(ii) \((C,A_1)\) is a detectable pair,

where \(A_1 = A - E \left[(CE)^T CE\right]^{-1}(CE)^T CA\).

A flow chart of UIO design procedure is shown in Figure 4.10.

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**Figure 4.10 Flow chart of UIO design procedure**
4.3.4 Sliding Mode Observer

Sliding mode observers are different from traditional observers mainly due to the injection of non-linear discontinuous term into observer based on the output estimation error. One of the main advantages of using sliding mode observers over their linear counterparts is that during sliding phase, they are insensitive to the unknown inputs. Moreover, they can also be used to reconstruct unknown inputs which could be a combination of system disturbances, faults or nonlinearities. In the design of a sliding mode observer, first a sliding manifold \( \{ x | \sigma(x) = 0 \} \) is to be defined. When the system states reach the sliding manifold from arbitrary initial states, the system states begin to be confined on the manifold and the system dynamics is decided based on the sliding manifold. In addition, if the matching condition is satisfied, the sliding mode control is robust to the model uncertainty, parameter variation and disturbance. Therefore, the sliding mode control becomes stable after the ideal sliding mode begins to operate on the stable manifold. Assume that the sliding manifold \( \sigma(x) : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear function as given below.

\[
\sigma(x) = s x = \alpha_0 x_n + \alpha_{n-1} x_{n-1} + \cdots + \alpha_1 x_1 + \alpha_0 \tag{4.44}
\]

where \( s \in \mathbb{R}^{m \times n} \) and the coefficients of \( \alpha_0, \alpha_1, \ldots, \alpha_n \) should be chosen so as to make the sliding manifold to be stable. Next, the sliding mode control which satisfies the reaching condition is to be determined. For reaching condition, the Lyapunov function can be defined as:

\[
\mathcal{V} = \frac{1}{z} \sigma^T \sigma \tag{4.45}
\]

If the reaching condition is satisfied, sliding mode control guarantees the stability and it confines the states on the sliding manifold. Furthermore, the derivative of the Lyapunov function should be negative definite:
The trajectory of the switching function $\sigma(x,t)$ can be given as

$$\dot{\sigma} = \sigma^T \sigma < 0$$  \hspace{1cm} (4.46)$$

The configuration of the sliding surface is illustrated in Figure 4.11, which indicates that the arbitrary initial condition reaches the sliding manifold in finite time. During the reaching phase, the system states are transferred to the sliding manifold. However, since the tracking error is not compensated in this region, the system becomes very sensitive. Unfortunately, the reaching condition is just a necessary condition to guarantee the ideal sliding mode. The reaching condition ensures only the asymptotic approach to the sliding manifold. After the time $t_r$, $\sigma(t)$ lies on the sliding manifold so called sliding mode. As long as the sliding mode is maintained, the system becomes robust to model uncertainty and external noise.

**Figure 4.11 Configuration of ideal sliding mode**

A brief review of Utkin observer is presented here. Consider a continuous time linear system described by
\[ x(t) = Ax(t) + Bu(t) \]  
(4.48)

\[ y(t) = Cx(t) \]  
(4.49)

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \) and \( p \leq m \). Assume that the matrices \( B \) and \( C \) are of full rank and \( \text{pair}(A, C) \) is observable. Reconstructing the state variables from the measured outputs is the prime reason for the observer design. In this the observed output vector can be represented as

\[ y = C_0 x_0 + C_{\delta} x_{\delta}, \quad x = (x_0, x_{\delta}) \]  
(4.50)

\[ C_0 \in \mathbb{R}^{p \times (n - p)}, C_{\delta} \in \mathbb{R}^{p \times p}, \det(C_0) \neq 0 \]

Using the following linear transformation of state variable

\[ T_1 = \begin{bmatrix} I_{n-p} & 0 \\ C_0 & C_{\delta} \end{bmatrix} \]  
(4.51)

the system represented by Equations (4.48) and (4.49) can be written in the form

\[ \dot{x}_0 = A_1 x_0 + A_1 \hat{y} + B_1 u \]  
(4.52)

\[ \dot{\hat{y}} = A_2 x_0 + A_2 \hat{y} + B_2 u \]  
(4.53)

Then the corresponding observer proposed by Utkin is given by

\[ \dot{\hat{x}} = A_1 \hat{x}_0 + A_1 \hat{\hat{y}} + B_1 u + LM \text{sgn}(\hat{y} - y) \]  
(4.54)

\[ \dot{\hat{y}} = A_2 \hat{x}_0 + A_2 \hat{\hat{y}} + B_2 u - M \text{sgn}(\hat{y} - y) \]  
(4.55)

where \((\hat{x}_0, \hat{y})\) are the estimates of \((x_0, y)\), \(L \in \mathbb{R}^{(n-p) \times p} \) is a constant nonsingular feedback gain matrix and \(\text{sgn} \) is the signum function and \(M \) is a
positive gain. If the error vector is defined as \( \varepsilon_y = \hat{y} - y \) and \( \varepsilon_a = \hat{x}_a - x_a \), then the error system is defined as

\[
\begin{align*}
\varepsilon_a &= A_1 \varepsilon_a + A_2 \varepsilon_y + L \text{sgn}(\varepsilon_y) \\
\varepsilon_y &= A_2 \varepsilon_a + A_2 \varepsilon_y - M \text{sgn}(\varepsilon_y)
\end{align*}
\] (4.56) (4.57)

For large value of \( M \) a sliding motion can be induced on the output error state.

