Chapter 4

Distance Two Labeling of Graphs
4.1 Introduction

A swift and reliable communication is the demand of the current era. Any electronic communication system depends on various parameters like the set up of transmitters, frequency assignment to these transmitters and the spectrum (set of real numbers from which frequency is assigned). The quality of communication is affected by the interference. The arrangement of transmitters and frequency assignment are responsible for the interference. If we focus on the interference which occurs only due to the distance between two transmitters, then it is observed that closer the transmitters, higher the interference and farther the transmitters, lower the interference. Hale [38] initiated this problem as a distance constrained frequency assignment problem which is widely known as channel assignment problem. This problem was first formulated as a graph labeling problem by Hale [38] and he introduced the notion of $T$-coloring of a graphs (the vertices of the graph are assigned values subject to certain conditions is known as graph labeling). Griggs and Yeh [36] considered a more general problem and defined a distance two labeling or $L(2, 1)$-labeling. In succeeding sections, distance two labeling of some graphs is briefly discussed.

4.2 Channel Assignment Problem

The channel assignment problem is the problem to assign a channel (non negative integer) to each TV or radio transmitters located at various places such that communication does not interfere. The interference between two transmitters mainly depends on efficiency of transmitters, available spectrum for channels and geographic location of transmitters. But if we assume that all the transmitters are equally efficient and available spectrum is fixed then the interference between two transmitters is closely related with the geographic location of the transmitters. If we consider two level interference namely major and minor then two transmitters are very close if the interference is major while close if the interference is minor.
4.3 Graph Model for Channel Assignment Problem

In 1980, Hale [38] developed a graph model for channel assignment problem in which the transmitters are represented by the vertices of a graph; two vertices are very close if they are adjacent in the graph and close if they are at distance two apart in the graph. The channels are related with non-negative integers known as labels of vertices in graph model.

4.4 T-coloring of a graph

Suppose $G$ is a graph and $T$ is a set of non-negative integers. A $T$-coloring of $G$ is an assignment of a positive integer $f(v)$ to each vertex $v$ of $G$ so that if $u$ and $v$ are joined by an edge of $G$, then $|f(u) - f(v)|$ is not in $T$. The concept of $T$-coloring has been extensively studied in recent past by many researches like Cozzens and Roberts [21, 22], Cozzens and Wang [23], Furedi et al. [29], Liu [49–52], Raychaudhuri [58, 59], Roberts [60, 61], Smith [68] and Tesman [69].

4.5 Distance Two Labeling and $\lambda$-number

In a private communication with Griggs during 1988 Roberts proposed a variant of channel assignment problem in which close transmitters must receive different channel and very close transmitters must receive channels that are at least two apart. Motivated by this problem Griggs and Yeh [36] introduced $L_{d}(2, 1)$-labeling and $\lambda(G, d)$-number:

**Definition 4.5.1.** Given a real number $d > 0$, an $L_{d}(2, 1)$-labeling of $G$ is a non-negative real-valued function $f : V(G) \to [0, \infty]$ such that the following conditions are satisfied:

1. $|f(u) - f(v)| \geq 2d$ if $d(u, v) = 1$
2. $|f(u) - f(v)| \geq d$ if $d(u, v) = 2$
For a graph $G$, if $f$ is an $L_d(2,1)$-labeling of $G$, then we say that $f \in L_d(2,1)(G)$. Define $\|f(G)\| = \max\{f(v) : v \in V(G)\}$. Then $\lambda(G,d) = \min\|f(G)\|$, where the minimum runs over all $f \in L_d(2,1)(G)$. In other word, the $L_d(2,1)$-labeling number of $G$ is the smallest number $k$ such that $G$ has an $L_d(2,1)$-labeling with no label greater than $k$. Griggs and Yeh [36] characterized $\lambda(G,d)$ in terms of $\lambda(G,1)$ as $\lambda(G,d) = d\lambda(G,1)$. They also showed that for $\lambda(G,1)$ it suffices to consider integral valued labeling in which 0 is allow as a label. Thereafter the $L_d(2,1)$-labeling has becomes popular as the $L(2,1)$-labeling or distance two labeling while $\lambda(G,1)$-number is known as $\lambda$-number for graph $G$ which defined as follows:

**Definition 4.5.2.** A distance two labeling ($L(2,1)$-labeling) of a graph $G = (V(G), E(G))$ is a function $f$ from the vertex set $V(G)$ to the set of nonnegative integers such that the following conditions are satisfied:

1. $|f(u) - f(v)| \geq 2$ if $d(u,v) = 1$
2. $|f(u) - f(v)| \geq 1$ if $d(u,v) = 2$

The span of $f$ is defined as $\max\{|f(u) - f(v)| : u,v \in V(G)\}$. The $\lambda$-number for a graph $G$, denoted by $\lambda(G)$, is the minimum span of a distance two labeling for $G$. A $k-L(2,1)$-labeling is an $L(2,1)$-labeling such that no label is grater than $k$. The $L(2,1)$-labeling number of $G$, denoted by $\lambda(G)$ or $\lambda$, is the smallest number $k$ such that $G$ has a $k-L(2,1)$-labeling.

A distance two labeling ($L(2,1)$-labeling) of various graph families is studied by many researchers like Calamoneri et al. [3], Chang and Kuo [4], Chang et al. [5, 6], Chen and Wang [19], Damei [24], Fishburn and Roberts [27, 28], Georges et al. [31–35], Jha [41], Jha et al. [42, 43], Jonas [44], Kang [45], Kuo and Yan [48], Lu et al. [57], Sakai [63], Schwarz and Troxell [64], Shao and Yeh [65], Shiu et al. [67], Vaidya et al. [72], Wang [74], Whittlesey et al. [76], Yeh [78, 79] and Zhou [80].

**Illustration 4.5.3.** In Figure 4.1, an optimal $L(2,1)$-labeling of $P_4$, $C_4$, $K_{1,5}$ and $K_5$ is shown for which $\lambda(P_4) = 3$, $\lambda(C_4) = 4$, $\lambda(K_{1,5}) = 6$ and $\lambda(K_5) = 8$. 
4.6 Some Known Results

In this section, we list some known results of distance two labeling and $\lambda$-number for various graph families.

Griggs and Yeh [36] introduced the concept of distance two labeling and prove some fundamental results in the form of following results.

**Proposition 4.6.1.** The $\lambda$-number of a $K_{1,\Delta}$ is $\Delta + 1$, where $\Delta$ is the maximum degree.

**Proposition 4.6.2.** The $\lambda$-number of a complete graph $K_n$ is $2n - 2$.

**Proposition 4.6.3.** Let $G$ be a graph with maximum degree $\Delta \geq 2$. If $G$ contains three vertices of degree $\Delta$ such that one of them is adjacent to the other two, then $\lambda(G) \geq \Delta + 2$.

**Proposition 4.6.4.** Let $P_n$ be a path on $n$ vertices. Then (i) $\lambda(P_2) = 2$, (ii) $\lambda(P_3) = \lambda(P_4) = 3$, and (iii) $\lambda(P_n) = 4$, for $n \geq 5$.

**Proposition 4.6.5.** Let $C_n$ be a cycle of length $n$. Then $\lambda(C_n) = 4$, for any $n$.

**Theorem 4.6.6.** Let $Q_n$ be the $n$-cube. Then, for all $n \geq 5$, $n + 3 \leq \lambda(Q_n) \leq 2n + 1$.

**Theorem 4.6.7.** Let $T$ be a tree with maximum degree $\Delta \geq 1$. Then $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$.

**Theorem 4.6.8.** Let $G$ be a graph with maximum degree $\Delta$. Then $\lambda(G) \leq \Delta^2 + 2\Delta$.

**Theorem 4.6.9.** If $G$ is a graph with diameter 2, then $\lambda(G) \leq \Delta^2$. 

\[ 
\begin{array}{c}
\text{Figure 4.1: } \lambda(P_4) = 3, \lambda(C_4) = 4, \lambda(K_{1,5}) = 6 \text{ and } \lambda(K_5) = 8. \\
\end{array} \]
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**Conjecture 4.6.10.** For any graph $G$ with maximum degree $\Delta \geq 2$, $\lambda(G) \leq \Delta^2$.

Sakai [63] showed that chordal graphs satisfy the Conjecture 4.6.10 and established the following result.

**Theorem 4.6.11.** Let $G$ be a chordal graph with maximum degree $\Delta(G)$. Then $\lambda(G) \leq \frac{(\Delta(G)+3)^2}{4}$.

Chang and Kuo [4] improved the bound for $\lambda$-number which is given in Theorem 4.6.8 by Griggs and Yeh [36] for arbitrary graph and discuss some important results.

**Proposition 4.6.12.** $\lambda(H) \leq \lambda(G)$, for any subgraph $H$ of a graph $G$.

**Lemma 4.6.13.** $\lambda(G \cup H) = \max\{\lambda(G), \lambda(H)\}$, for any two graphs $G$ and $H$.

**Theorem 4.6.14.** $\lambda(G) \leq \Delta^2 + \Delta$, for any graph $G$ with maximum degree $\Delta$.

Chang and Kuo [4] have also presented a polynomial time algorithm to determine $\lambda(T)$ of a tree $T$.

Georges and Mauro [31] studied $L(j,k)$-labeling and investigated the following results.

**Theorem 4.6.15.** If $3 \leq q$ and $p$ is prime. Then
(1) $\lambda^2_1(K^q_p) = p^{2r} - 1$ if $r > 1$ and $q \leq p$ and
(2) $\lambda^2_1(K^q_p) = p^2 - 1$ if $q < p$.

**Theorem 4.6.16.** Let $j, k, n$ and $m$ be integer where $2 \leq n \leq m$ and $j \geq k$. Then
(1) $\lambda^j_k(K_n \times K_m) = (m-1)j + (n-1)k$ if $\frac{j}{k} > n$ and
(2) $\lambda^j_k(K_n \times K_m) = (mn-1)$ if $\frac{j}{k} \leq n$.

**Theorem 4.6.17.** Let $j, k$ and $n$ be integers where $2 \leq n$ and $j \geq k$. Then
(1) $\lambda^j_k(K^2_n) = (n-1)j + (2n-2)k$ if $\frac{j}{k} > n - 1$ and
(2) $\lambda^j_k(K^2_n) = (n^2 - 1)k$ if $\frac{j}{k} \leq n - 1$.

Kang [45] settled the Conjecture 4.6.10 for Hamiltonian graphs with maximum degree 3 by investigating the following result.
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**Theorem 4.6.18.** If $G$ is a Hamiltonian graph with maximum degree 3, then $G$ has a $L(2,1)$-labeling of $\lambda(G) \leq 9$.

Jha et al. [42] discuss $L(2,1)$-labeling of direct product of paths and cycles and they proved that

**Theorem 4.6.19.** If $m \equiv 0 \pmod{7}$ and $n \equiv 0 \pmod{7}$, then $\lambda(C_m \times C_n) = 6$.

**Corollary 4.6.20.** If $m \geq 5$, $n \geq 4$ and $i \geq 1$, then $\lambda(P_m \times P_n) = \lambda(C_{7i} \times P_n) = 6$.

**Theorem 4.6.21.** If $r \equiv 0 \pmod{11}$, $s \equiv 0 \pmod{11}$ and $t \equiv 0 \pmod{11}$, then $\lambda(C_r \times C_s \times C_t) = 10$.

**Theorem 4.6.22.** For any $m \geq 3$, $\lambda(P_4 \times C_m) = 6$.

**Theorem 4.6.23.** Let $m \geq 3$. Then $\lambda(P_s \times C_m) = 7$ if $m = 3, 4, 5, 6, 8, 9, 10, 12, 13, 17, 18, 20, 24, 26, 34, 40$ and $\lambda(P_s \times C_m) = 6$; otherwise.

**Theorem 4.6.24.** Let $n \geq 6$ and $m \geq 7$. Then $\lambda(P_n \times C_m) = 6$ if and only if $m = 7k$, $k \geq 1$.

Schwartz and Sakai [64] completely determine the $\lambda$-number of cartesian product of two cycles as follows.

**Theorem 4.6.25.** If $n, m \geq 3$ then

$$\lambda(C_n \times C_m) = \begin{cases} 
6, & \text{if } n, m \equiv 0 \pmod{3} \\
8, & \text{if } \{n,m\} \in A \\
7, & \text{otherwise.}
\end{cases}$$

where $A = \{\{3,i\} : i \geq 3, i \text{ odd or } i = 4, 10\} \cup \{\{5,i\} : i = 5, 6, 9, 10, 13, 17\} \cup \{\{6,7\}, \{6,11\}, \{7,9\}, \{9,10\}\}$.

Shao and Yeh [65] proved that

**Theorem 4.6.26.** Let $\Delta$ be the maximum degree of $G \times H$. Then $\lambda\{G \times H\} \leq \Delta^2 + 1$ if one of $\Delta(G)$ and $\Delta(H)$ is 1. Otherwise $\Delta(G \times H) \leq \Delta^2$. 
Jha [41] discussed optimal $L(2, 1)$-labeling of strong products of cycles as follows.

**Theorem 4.6.27.** If $k \geq 1$ and $m_0, \ldots, m_{k-1}$ are each a multiple of $3^k + 2$, then $\lambda(C_{m_0} \boxtimes \ldots \boxtimes C_{m_{k-1}}) = 3^{k} + 1$.

### 4.7 Distance Two Labeling of Cactus Graph

In the discussion of $\lambda$-number of graphs, much attention has been paid to a connected acyclic graphs, that is, tree $T$ because the maximum degree determines the distance two labeling number of trees. Griggs and Yeh [36] have proved that the $\lambda$-number of any tree is $\Delta + 1$ or $\Delta + 2$. As time passed, the trees were classified depending upon their $\lambda$-number. The trees $T$ with $\lambda$-number $\Delta + 1$ are classified as type 1 while the trees $T$ with $\lambda$-number $\Delta + 2$ are classified as type 2. This concept was remained the focus of many research papers. We present here a graph families whose $\lambda$-number is $\Delta + 1$ or $\Delta + 2$ but they are not trees.

**Theorem 4.7.1.** For $P_m(K_n)$,

$$\lambda(P_m(K_n)) = \begin{cases} 
\Delta + 1, & \text{if } m = 3 \\
\Delta + 2, & \text{if } m \geq 4.
\end{cases}$$

**Proof.** Let $P_m(K_n)$ be the linear cacti whose vertex set is $V(P_m(K_n)) = \{v_i^j, v_{m+1}^j : 1 \leq i \leq m, 1 \leq j \leq n-1\}$ and $E(P_m(K_n)) = \{v_i^jv_{i+1}^j, v_i^jv_k^j : 1 \leq i \leq m, 1 \leq j, k \leq n-1, j \neq k\}$. The graph $K_1^\Delta$ is a subgraph of $P_m(K_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, it follows that $\lambda(P_m(K_n)) \geq \Delta + 1$. For $m = 3$, define $f$ by $f(v_1^1) = 3, f(v_1^2) = 2l + 1, f(v_1^2) = 0, f(v_2^1) = 2l - 2, f(v_2^1) = \Delta + 1, f(v_3^1) = 2l - 3, f(v_3^1) = 2n - 3$, where $2 \leq l \leq n - 1$ which is an $L(2, 1)$-labeling of $P_3(K_n)$ and hence $\lambda(P_3(K_n)) = \Delta + 1$. For $m \geq 4$, in the graph $P_m(K_n)$, the close neighborhood of each $v_i^j$ where $i = 3, \ldots, m - 1$ contains three vertices with degree $\Delta$ and hence by Proposition 4.6.3, $\lambda(P_m(K_n)) \geq \Delta + 2$. Now for each $i = 1, 2, \ldots, m + 1$ and $j = 1, 2, \ldots, n - 1$ define $f : V(P_m(K_n)) \rightarrow \{0, 1, 2, \ldots, \Delta + 2\}$ as follows:
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\[ f(v_1^2) = 0 \quad \text{if } i \equiv 1 \pmod{4} \]
\[ f(v_1^2) = \Delta + 1 \quad \text{if } i \equiv 2 \pmod{4} \]
\[ f(v_1^3) = 1 \quad \text{if } i \equiv 3 \pmod{4} \]
\[ f(v_1^4) = \Delta + 2 \quad \text{if } i \equiv 0 \pmod{4} \]
\[ f(v_j^2) = 2j - 2 \quad \text{if } i \equiv 1 \pmod{2} \]
\[ f(v_j^2) = 2j - 1 \quad \text{if } i \equiv 0 \pmod{2} \]

In above defined function, redefine \( f \) at \( f(v_2^i) \), where \( i \equiv 3 \pmod{4} \) as \( f(v_2^i) = \Delta - 2 \) then \( f \) is an \( L(2,1) \)-labeling for \( P_m(K_n) \) and from the definition of \( f \) it is clear that for \( m \geq 4 \), \( \lambda(P_m(K_n)) \leq \Delta + 2 \).

Thus, we have

\[
\lambda(P_m(K_n)) = \begin{cases} 
\Delta + 1, & \text{if } m = 3 \\
\Delta + 2, & \text{if } m \geq 4. 
\end{cases}
\]

Illustration 4.7.2. In Figure 4.2, an optimal \( L(2,1) \)-labeling of linear cactus \( P_7(K_4) \) is shown for which \( \lambda(P_7(K_4)) = \Delta + 2 = 6 + 2 = 8 \).

\[ \text{Figure 4.2: } \lambda(P_7(K_4)) = 8. \]

Theorem 4.7.3. For \( S_{n_1,n_2,...,n_k}(K_n) \), \( \lambda(S_{n_1,n_2,...,n_k}(K_n)) = \Delta + 1 \).