### 4.4 PERFORMANCE ANALYSIS OF OBSERVERS

The simulation results of all four observer algorithms are presented for a chosen MIMO system. The model of the system, which is obtained by linearizing the system around stable equilibrium point, is considered for the design procedures. The system matrices are

\[
A = \begin{bmatrix} -22287 & -1839 \\ 17.19 & -0.012 \end{bmatrix}, \quad B = \begin{bmatrix} 146.63 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}
\]

The observability test reveals that both armature current and speed are observable. The location of the Luenberger observer poles are chosen to be 10 times faster than the system poles. The design procedure of UIO begins with the verification of the condition on the rank of the system matrices. The first condition to be checked is \( \text{rank}(E) = \text{rank}(CE) = I \). Then the matrices \( H \), \( T \) and \( A_1 \) are computed as,

\[
H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T = \begin{bmatrix} 0.331 & -0.551 \\ -0.222 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.343 & -0.437 \\ 0.033 & 0.034 \end{bmatrix}
\]
The observability matrix of \((C, A_1)\) is a full rank matrix thus it satisfies the condition (ii) of UIO design. Therefore the UIO can be designed using the following values of \(F\) and \(K\) matrices.

\[
F = \begin{bmatrix}
-0.048 & -0.055 \\
0.154 & -0.765
\end{bmatrix} \quad K = \begin{bmatrix}
-0.035 & -0.472 \\
-0.222 & 0.111
\end{bmatrix}
\]

4.4.1 With Additive Fault

![Residual during additive fault](image)

Figure 4.12 Residual during additive fault (a) LO (b) KO (c) UIO (d) SMO
4.4.2 With Disturbance (White Noise)

Figure 4.13 Residual during disturbance (a) LO (b) KO (c) UIO (d) SMO

Figure 4.12 shows the residual of all four observers for additive fault introduced into the system at the 100th sampling instant. At the instant of occurrence of fault, the magnitude of residual is expected to rise to nonzero value, and the magnitude of residual is used to determine the severity of the fault. Figure 4.13 shows the residual of Luenberger observer, Kalman observer, Unknown input observer and Sliding mode observer respectively for white noise disturbance present in the system. The effectiveness of each observer design is assessed by comparing the method of gain matrix, residual amplitude and deviation. Table 4.2 shows the residual of the observers for additive fault and white noise disturbance. In case of additive fault, range of
the residual of LO is the least one while the residual of KO is the highest, whereas during the disturbance signal injected into the system in form of white noise, the range of residual of SMO is the least one while the residual of LO is the highest, which suggests that the performance of SMO is superior in terms of rejecting the disturbance.

Table 4.2 Residual of observers

<table>
<thead>
<tr>
<th>Observers</th>
<th>Design of Gain Matrix</th>
<th>Residual (with Additive Fault)</th>
<th>Residual (with Disturbance)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LO</td>
<td>Pole placement</td>
<td>-0.002 to 0.004</td>
<td>-0.04 to 0.06</td>
</tr>
<tr>
<td>KO</td>
<td>Using correction matrix</td>
<td>-0.01 to 0.025</td>
<td>-0.05 to 0.05</td>
</tr>
<tr>
<td>UIO</td>
<td>Pole placement</td>
<td>-0.015 to 0.01</td>
<td>-0.015 to 0.02</td>
</tr>
<tr>
<td>SMO</td>
<td>Pole placement</td>
<td>-0.015 to 0.05</td>
<td>-0.01 to 0.01</td>
</tr>
</tbody>
</table>

The lowest residual deviation of SMO in case of white noise disturbance proves that the SMO is much superior in dealing with the disturbances present in the system. Furthermore, the performance of SMO can be improved by choosing suitable sliding mode manifold function. In the next section a modified SMO which can accommodate the uncertainty of the model in estimating the state variables is formulated.