Proof. Let \( S_{n_1,n_2,...,n_k}(K_n) \) be the spider cactus whose vertex set is \( V(S_{n_1,n_2,...,n_k}(K_n)) = \{v_{i,j}^l, v_{n_l+1,1}^i : 1 \leq l \leq k, 1 \leq i \leq n_l, 1 \leq j \leq n - 1 \text{ where } v_{1,1}^1 = v_{1,1}^2 = ... = v_{1,1}^k = v_0 \} \) and \( E(S_{n_1,n_2,...,n_k}(K_n)) = \{v_{i,j}^l, v_{i+1,1}^l, v_{i,j}^l, v_{i,m}^l : 1 \leq l \leq k, 1 \leq i \leq n_l, 1 \leq j, m \leq n - 1 \} \).
$n - 1, j \neq m$. The graph $K_{1,\Delta}$ is a subgraph of $S_{n_1,n_2,...,n_k}(K_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, it follows that $\lambda(S_{n_1,n_2,...,n_k}(K_n)) \geq \Delta + 1$. Now define $f : V(S_{n_1,n_2,...,n_k}(K_n)) \rightarrow \{0, 1, 2, ..., \Delta + 1\}$ as follows:

$$f(v_0) = \Delta + 1$$

$$f(v_{1,2}) = 0$$

$$f(v_{2,2}) = 1$$

$$... ... ...$$

$$f(v_{k,2}) = k - 1$$

$$f(v_{1,3}) = f(v_{k,2}) + 1$$

$$f(v_{2,3}) = f(v_{1,3}) + 1$$

$$... ... ...$$

$$f(v_{1,3}) = f(v_{1,3}^{k-1}) + 1$$

$$... ... ...$$

$$f(v_{1,n-1}) = f(v_{1,n-2}) + 1$$

$$f(v_{2,n-1}) = f(v_{1,n-1}) + 1$$

$$... ... ...$$

$$f(v_{k,n-1}) = f(v_{1,n-1}^{k-1}) + 1$$

$$f(v_{i,j}) = (f(v_{i-1,j}) + 2(n - 1)) \mod (\Delta + 1)$$

The above defined function is an $L(2, 1)$-labeling for $S_{n_1,n_2,...,n_k}(K_n)$ and from the definition of $f$ it is clear that $\lambda(S_{n_1,n_2,...,n_k}(K_n)) \leq \Delta + 1$.

Thus, we have $\lambda(S_{n_1,n_2,...,n_k}(K_n)) = \Delta + 1$. \hfill $\blacksquare$
Illustration 4.7.4. In Figure 4.3, an optimal \(L(2, 1)\)-labeling of spider cactus \(S_{4,3,2,1}(K_4)\) is shown for which \(\lambda(S_{4,3,2,1}(K_4)) = \Delta + 1 = 12 + 1 = 13\).

\[\begin{array}{c}
\includegraphics[width=0.7\textwidth]{fig4_3.png}
\end{array}\]

Figure 4.3: \(\lambda(S_{4,3,2,1}(K_4)) = 13\).

Corollary 4.7.5. For \(K'_t\) (one point union of \(t\) complete graph \(K_n\)), \(\lambda(K'_t) = \Delta + 1\).

Illustration 4.7.6. In Figure 4.4, an optimal \(L(2, 1)\)-labeling of \(K^3_4\) is shown for which \(\lambda(K^3_4) = \Delta + 1 = 9 + 1 = 10\).

\[\begin{array}{c}
\includegraphics[width=0.7\textwidth]{fig4_4.png}
\end{array}\]

Figure 4.4: \(\lambda(K^3_4) = 10\).

Theorem 4.7.7. Let \(G\) be a lobster cactus then \(\lambda(G) = \Delta + 1\) or \(\Delta + 2\), where \(\Delta\) is the maximum degree of the vertex.
Proof. Let $G$ be a lobster cactus having a vertex with maximum degree $\Delta$. The graph $K_{1,\Delta}$ is a subgraph of $G$ and hence by Proposition 4.6.1 and Proposition 4.6.12, it follows that $\lambda(G) \geq \Delta + 1$. The following algorithm establishes the upper bound.

**Algorithm 4.7.8.** An optimal $L(2,1)$-labeling of given lobster cactus.

**Input:** A lobster cactus with maximum degree $\Delta$.

**Idea:** Identify the vertices which are at distance one and two apart.

**Initialization:** Let $v_0$ be the vertex of degree $\Delta$. Label the vertex $v_0$ by 0 and take $S = \{v_0\}$.

**Iteration:** Define $f : V(G) \to \{0, 1, 2, \ldots\}$

**Step 1:** Find $N(v_0)$. If $N(v_0) = \{v_1, v_2, \ldots, v_\Delta\}$ then partition $N(v_0)$ into $k$ sets $V_1, V_2, \ldots, V_k$ such that for each $i = 1, 2, \ldots, k$ the graph induced by $V_i \cup \{v_0\}$ forms a complete subgraph of $G$. The definition of $G$ itself confirms the existence of such partition with the characteristic that for $i \neq j, u \in V_i, v \in V_j, d(u, v) = 2$.

**Step 2:** Choose a vertex $v_1 \in N(v_0)$ and define $f(v_1) = 2$. Find a vertex $v_2 \in N(v_0)$ such that $d(v_1, v_2) = 2$ and define $f(v_2) = 3$. Continue this process till all the vertices of $N(v_0)$ are labeled. Take $S = \{v_0\} \cup \{v \in V(G) / f(v) \text{ is a label of } v\}$.

**Step 3:** For $f(v_i) = i$. Find $N(v_i)$ and define $f(v) = \text{the smallest number from the set } \{0, 1, 2, \ldots\} - \{i - 1, i, i + 1\}$, where $v \in N(v_i) - S$ such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. Denote $S = S^1 \cup \{v \in V(G) / f(v) \text{ is a label of } v\}$.

**Step 4:** Continue this recursive process till $S^n = V(G)$, where $S^n = S^{n-1} \cup \{v \in V(G) / f(v) \text{ is a label of } v\}$.

**Output:** $\max\{f(v) / v \in V(G)\} = \Delta + 2$.

Hence, $\lambda(G) \leq \Delta + 2$.

Thus, $\lambda(G)$ is either $\Delta + 1$ or $\Delta + 2$. ■
Illustration 4.7.9. In FIGURE 4.5, an optimal $L(2, 1)$-labeling of lobster cactus is shown for which $\lambda(G) = \Delta + 1 = 12 + 1 = 13$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{lobster_cactus_labeled}
\caption{\(\lambda(G) = 13\).}
\end{figure}

Corollary 4.7.10. Let $G$ be a caterpillar cactus with maximum degree of vertex $\Delta$ then $\lambda(G) = \Delta + 1$ or $\Delta + 2$.

Illustration 4.7.11. In FIGURE 4.6, an optimal $L(2, 1)$-labeling of caterpillar graph is shown for which $\lambda(G) = \Delta + 1 = 12 + 1 = 13$.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{caterpillar_cactus_labeled}
\caption{\(\lambda(G) = 13\).}
\end{figure}
Theorem 4.7.12. Let $C(K_n(k))$ be an $n$-complete $k$-regular cactus with maximum degree $\Delta$ and $k \geq 3$. Then $\lambda(C(K_n(k)))$ is either $\Delta + 1$ or $\Delta + 2$.

**Proof.** Let $C(K_n(k))$ be an $n$-complete $k$-regular cactus with maximum degree $\Delta$. The star $K_{1,\Delta}$ is a subgraph of $C(K_n(k))$ and hence $\lambda(C(K_n(k))) \geq \Delta + 1$.

For upper bound, we apply the following Algorithm:

**Algorithm 4.7.13.** An optimal $L(2,1)$-labeling of given $n$-complete $k$-regular cactus.

**Input:** An $n$-complete $k$-regular cactus graph with maximum degree $\Delta$.

**Idea:** Identify the vertices which are at distance one and two apart.

**Initialization:** Let $v_0$ be the vertex of degree $\Delta$. Label the vertex $v_0$ by 0 and take $S = \{v_0\}$

**Iteration:** Define $f : V(G) \to \{0, 1, 2, \ldots\}$ as follows.

**Step 1:** Find $N(v_0)$. If $N(v_0) = \{v_1, v_2, \ldots, v_\Delta\}$ then partition $N(v_0)$ into $k$ sets $V_1, V_2, \ldots, V_k$ such that for each $i = 1, 2, \ldots, k$ the graph induced by $V_i \cup \{v_0\}$ forms a complete subgraph of $C(K_n(k))$. The definition of $C(K_n(k))$ itself confirms the existence of such partition with the characteristic that for $i \neq j, u \in V_i, v \in V_j, d(u, v) = 2$.

**Step 2:** Choose a vertex $v_1 \in N(v_0)$ and define $f(v_1) = 2$. Find a vertex $v_2 \in N(v_0)$ such that $d(v_1, v_2) = 2$ and define $f(v_2) = 3$. Continue this process till all the vertices of $N(v_0)$ are labeled. Take $S = \{v_0\} \cup \{v \in V(G) / f(v) \text{ is a label of } v\}$.

**Step 3:** For $f(v_i) = i$. Find $N(v_i)$ and define $f(v) = \{0, 1, 2, \ldots\} - \{i - 1, i, i + 1\}$, where $v \in N(v_i) - S$ such that $|f(u) - f(v)| \geq 2$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq 1$ if $d(u, v) = 2$. Denote $S \cup \{v \in V(G) / f(v) \text{ is a label of } v\} = S^n$.

**Step 4:** Continue this recursive process till $S^n = V(G)$, where $S^n = S^{n-1} \cup \{v \in V(G) / f(v) \text{ is a label of } v\}$.

**Output:** $\max\{f(v) / v \in V(G)\} = \Delta + 2$. Hence, $\lambda(C(K_n(k))) \leq \Delta + 2$

Thus, $\lambda(C(K_n(k)))$ is either $\Delta + 1$ or $\Delta + 2$. □
Illustration 4.7.14. In Figure 4.7, an optimal $L(2,1)$-labeling of 3-complete 3-regular cactus $C(K_3(3))$ is shown for which $\lambda(C(K_3(3))) = \Delta + 2 = 8$.

Griggs and Yeh [36] have proved that (1) $\lambda(P_2) = 2$, (2) $\lambda(P_3) = \lambda(P_4) = 3$, and (3) $\lambda(P_n) = 4$, for $n \geq 5$. This can be verified by Proposition 4.6.3 and using our Algorithm 4.7.13. In fact, any path $P_n$ is 2-complete 2-regular cactus $C(K_2(2))$. Thus a single Algorithm will work to determine the $\lambda$-number of path $P_n$. Using Algorithm 4.7.13 the $L(2,1)$-labeling of $P_2$, $P_3$, $P_4$ and $P_5$ is demonstrated in Figure 4.8.

Figure 4.7: $\lambda(C(K_3(3))) = \Delta + 2 = 8$.

Figure 4.8: $\lambda(P_2) = 2$, $\lambda(P_3) = \lambda(P_4) = 3$ and $\lambda(P_5) = 4$. 
**Theorem 4.7.15.** Let $C(K_n)$ be an $n$-complete cactus with at least one cut vertex which belongs to at least three blocks. Then $\lambda(C(K_n))$ is either $\Delta + 1$ or $\Delta + 2$.

**Proof.** Let $C(K_n)$ be the arbitrary an $n$-complete cactus with maximum degree $\Delta$. The graph $K_{1,\Delta}$ is a subgraph of $C(K_n)$ and hence by Propositions 4.6.1 and Proposition 4.6.12, $\lambda(C(K_n)) \geq \Delta + 1$. Moreover, $C(K_n)$ is a subgraph of $C(K_n(k))$ (where $k = \frac{\Delta}{n-1}$) and hence by Proposition 4.6.12 and Theorem 4.7.12, $\lambda(C(K_n)) \leq \Delta + 2$.

Thus, we proved that $\lambda(C(K_n))$ is either $\Delta + 1$ or $\Delta + 2$. 

**Corollary 4.7.16.** Let $T$ be a tree with maximum degree $\Delta \geq 2$. Then $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$.

**Proof.** Let $T$ be a tree with maximum degree $\Delta \geq 2$. If $\Delta = 2$ then $T$ is a path and problem is settled. But if $\Delta > 2$ then $\lambda(T) \geq \Delta + 1$ as $K_{1,\Delta}$ is a subgraph of $T$. The upper bound of $\lambda$-number is $\Delta + 2$ according to Theorem 4.7.15 as any tree $T$ is a 2-complete cactus $C(K_2)$.

Thus, we proved that $\lambda(T)$ is either $\Delta + 1$ or $\Delta + 2$.

**Illustration 4.7.17.** In Figure 4.9, an optimal $L(2,1)$-labeling of tree $T$ is shown which is 2-complete cactus $C(K_2)$ with maximum degree $\Delta = 3$ for which $\lambda(T) = \lambda(C(K_2)) = \Delta + 2 = 5$.

![Figure 4.9: $\lambda(T) = \lambda(C(K_2)) = \Delta + 2 = 5.$](image)

**Corollary 4.7.18.** $\lambda(K_{1,n}) = n + 1$.

**Proof.** The star $K_{1,n}$ is a 2-complete $n$-regular cactus. Then by Theorem 4.7.12, $\lambda(K_{1,n}) = n + 1$. 


Corollary 4.7.19. For the Friendship graph $F_n$, $\lambda(F_n) = 2n + 1$.

Proof. The Friendship graph $F_n$ is a 3-complete 2n-regular cactus. Then by Theorem 4.7.12, $\lambda(F_n) = 2n + 1$.

Illustration 4.7.20. In Figure 4.10 and Figure 4.11, an optimal $L(2, 1)$-labeling of star $K_{1,4}$ and Friendship graph $F_4$ are shown for which $\lambda$-number is 5 and 9 respectively.

Definition 4.7.21. An $n$-ary $k$-regular cactus is a connected graph whose all blocks are isomorphic to $C_n$ and block cutpoint graph is a tree having each block vertex $b_i$ has degree $n$ except leaf blocks and each cut vertex has degree $\frac{k}{2}$. We will denote it by $C_n(k)$.

Observation 4.7.22. For an $n$-ary $k$-regular cactus we notice that

- $n \geq 3$.
- $k \geq 4$ and $k = \text{even}$.
- The maximum degree of a vertex is $k$. 
Illustration 4.7.23. In the following Figure 4.12, the 4-ary 4-regular cactus is presented while Figure 4.13 shows its block cutpoint graph.

\[\text{Figure 4.12: } C_4(4)\]

\[\text{Figure 4.13: Block cut point graph of } C_4(4)\]

Theorem 4.7.24. The \(\lambda\)-number of an \(n\)-ary \(\Delta\)-regular cactus is \(\Delta + 1\) or \(\Delta + 2\).

Proof. Let \(C_n(\Delta)\) be an \(n\)-ary \(\Delta\)-regular cactus. The graph \(K_{1,\Delta}\) is a subgraph of \(G\) and as \(\lambda(K_{1,\Delta}) = \Delta + 1\) by Proposition 4.6.1 and Proposition 4.6.12, it follows that \(\lambda(C_n(\Delta)) \geq \Delta + 1\). We now show that there exists an \(L(2,1)\)-labeling of \(C_n(\Delta)\) with labels from the set \(S = \{0, 1, \ldots, \Delta + 2\}\).
Let \( v_0 \) be the vertex with degree \( \Delta \). Label the vertex \( v_0 \) by 0 and its adjacent vertices from the set \( \{2, 3, \ldots, \Delta + 2\} \). Let \( v_{0i} \) be the adjacent vertex to \( v_0 \) and it has label \( i \), for some \( i \in \{2, 3, \ldots, \Delta + 1\} \). Now consider \( v_{0ij} \) which is a vertex adjacent to \( v_{0i} \). In \( C_n(\Delta) \), the vertex \( v_{0i} \) is adjacent to at most \( \Delta - 1 \) vertices in the graph. Hence \( v_{0ij} \) can be assigned a label that differ from those assigned to at most \( \Delta - 1 \) vertices and differ from any label within 1 of the labels assigned to \( v_{0i} \). Hence at most \( (\Delta - 1) + 3 = \Delta + 2 \) labels cannot be used to label \( v_{0ij} \) leaving at least one available label in \( S \) to label \( v_{0ij} \) and obtain \( \lambda(C_n(\Delta)) \leq \Delta + 2 \)

Thus, \( \lambda(C_n(\Delta)) = \Delta + 1 \) or \( \Delta + 2 \).

\[ \square \]

On dropping the \( k \) regularity of cut vertices in an \( n \)-ary \( k \)-regular cactus we prove a general result as a corollary.

**Corollary 4.7.25.** The \( \lambda \)-number of an \( n \)-ary cactus with maximum degree \( \Delta \) is \( \Delta + 1 \) or \( \Delta + 2 \).

**Proof.** Let \( G \) be the arbitrary an \( n \)-ary cactus with maximum degree \( \Delta \). The graph \( K_{1,\Delta} \) is a subgraph of \( G \) and hence by Proposition 4.6.1 and Proposition 4.6.12, \( \lambda(G) \geq \Delta + 1 \). Note that \( G \) is a subgraph of \( C_n(\Delta) \) and hence by Theorem 4.7.24, \( \lambda(G) \leq \Delta + 2 \).

Thus, \( \lambda(G) = \Delta + 1 \) or \( \Delta + 2 \).

\[ \square \]

First we present some regular cacti whose \( \lambda \)-number is precisely \( \Delta + 1 \) and later we give an example of regular cactus whose \( \lambda \)-number is precisely \( \Delta + 2 \). A one point union of \( k \)-copies of cycle \( C_n \) is the graph obtained by taking \( v \) as a common vertex such that any two cycles are edge disjoint and do not have any vertex in common except \( v \). We will denote it by \( C_n^{(k)} \). The Theorem 4.7.26 deals with one point union of two cycles where exact label assignment is carried out while to prove Theorem 4.7.27 we choose analytical approach.

**Theorem 4.7.26.** \( \lambda(C_n^{(2)}) = \lambda(C_n(4)) = 5 \).

**Proof.** Let \( C_n^{(2)} \) be the one point union of two cycles \( C_n \) with \( n \) vertices respectively. Let \( v_j^1, 0 \leq j \leq n - 1 \) and \( v_j^2, 0 \leq j \leq n - 1 \) be the vertices of \( C_n^{(2)} \). Without loss of
generality assume that $v_0 = v_0^1 = v_0^2$. The graph $K_{1,4}$ is a subgraph of one point union of two cycles and hence by Proposition 4.6.1 and Proposition 4.6.12, $\lambda(C_{n}^{(2)}) \geq 5$.