4.5 ROBUST SMO FOR FAULT DETECTION

A robust sliding mode observer is first established based on constrained Lyapunov equation and then, the equivalent output error injection signal is employed to reconstruct the fault signal using the characteristics of Robust Sliding Mode Observer (RSMO). The basic idea behind the use of observers for fault detection is to form residuals from the difference between
the actual system outputs and the estimated outputs using an observer. Once a fault occurs, the residuals are expected to react by becoming greater than a prespecified threshold. When the system under consideration is subject to unknown disturbances or unknown inputs, their effect has to be decoupled from the residuals to avoid false alarms. By conducting proper residual analysis, a fault in the system can be identified. Simulations are carried out to verify the effectiveness of the proposed framework for sensor fault detection and estimation.

Consider the following state equation of RSMO

\[ \dot{x} = A\hat{x}(t) + Bu(t) - C(g\hat{x}(t) - y(t)) + Bu \]  

where \( \hat{x}(t) \) represents the state estimates for the system states, \( G \) is a constant design parameter matrix, and the discontinuous vector \( v \) is observer input. From Equation (4.58), the error of the system is determined as,

\[ \dot{e}(t) = A_e e(t) - B\xi(t, x, u) + Bu \]  

where \( A_e = A - GC \). The sliding mode manifold of the observer is designed as follows,

\[ s = Me = FCe \]  

where \( M = FC \). Matrix \( C \) in Equation (4.60) is the output matrix of the system, so the design of the sliding mode manifold depends on the design of parameter matrix \( F \). The sliding mode control strategy \( v \) is designed as

\[
v = \begin{cases} 
\frac{(sM^TB)^T}{\|sM^TB\|^2} |s| |(\rho|MB|) + \eta \| & \text{if } \|sM^TB\| \neq 0 \\
0 & \text{otherwise}
\end{cases}
\]  

(4.61)
For the positive scalar value of \( \eta \), the design parameter \( \rho \) can be chosen as

\[
\rho = r_1 \|u\| + \alpha(t, y) \quad (4.62)
\]

The error vector of the system can be decomposed as follows,

\[
e_1(t) = A_{011}e_1(t) + A_{012}e_2(t) \quad (4.63)
\]
\[
e_2(t) = A_{021}e_1(t) + A_{022}e_2(t) - B_2\xi(t, x, u) + B_2v \quad (4.64)
\]

where \( A_0 = \begin{bmatrix} A_{011} & A_{012} \\ A_{021} & A_{022} \end{bmatrix} \) \( B = \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \) Rewriting the sliding mode manifold in decomposed form,

\[
s = M_1e_1 + M_2e_2 \quad (4.65)
\]

where \( M = \begin{bmatrix} M_1 & M_2 \end{bmatrix} \). For convenience define two matrices as follows,

\[
A_s = M^TMA_0 \quad (4.66)
\]
\[
A_m = A_{011} - A_{012}M_2^{-1}M_1 \quad (4.67)
\]

To design the robust sliding mode observer to nonlinear uncertain system, a sliding mode manifold is chosen as in Equation (4.60). The sliding mode control strategy is adopted as in Equation (4.61) if the parameter \( \rho \) satisfies the matched condition in Equation (4.62) and design parameter matrix \( G \) makes the matrix \( A_0 \) in Equation (4.59) a Hurwitz matrix. When a design parameter matrix \( F \) makes \( A_m \) in Equation (4.67) a Hurwitz matrix, the following condition is satisfied.

\[
\lambda_{\text{max}}(A_s) \leq 0 \quad (4.68)
\]

where \( \lambda_{\text{max}}(A_s) \) represents the maximum eigenvalue of matrix. If the above condition holds good, then the sliding mode observer is robust to the
nonlinearities and/or uncertainties $f(t, x, u)$ present in system, and it can estimate the states of the system asymptotically. Consider the following Lyapunov function

$$V(s) = \frac{1}{2} s^T s = \frac{1}{2} (Me)^T Me = \frac{1}{2} e^T M^T Me$$

(4.69)

The derivative of $V(s)$ along the error system given in Equation (4.65) can be obtained as

$$\dot{V}(s) = s^T \dot{s} = e^T M^T Mb = e^T M^T M(\dot{A}e - Be \xi + Bu)$$

(4.70)

$$= e^T M^T MA_v e - s^T MB \xi + s^T MBv$$

(4.71)

$$= e^T A_v e - s^T MB \xi + s^T MBv$$

(4.72)

According to Rayleigh principle, it can be obtained,

$$\dot{V}(s) \leq \lambda_{\text{max}}(A_v) \|e\|^2 - s^T MB \xi + s^T MBv$$

(4.73)

Substituting Equation (4.64) and sliding mode control strategy given in Equation (4.61) into Equation (4.69) yields

$$\dot{V}(s) \leq -s^T MB \xi + s^T MBv$$

(4.74)

$$= -s^T MB \xi - \rho \|s\| \|MB\| - \eta \|s\|$$

(4.75)

$$\leq \|s\| \|MB\| \|\xi\| - \rho \|s\| \|MB\| - \eta \|s\|$$

(4.76)

$$= -\rho \|s\| \|MB\| - \eta \|s\|$$

(4.77)