Now we want to show that $\lambda(C_{n}^{(2)}) \leq 5$. Define $f : V(C_{n}^{(2)}) \rightarrow \{0, 1, 2, ..., 5\}$ as follows:

1. $n \equiv 0 \ (mod\ 3)$
   \[
   \begin{align*}
   f(v_j^1) &= 0 \text{ if } j \equiv 0 \ (mod\ 3) \\
   f(v_j^1) &= 2 \text{ if } j \equiv 1 \ (mod\ 3) \\
   f(v_j^1) &= 4 \text{ if } j \equiv 2 \ (mod\ 3) \\
   f(v_j^2) &= 0 \text{ if } j \equiv 0 \ (mod\ 3) \\
   f(v_j^2) &= 3 \text{ if } j \equiv 1 \ (mod\ 3) \\
   f(v_j^2) &= 5 \text{ if } j \equiv 2 \ (mod\ 3)
   \end{align*}
   \]

2. $n \equiv 1 \ (mod\ 3)$ accept for $n = 4$, redefine the above $f$ of (1) at $v_{n-3}^1, v_{n-2}^1, v_{n-1}^1, v_{n-2}^2, v_{n-1}^2$ as
   \[
   \begin{align*}
   f(v_j^1) &= 3 \text{ if } j = n-3 \\
   f(v_j^1) &= 1 \text{ if } j = n-2 \\
   f(v_j^1) &= 4 \text{ if } j = n-1 \\
   f(v_j^2) &= 1 \text{ if } j = n-2 \\
   f(v_j^2) &= 5 \text{ if } j = n-1
   \end{align*}
   \]

   For $n = 4$, $f$ is, $f(v_0) = 0$, $f(v_1^1) = 2$, $f(v_2^1) = 5$, $f(v_3^1) = 3$, $f(v_1^2) = 4$, $f(v_2^2) = 1$, $f(v_3^2) = 5$.

3. $n \equiv 2 \ (mod\ 3)$ then redefine the above $f$ of (1) at $v_{n-2}^1, v_{n-1}^1, v_{n-2}^2, v_{n-1}^2$ as
   \[
   \begin{align*}
   f(v_j^1) &= 1 \text{ if } j = n-2 \\
   f(v_j^1) &= 5 \text{ if } j = n-1 \\
   f(v_j^2) &= 1 \text{ if } j = n-2 \\
   f(v_j^2) &= 4 \text{ if } j = n-1
   \end{align*}
   \]

   Thus, $\lambda(C_{n}^{(2)}) = \lambda(C_n(4)) = 5$. 

\[ \blacksquare \]
Theorem 4.7.27. \( \lambda(C_n^{(k)}) = \lambda(C_n(2k)) = 2k + 1. \)

**Proof.** Let \( C_n^{(k)} \) be the one point union of \( k \) cycles \( C_n \). If \( k = 2 \) then the result follows by Theorem 4.7.26 and hence assume \( k \geq 3 \). Without loss of generality assume that \( v_0 \) is the common vertex of all cycles. The graph \( K_{1,2k} \) is a subgraph of \( C_n^{(k)} \) and hence by Proposition 4.6.1 and Proposition 4.6.12, \( \lambda(C_n^{(k)}) \geq 2k + 1 \).

Now we want to prove that \( \lambda(C_n^{(k)}) \leq 2k + 1 \). In a graph \( C_n^{(k)} \), one vertex is of degree \( 2k \) which is a common vertex of all cycles and remaining vertices of degree 2. Label the common vertex \( v_0 \) by 0 or \( 2k + 1 \) and its adjacent vertices from the set \( \{2, 3, \ldots, 2k + 1\} \) or \( \{0, 1, \ldots, 2k - 1\} \). For remaining vertices, observe that the enough number of labels are available in the set \( \{0, 1, \ldots, 2k + 1\} \) as they have degree 2. Thus, \( \lambda(C_n^{(k)}) = \lambda(C_n(2k)) = 2k + 1. \)

**Corollary 4.7.28.** The \( \lambda \)-number of a Friendship graph \( F_k (= C_3(2k)) \) is \( 2k + 1 \).

**Illustration 4.7.29.** In **Figure 4.14**, an optimal \( L(2,1) \)-labeling of Friendship graph \( F_4 \) is shown in which \( \lambda(F_4) = 9 \).

Thus, we have investigated a graph family whose \( \lambda \)-number is precisely \( \Delta + 1 \). But there are some graphs whose \( \lambda \)-number is precisely \( \Delta + 2 \). Following is an example of such graph.

**Illustration 4.7.30.** In **Figure 4.15**, an optimal \( L(2,1) \)-labeling of 4-ary 4-regular cactus is shown in which \( \lambda(C_4(4)) = 6 \) using Proposition 4.6.3.
The remaining sections are devoted to the discussion of larger graphs obtained from the given graphs.

### 4.8 Distance Two Labeling for Middle Graph of Some Graphs

**Theorem 4.8.1.** For the middle graph $M(P_n)$ of path $P_n$,

$$
\lambda(M(P_n)) = \begin{cases} 
  3, & \text{if } n = 2 \\
  4, & \text{if } n = 3 \\
  5, & \text{if } n = 4, 5 \\
  6, & \text{if } n \geq 6.
\end{cases}
$$

**Proof.** Let $v_0, v_1, \ldots, v_{n-1}$ and $e_0 = (v_0, v_1), \ldots, e_{n-1} = (v_{n-2}, v_{n-1})$ are the vertices and edges of path $P_n$ then $V(M(P_n)) = \{v_0, v_1, \ldots, v_{n-1}, e_0, e_1, \ldots, e_{n-2}\}$. The result is easy to verify for $n = 2, 3, 4$ and 5 so we consider $n \geq 6$. The graph $K_{1,4}$ is a subgraph of $M(P_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, $\lambda(M(P_n)) \geq 5$. In the graph $M(P_n)$, the close neighborhood of each $e_i$ where $i = 0, 1, \ldots, n-2$ contains three vertices.
with degree $\Delta$. Hence by Proposition 4.6.3, $\lambda(M(P_n)) \geq 6$. Now define $f : V(M(P_n)) \to \{0, 1, \ldots, 6\}$ as follows.

\[
\begin{align*}
  f(v_i) &= 0, \quad f(e_i) = 2 \text{ if } i \equiv 0 \pmod{7} \\
  f(v_i) &= 4, \quad f(e_i) = 6 \text{ if } i \equiv 1 \pmod{7} \\
  f(v_i) &= 1, \quad f(e_i) = 3 \text{ if } i \equiv 2 \pmod{7} \\
  f(v_i) &= 5, \quad f(e_i) = 0 \text{ if } i \equiv 3 \pmod{7} \\
  f(v_i) &= 2, \quad f(e_i) = 4 \text{ if } i \equiv 4 \pmod{7} \\
  f(v_i) &= 6, \quad f(e_i) = 1 \text{ if } i \equiv 5 \pmod{7} \\
  f(v_i) &= 3, \quad f(e_i) = 5 \text{ if } i \equiv 6 \pmod{7}
\end{align*}
\]

Thus,

\[
\lambda(M(P_n)) = \begin{cases}
  3, & \text{if } n = 2 \\
  4, & \text{if } n = 3 \\
  5, & \text{if } n = 4, 5 \\
  6, & \text{if } n \geq 6.
\end{cases}
\]

**Illustration 4.8.2.** In Figure 4.16, an optimal L(2,1)-labeling of graph $M(P_6)$ is shown in which $\lambda(M(P_6)) = 6$.

![Figure 4.16: $\lambda(M(P_6)) = 6$.](image)

**Theorem 4.8.3.** For the middle graph $M(C_n)$ of cycle $C_n$,

\[
\lambda(M(C_n)) = \begin{cases}
  6, & \text{if } n \equiv 0, 1 \pmod{3} \\
  7, & \text{if } n \equiv 2 \pmod{3}
\end{cases}
\]
Proof. Let $v_0, v_1, \ldots, v_{n-1}$ and $e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_{n-1} = (v_{n-1}, v_0)$ are the vertices and edges of cycle $C_n$ then $V(M(C_n)) = \{v_0, v_1, \ldots, v_{n-1}, e_0, e_1, \ldots, e_{n-1}\}$. The graph $K_{1,4}$ is a subgraph of $M(C_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, $\lambda(M(C_n)) \geq 5$. In the graph $M(C_n)$, the close neighborhood of each $e_i$ where $i = 0, 1, \ldots, n-1$ contains three vertices with degree $\Delta$. Hence by Proposition 4.6.3, $\lambda(M(C_n)) \geq 6$. Now define labeling as follows.

1. $n \equiv 0 \text{ (mod 3)}$

   \[
   f(v_i) = \begin{cases} 
   3 & \text{if } i \equiv 0 \text{ (mod 3)} \\
   2 & \text{if } i \equiv 1 \text{ (mod 3)} \\
   1 & \text{if } i \equiv 2 \text{ (mod 3)}
   \end{cases}
   \]

   \[
   f(e_i) = \begin{cases} 
   0 & \text{if } i \equiv 0 \text{ (mod 3)} \\
   4 & \text{if } i \equiv 1 \text{ (mod 3)} \\
   6 & \text{if } i \equiv 2 \text{ (mod 3)}
   \end{cases}
   \]

2. $n \equiv 1 \text{ (mod 3)}$ then redefine the above $f$ at $v_{n-2}, v_{n-1}, e_{n-3}, e_{n-2}, e_{n-1}$ as

   \[
   f(v_i) = \begin{cases} 
   3 & \text{if } i = n-2 \\
   4 & \text{if } i = n-1 \\
   5 & \text{if } i = n-3
   \end{cases}
   \]

   \[
   f(e_i) = \begin{cases} 
   1 & \text{if } i = n-2 \\
   6 & \text{if } i = n-1
   \end{cases}
   \]

3. $n \equiv 2 \text{ (mod 3)}$ then redefine the above $f$ at $v_{n-2}, v_{n-1}, e_{n-2}, e_{n-1}$ as

   \[
   f(v_i) = \begin{cases} 
   0 & \text{if } i = n-2 \\
   5 & \text{if } i = n-1 \\
   2 & \text{if } i = n-2
   \end{cases}
   \]

   \[
   f(e_i) = \begin{cases} 
   7 & \text{if } i = n-1
   \end{cases}
   \]
Thus,
\[
\lambda(M(C_n)) = \begin{cases} 
6, & \text{if } n \equiv 0, 1 \pmod{3} \\
7, & \text{if } n \equiv 2 \pmod{3} 
\end{cases}
\]

**Illustration 4.8.4.** In Figure 4.17, an optimal $L(2,1)$-labeling of graph $M(C_4)$ is shown in which $\lambda(M(C_n)) = 6$.

![Figure 4.17: $\lambda(M(C_n)) = 6$.](image)

**Theorem 4.8.5.** The $\lambda$-number of $M(K_{1,n})$ is $2n$.

**Proof.** Let $v_0, v_1, ..., v_n$ and $e_1 = (v_0, v_1), ..., e_n = (v_0, v_n)$ are the vertices and edges of star $K_{1,n}$ then $V(M(K_{1,n})) = \{v_0, ..., v_n, e_1, ..., e_n\}$. The complete graph $K_{n+1}$ formed by the vertices $v_0, e_1, ..., e_n$ is a subgraph of $M(K_{1,n})$ and hence by Proposition 4.6.2 and 4.6.12, $\lambda(M(K_{1,n})) \geq 2(n+1) - 2 = 2n$. Now define $f : V(M(K_{1,n})) \to \{0, 1, ..., 2n\}$ as follows.

- $f(v_0) = 0$
- $f(e_i) = 2i, i = 1,2,...,n$
- $f(v_1) = 2n - 1$
- $f(v_{i+1}) = 2i - 1, i = 1,2,...,(n-1)$

Thus, $\lambda(M(K_{1,n})) = 2n$.

**Illustration 4.8.6.** In Figure 4.18, an optimal $L(2,1)$-labeling of graph $M(K_{1,3})$ is shown in which $\lambda(M(K_{1,3})) = 6$. 
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Theorem 4.8.7. \( \lambda(M(F_n)) = 4n \) where \( F_n \) is a friendship graph (A friendship graph is a one point union of \( n \) copies of cycle \( C_3 \)).

Proof. Let \( F_n \) be the friendship graph formed by \( n \) triangles \( C_1, \ldots, C_n \). Let \( v_0, v_1, \ldots, v_{2n} \) and \( e_1 = (v_0, v_1), \ldots, e_{2n} = (v_0, v_{2n}), e_1' = (v_1, v_2), e_2' = (v_3, v_4), \ldots, e_n' = (v_{2n-1}, v_{2n}) \) be the vertices and edges of \( F_n \) then \( V(M(F_n)) = \{v_0, \ldots, v_{2n}, e_1, \ldots, e_{2n}, e_1', \ldots, e_n'\} \). The complete graph \( K_{2n+1} \) form by vertices \( v_0, e_1, \ldots, e_{2n} \) is a subgraph of \( M(F_n) \) and hence by Proposition 4.6.2 and Proposition 4.6.12, \( \lambda(M(F_n)) \geq 2(2n + 1) - 2 = 4n \). Now define \( f : V(M(F_n)) \rightarrow \{0, 1, \ldots, 4n\} \) as follows.

\[
\begin{align*}
f(v_0) &= 0 \\
f(e_i) &= 2i, i = 1, 2, \ldots, n \\
f(v_1) &= 4n - 3 \\
f(v_2) &= 4n - 1 \\
f(v_{i+1}) &= 2i - 1, i = 1, 2, \ldots, (2n - 2) \\
f(e_i') &= 4i + 3, i = 1, \ldots, (n - 1) \\
f(e_n') &= 3
\end{align*}
\]

Thus, \( \lambda(M(F_n)) = 4n \). ■

Illustration 4.8.8. In Figure 4.19, an optimal \( L(2, 1) \)-labeling of graph \( M(F_3) \) is shown in which \( \lambda(M(F_3)) = 12 \).
Theorem 4.8.9. For the middle graph \( M(W_n) \) of wheel \( W_n \),

\[
\lambda(M(W_n)) = \begin{cases} 
9, & \text{if } n = 3 \\
10, & \text{if } n = 4 \\
11, & \text{if } n = 5 \\
2n, & \text{if } n \geq 6 
\end{cases}
\]

Proof. Let \( v_0, v_1, \ldots, v_n \) and \( e_1 = (v_0, v_1), \ldots, e_n = (v_0, v_n), e'_1 = (v_1, v_2), \ldots, e'_n = (v_n, v_1) \) are the vertices and edges of \( W_n \) then \( V(M(W_n)) = \{v_0, v_1, \ldots, v_n, e_1, \ldots, e_n, e'_1, \ldots, e'_n\} \).

The result is easy to verify for \( n = 3, 4 \) and \( 5 \) so consider \( n \geq 6 \). The complete graph \( K_{n+1} \) formed by the vertices \( v_0, e_1, \ldots, e_n \) is a subgraph of \( M(W_n) \) and hence by Proposition 4.6.2 and Proposition 4.6.12, \( \lambda(M(W_n)) \geq 2(n + 1) - 2 = 2n \).

Now define \( f : V(M(W_n)) \to \{0, 1, \ldots, 2n\} \) as follows.

- \( f(v_0) = 0 \)
- \( f(e_i) = 2i, i = 1, 2, \ldots, n \)
- \( f(e'_1) = 2n - 1 \)
- \( f(e'_{i+1}) = 2i - 1, i = 1, 2, \ldots, (n - 1) \)
- \( f(v_{i-1}) = 2i + 1, i = 2, 3, \ldots, (n-1) \)
- \( f(v_{n-1}) = 1 \)
- \( f(v_n) = 3 \)
Thus,

\[
\lambda(M(W_n)) = \begin{cases} 
9, & \text{if } n = 3 \\
10, & \text{if } n = 4 \\
11, & \text{if } n = 5 \\
2n, & \text{if } n \geq 6 
\end{cases}
\]

**Illustration 4.8.10.** In Figure 4.20, an optimal $L(2,1)$-labeling of graph $M(W_6)$ is shown in which $\lambda(M(W_6)) = 12.$

**Figure 4.20:** $\lambda(M(W_6)) = 12.$

In Table 4.1, the $\lambda$-number of some standard graphs and their middle graphs are given.

**Table 4.1**

<table>
<thead>
<tr>
<th>$P_n$</th>
<th>$\lambda(G)$</th>
<th>$M(G)$</th>
<th>$\lambda(M(G))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 2$</td>
<td>2</td>
<td>$n = 2$</td>
<td>3</td>
</tr>
<tr>
<td>$n = 3, 4$</td>
<td>3</td>
<td>$n = 3$</td>
<td>4</td>
</tr>
<tr>
<td>$n \geq 5$</td>
<td>4</td>
<td>$n \geq 4, 5$</td>
<td>5</td>
</tr>
<tr>
<td>$n \geq 6$</td>
<td>6</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$C_n$</th>
<th>$\lambda(G)$</th>
<th>$M(G)$</th>
<th>$\lambda(M(G))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n \geq 3$</td>
<td>4</td>
<td>$n \equiv 0, 1(\text{mod}3)$</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$n \equiv 2(\text{mod}3)$</td>
<td>7</td>
</tr>
</tbody>
</table>
4.9 Distance Two Labeling for Total Graph of Some Graphs

Theorem 4.9.1. For the total graph $T(P_n)$ of path $P_n$,

$$
\lambda(T(P_n)) = \begin{cases} 
4, & \text{if } n = 2 \\
5, & \text{if } n = 3 \\
6, & \text{if } n \geq 4.
\end{cases}
$$

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ and $e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_{n-2} = (v_{n-2}, v_{n-1})$ are the vertices and edges of path $P_n$ then $V(T(P_n)) = \{v_0, v_1, \ldots, v_{n-1}, e_0, e_1, \ldots, e_{n-2}\}$.