According to Equation (4.60) the design parameter $\rho$ satisfies

$$\rho \geq \|\xi\|$$

so it can be obtained as
\[
\dot{V}(s) \leq -\eta ||s||
\] (4.78)

\[
\dot{V}(s) < 0 \quad \text{for} \quad ||s|| \neq 0
\] (4.79)

After the error system represented in Equation (4.59) reaches the sliding mode manifold surface \( s = 0 \), the dynamic behavior of error system will depend on the linear sliding mode manifold given in (4.60). When the error system reaches the sliding mode manifold surface \( s = 0 \),

\[
s = Me = M_1e_1 + M_2e_2 = 0
\] (4.80)

Error \( e_2 \) can be obtained as

\[
e_2 = -M_2^{-1}M_1e_1
\] (4.81)

Substituting Equation (4.81) into the error system Equation (4.63), the dynamics of the error can be obtained as

\[
\dot{e}_1(t) = (A_{011} - A_{012}M_2^{-1}M_1)e_1(t) = A_Me_1(t)
\] (4.82)

After reaching the sliding mode manifold surface \( s = 0 \), the error system will converge to the equilibrium point \( e = 0 \) asymptotically. If the sliding mode parameter matrix \( F \) is chosen such that it makes \( A_M \) a Hurwitz matrix, then the system will be asymptotically stable.

### 4.6 RESIDUAL GENERATION

The common procedure for fault detection and isolation using residuals is made of two main steps: residual generation, and residual evaluation. Residual generation is the core element of an observer based fault detection system. The residuals are generated using the difference between the real and the estimated outputs of the system. This difference is usually
computed using the norm of the output estimation error vector. RSMO approach for fault detection makes use of the disturbance decoupling principle, in which the residual is computed assuming the decoupling of the effects of faults on different inputs. For the purpose of fault isolation, it is also assumed that the effect of a fault is decoupled from the effects of other faults. A well designed residual signal is defined such that it is equal or near zero in the fault free case and is clear from zero when the system is faulty.

\[ r(t) = 0 \quad \text{or} \quad \{ r(t) \approx 0 \} \quad \text{Fault free case} \]

\[ r(t) \neq 0 \quad \text{Faulty case} \]

### 4.7 SIMULATION RESULTS AND DISCUSSION

The simulation results of the robust sliding mode observer scheme applied to a DC servo system is demonstrated. The model of the system, which is obtained by linearizing the system around stable equilibrium point is considered for the design procedures. The nonlinearity function \( \xi(t, x, u) \) which satisfies the match condition is given by

\[
\xi(t, x, u) = \begin{bmatrix}
0.3 \cos(2\pi t) \\
0.3 \sin(2\pi t)
\end{bmatrix}
\]

The upper limitation of matched nonlinearities and/or uncertainties is given by \( \|\xi(t, x, u)\| \leq 0.3 \). The estimated and actual values of state variables are shown in Figure 4.14 and Figure 4.15. Selecting the eigen values of the observer greater than the system makes the RSMO to estimate the state variable quite accurately even in the presence of nonlinearities. Result of speed sensor fault is shown in Figure 4.16. A sudden rise of the second state variable (abrupt fault) at 500 msec produces a residual greater than the threshold value. The residual of the system at 500 msec is greater than the threshold value as shown in Figure 4.17. The observer rejects the unknown
disturbance in the form of load disturbance which is represented as load torque (M_L) in the state equation. When the system is in normal working condition the generated residual is equal to zero or near zero implies the fault free case and in case of sensor fault the residual generated is greater than the prespecified value which clearly indicates the malfunction in the sensor which measures the speed of the motor.

Figure 4.14 Estimated state variable \( (X_1) \)

Figure 4.15 Estimated state variable \( (X_2) \)
Figure 4.16 Output variable during fault in speed sensor

Figure 4.17 Residual of RSMO during sensor fault
4.8 CONCLUSION

The performance of four observer algorithms, including Luenberger observer, Kalman observer, Unknown input observer, Sliding mode observer, to estimate the states of a system during additive fault and external disturbance has been assessed. A DC servo system, which has multiple inputs and multiple outputs, is taken for the implementation of the observer algorithms. Simulation study reveals that the performance of SMO is superior in terms of rejecting the disturbance present in the system. To improve performance of conventional SMO, a modified form of SMO has been formulated for fault detection and identification of a nonlinear system. Single sensor fault present in the system was identified by residual evaluation technique and the performance of the RSMO to decouple the disturbance from the fault was also highlighted. By selecting a suitable sliding mode manifold function, the convergence rate was altered to estimate the state variables quickly even in the presence of nonlinearities. The simulation results prove that the RSMO can quickly estimate the state variables of the system even in the presence of unknown disturbance.