For $n = 2$, the graph $T(P_2)$ is a graph $K_3$ and hence by Proposition 4.6.2, $\lambda(T(P_2)) = 2(3) - 2 = 4$. For $n \geq 3$, the graph $K_{1,4}$ is a subgraph of $T(P_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, $\lambda(T(P_n)) \geq 5$. For $n \geq 4$, in the graph $T(P_n)$, the close neighborhood of each $e_i$ where $i = 1, 2, \ldots, n - 3$ contains three vertices with degree $\Delta$.

Hence by Proposition 4.6.3, $\lambda(T(P_n)) \geq 6$. Now define $f : V(T(P_n)) \to \{0, 1, \ldots, 6\}$ as follows.

$$
f(v_i) = 4, \quad f(e_i) = 2 \text{ if } i \equiv 0 \pmod{7} \\
f(v_i) = 0, \quad f(e_i) = 5 \text{ if } i \equiv 1 \pmod{7} \\
f(v_i) = 3, \quad f(e_i) = 1 \text{ if } i \equiv 2 \pmod{7} \\
f(v_i) = 6, \quad f(e_i) = 4 \text{ if } i \equiv 3 \pmod{7} \\
f(v_i) = 2, \quad f(e_i) = 0 \text{ if } i \equiv 4 \pmod{7} \\
f(v_i) = 5, \quad f(e_i) = 3 \text{ if } i \equiv 5 \pmod{7} \\
f(v_i) = 1, \quad f(e_i) = 6 \text{ if } i \equiv 6 \pmod{7}
$$
The above defined function provides $L(2,1)$-labeling for $T(P_n)$ and from the definition of $f$ it is clear that $\lambda(T(P_3)) \leq 5$ and for $n \geq 4$, $\lambda(T(P_n)) \leq 6$.

Thus, we have

$$\lambda(T(P_n)) = \begin{cases} 4, & \text{if } n = 2 \\ 5, & \text{if } n = 3 \\ 6, & \text{if } n \geq 4. \end{cases}$$

Illustration 4.9.2. In Figure 4.21, an optimal $L(2,1)$-labeling of graph $T(P_8)$ is shown where $\lambda(T(P_8)) = 6$.

![Figure 4.21: $\lambda(T(P_8))=6.$](image)

Theorem 4.9.3. For the total graph $T(C_n)$ of cycle $C_n$, $\lambda(T(C_n)) \in [6,8]$.

Proof. Let $v_0, v_1, \ldots, v_{n-1}$ and $e_0 = (v_0, v_1), e_1 = (v_1, v_2), \ldots, e_{n-1} = (v_{n-1}, v_0)$ are the vertices and edges of cycle $C_n$ then $V(T(C_n)) = \{v_0, v_1, \ldots, v_{n-1}, e_0, e_1, \ldots, e_{n-1}\}$. The graph $K_{1,4}$ is a subgraph of $T(C_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, $\lambda(T(C_n)) \geq 5$. In the graph $T(C_n)$, the close neighborhood of each $e_i$ where $i = 0,1,\ldots,n-1$ contains three vertices with degree $\Delta$. Hence by Proposition 4.6.3, $\lambda(T(C_n)) \geq 6$. Now define labeling as follows.

Case 1: $n = 7k$

$$f(v_i) = 4, f(e_i) = 2 \text{ if } i \equiv 0 \pmod{7}$$
$$f(v_i) = 0, f(e_i) = 5 \text{ if } i \equiv 1 \pmod{7}$$
$$f(v_i) = 3, f(e_i) = 1 \text{ if } i \equiv 2 \pmod{7}$$
$$f(v_i) = 6, f(e_i) = 4 \text{ if } i \equiv 3 \pmod{7}$$
$$f(v_i) = 2, f(e_i) = 0 \text{ if } i \equiv 4 \pmod{7}$$
\[
f(v_i) = 5, \ f(e_i) = 3 \text{ if } i \equiv 5 \pmod{7} \\
f(v_i) = 1, \ f(e_i) = 6 \text{ if } i \equiv 6 \pmod{7}
\]

**Case 2: \(n \neq 7k\)**

(1) \(n \equiv 0 \pmod{3}\)

\[
\begin{align*}
f(v_i) &= 0 \text{ if } i \equiv 0 \pmod{3} \\
f(v_i) &= 3 \text{ if } i \equiv 1 \pmod{3} \\
f(v_i) &= 6 \text{ if } i \equiv 2 \pmod{3} \\
f(e_i) &= 5 \text{ if } i \equiv 0 \pmod{3} \\
f(e_i) &= 8 \text{ if } i \equiv 1 \pmod{3} \\
f(e_i) &= 2 \text{ if } i \equiv 2 \pmod{3}
\end{align*}
\]

(2) \(n \equiv 1\) or \(2 \pmod{3}\) then redefine the above \(f(v_{n-1})\) and \(f(e_{n-1})\) as

\[
\begin{align*}
f(v_i) &= 4 \text{ if } i = n - 1 \\
f(e_i) &= 7 \text{ if } i = n - 1
\end{align*}
\]

The above defined function provides \(L(2,1)\)-labeling for \(T(C_n)\) and hence \(\lambda(T(C_n)) \leq 8\). Thus, we have \(\lambda(T(C_n)) \in [6, 8]\). □

**Illustration 4.9.4.** In Figure 4.22, an optimal \(L(2,1)\)-labeling of graphs \(T(C_3)\), \(T(C_6)\) and \(T(C_7)\) is shown where \(\lambda(T(C_n)) \in [6, 8]\).
Theorem 4.9.5. The $\lambda$-number of $T(K_{1,n})$ is $2n + 1$.

Proof. Let $v_0, v_1, \ldots, v_n$ and $e_1 = (v_0, v_1), \ldots, e_n = (v_0, v_n)$ are the vertices and edges of star $K_{1,n}$ then $V(T(K_{1,n})) = \{v_0, \ldots, v_n, e_1, \ldots, e_n\}$. The star $K_{1,2n}$ is a subgraph of $T(K_{1,n})$ and hence by Proposition 4.6.1 and Proposition 4.6.12, $\lambda(T(K_{1,n})) \geq 2n + 1$. Now define $f: V(T(K_{1,n})) \to \{0, 1, \ldots, 2n + 1\}$ as follows.

\begin{align*}
f(v_0) &= 0 \\
f(e_i) &= 2i, i = 1, 2, \ldots, n \\
f(v_i) &= 2i - 3, i = 3, 4, \ldots, n \\
f(v_1) &= 2n - 1 \\
f(v_2) &= 2n + 1
\end{align*}

The above defined function provides $L(2, 1)$-labeling for $T(K_{1,n})$ and hence $\lambda(T(K_{1,n})) \leq 2n + 1$.

Thus, we have $\lambda(T(K_{1,n})) = 2n + 1$. ■

Illustration 4.9.6. In Figure 4.23, an optimal $L(2, 1)$-labeling of graph $T(K_{1,3})$ is shown where $\lambda(T(K_{1,3})) = 7$.

![Figure 4.23: $\lambda(T(K_{1,3})) = 7$.](image)

Theorem 4.9.7. $\lambda(T(F_n)) = 4n + 1$ where $F_n$ is a friendship graph (A friendship graph is a one point union of $n$ copies of cycle $C_3$).

Proof. Let $F_n$ be a friendship graph form by $n$ triangles $C_1, \ldots, C_n$. Let $v_0, v_1, \ldots, v_{2n}$ and $e_1 = (v_0, v_1), \ldots, e_{2n} = (v_0, v_{2n})$, $e'_1 = (v_1, v_2)$, $e'_2 = (v_3, v_4), \ldots, e'_{2n} = (v_{2n-1}, v_{2n})$ are
the vertices and edges of $F_n$ then $V(T(F_n)) = \{v_0, \ldots, v_{2n}, e_1, \ldots, e_{2n}, e'_1, \ldots, e'_n\}$. The star $K_{1,4n}$ is a subgraph of $T(F_n)$ and hence by Proposition 4.6.1 and Proposition 4.6.12, \( \lambda(T(F_n)) \geq 4n + 1 \). Now define $f: V(T(F_n)) \rightarrow \{0, 1, \ldots, 4n + 1\}$ as follows.

$f(v_0) = 0$

$f(e_i) = 2i, i = 1, 2, \ldots, n$

$f(v_{2i+1}) = 4i - 1, i = 1, 2, \ldots, (n - 1)$

$f(v_{2i}) = 4i - 3, i = 2, 3, \ldots, n$

$f(e'_i) = 4i - 3, i = 1, 2, \ldots, n$

$f(v_1) = 4n - 1$

$F(v_2) = 4n + 1$

The above defined function provides $L(2, 1)$-labeling for $T(F_n)$ and hence \( \lambda(T(F_n)) \leq 4n + 1 \).

Thus, we have \( \lambda(T(F_n)) = 4n + 1 \). \[ \blacksquare \]

**Illustration 4.9.8.** In Figure 4.24, an optimal $L(2, 1)$-labeling of graph $T(F_3)$ is shown where \( \lambda(T(F_3)) = 13 \).

![Figure 4.24: \( \lambda(T(F_3)) = 13 \).](image)

In Table 4.2, the $\lambda$-numbers of some standard graphs and their total graphs are listed.
TABLE 4.2

<table>
<thead>
<tr>
<th>$G$</th>
<th>$\lambda(G)$</th>
<th>$T(G)$</th>
<th>$\lambda(T(G))$</th>
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<tbody>
<tr>
<td>$P_n$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 2$</td>
<td>2</td>
<td>$n = 2$</td>
<td>4</td>
</tr>
<tr>
<td>$n = 3, 4$</td>
<td>3</td>
<td>$n = 3$</td>
<td>5</td>
</tr>
<tr>
<td>$n \geq 5$</td>
<td>4</td>
<td>$n \geq 4$</td>
<td>6</td>
</tr>
<tr>
<td>$C_n$</td>
<td></td>
<td>$n \geq 3$</td>
<td>${6, 8}$</td>
</tr>
<tr>
<td>$K_{1,n}$</td>
<td></td>
<td>$n + 1$</td>
<td>$n \geq 3$</td>
</tr>
<tr>
<td>$F_n$</td>
<td></td>
<td>$2n + 1$</td>
<td>$n \geq 3$</td>
</tr>
</tbody>
</table>

4.10 Distance Two Labeling for Square Graph of Some Graphs

Theorem 4.10.1. For path $P_n$,

$$\lambda(P^2_n) = \begin{cases} 
4, & \text{if } n = 3 \\
5, & \text{if } n = 4, 5, 6 \\
6, & \text{if } n \geq 7.
\end{cases}$$

Proof. Let $P^2_n$ be the square graph of path $P_n$. The result is trivial for $n = 3, 4, 5$ and 6 so we consider $n \geq 7$. The graph $K_{1,4}$ is a subgraph of $P^2_n$ and hence by Proposition 4.6.1 and Proposition 4.6.12, we have $\lambda(P^2_n) \geq 5$. In the graph $P^2_n$, the close neighborhood of each $v_i$, where $i = 2, 3, ..., n - 3$ contains three vertices with degree $\Delta = 4$. Hence by Proposition 4.6.3, $\lambda(P^2_n) \geq 6$.

Now we prove $\lambda(P^2_n) \leq 6$. Define $f : V(P^2_n) \to \{0, 1, ..., 6\}$ as follows.

$$f(v_i) = 4 \text{ if } i \equiv 0 \text{ (mod 7)}$$

$$f(v_i) = 2 \text{ if } i \equiv 1 \text{ (mod 7)}$$

$$f(v_i) = 0 \text{ if } i \equiv 2 \text{ (mod 7)}$$

$$f(v_i) = 5 \text{ if } i \equiv 3 \text{ (mod 7)}$$

$$f(v_i) = 3 \text{ if } i \equiv 4 \text{ (mod 7)}$$

$$f(v_i) = 1 \text{ if } i \equiv 5 \text{ (mod 7)}$$

$$f(v_i) = 6 \text{ if } i \equiv 6 \text{ (mod 7)}$$
Thus, we proved that

\[
\lambda(P_n^2) = \begin{cases} 
4, & \text{if } n = 3 \\
5, & \text{if } n = 4, 5, 6 \\
6, & \text{if } n \geq 7.
\end{cases}
\]

**Illustration 4.10.2.** In Figure 4.25, an optimal \(L(2, 1)\)-labeling of graph \(P_7^2\) is shown where \(\lambda(P_7^2) = 6\).

![Figure 4.25: \(\lambda(P_7^2) = 6\).](image)

**Theorem 4.10.3.** For spider \(S_{l_1, l_2, \ldots, l_n}\),

\[
\lambda(S_{l_1, l_2, \ldots, l_n}^2) = \begin{cases} 
2n + 1, & \text{if all } l_i \geq 2, 1 \leq i \leq n \\
2n, & \text{otherwise.}
\end{cases}
\]

**Proof.** Let \(S_{l_1, l_2, \ldots, l_n}^2\) be the square graph of the spider \(S_{l_1, l_2, \ldots, l_n}\) with \(n\) legs. If \(n = 2\) then it is path and result is already settled. Now if at least one \(l_i = 1\) then \(K_{n+1}\) is a subgraph of \(S_{l_1, l_2, \ldots, l_n}\) and hence \(\lambda(S_{l_1, l_2, \ldots, l_n}^2) \geq 2n\) by Proposition 4.6.2 and Proposition 4.6.12 but if all \(l_i \geq 2\) then \(K_{1, 2n}\) is a subgraph of \(S_{l_1, l_2, \ldots, l_n}^2\) and hence \(\lambda(S_{l_1, l_2, \ldots, l_n}^2) \geq 2n + 1\) by Proposition 4.6.1 and Proposition 4.6.12.

Now we define \(f : S_{l_1, l_2, \ldots, l_n}^2 \rightarrow \{0, 1, 2, \ldots\}\),

\[
f(v_{0,0}) = 0 \\
f(v_{i,1}) = 2i, \ 1 \leq i \leq n \\
f(v_{i,2}) = 2(i + 1) + 1, \ 1 \leq i \leq n - 1 \\
f(v_{n,2}) = 3
\]

For all other vertices

\[
f(v_{1,3}) = f(v_{3,3}) = 7, \ f(v_{2,3}) = 1, \ f(v_{i,j}) = f(v_{i,j-1}) + 2 \ (mod 8), \ 1 \leq i \leq n, \ 4 \leq j \leq l_i \text{ if } n = 3 \text{ otherwise } f(v_{i,j}) = f(v_{i,j-1}) + 2 \ (mod 10), \ 1 \leq i \leq n, \ 3 \leq j \leq l_i \text{ if } n \geq 4
\]
Thus, we proved that

\[
\lambda(S_{l_1,l_2,\ldots,l_n}) = \begin{cases} 
2n + 1, & \text{if all } l_i \geq 2, 1 \leq i \leq n \\
2n, & \text{otherwise}.
\end{cases}
\]

**Illustration 4.10.4.** In Figure 4.26, an optimal \(L(2,1)\)-labeling of a spider \(S_{6,5,4,3,2}^2\) is shown where \(\lambda(S_{6,5,4,3,2}^2) = 11\) as \(n = 5\).

**Theorem 4.10.5.** For comb graph \(G = P_n \odot K_1\),

\[
\lambda(G) = \begin{cases} 
6, & \text{if } n = 3 \\
7, & \text{if } n = 4 \\
8, & \text{if } n = 5, 6 \\
9, & \text{if } n \geq 7.
\end{cases}
\]

**Proof.** Define the square graph of comb graph \(G = P_n \odot K_1\) as \(G^2\). The result is easy to verify for \(n = 3, 4, 5\) and 6 so we consider \(n \geq 7\). The graph \(K_{1,7}\) is a subgraph of \(G^2\) and hence by Proposition 4.6.2 and Proposition 4.6.12, we have \(\lambda(G^2) \geq 8\). In the graph
$G^2$, the close neighborhood of each $v_i$, where $i = 2, 3, ..., n - 3$ contains three vertices with degree $\Delta = 7$. Hence by Proposition 4.6.3, $\lambda(G^2) \geq 9$.

Now we prove $\lambda(G^2) \leq 9$. Define $f : V(G) \rightarrow \{0, 1, 2, ..., 9\}$ as follows:

\[
\begin{align*}
    f(v_i) &= 0 \text{ if } i \equiv 0 \pmod{5} \\
    f(v_i) &= 2 \text{ if } i \equiv 1 \pmod{5} \\
    f(v_i) &= 4 \text{ if } i \equiv 2 \pmod{5} \\
    f(v_i) &= 6 \text{ if } i \equiv 3 \pmod{5} \\
    f(v_i) &= 8 \text{ if } i \equiv 4 \pmod{5} \\
    f(v'_i) &= 5 \text{ if } i \equiv 0 \pmod{5} \\
    f(v'_i) &= 7 \text{ if } i \equiv 1 \pmod{5} \\
    f(v'_i) &= 9 \text{ if } i \equiv 2 \pmod{5} \\
    f(v'_i) &= 1 \text{ if } i \equiv 3 \pmod{5} \\
    f(v'_i) &= 3 \text{ if } i \equiv 4 \pmod{5}
\end{align*}
\]

Thus, we proved that

\[
\lambda(G^2) = \begin{cases} 
6, & \text{if } n = 3 \\
7, & \text{if } n = 4 \\
8, & \text{if } n = 5, 6 \\
9, & \text{if } n \geq 7.
\end{cases}
\]

**Illustration 4.10.6.** In Figure 4.27, an optimal $L(2, 1)$-labeling of square graph of $P_7 \odot K_1$ is shown where $\lambda((P_7 \odot K_1)^2) = 9$.

\[\text{Figure 4.27: } \lambda((P_7 \odot K_1)^2) = 9.\]
4.11 Further Scope of Research

It is possible to investigate $\lambda$-number for the larger graphs obtained from the standard graphs by using graph operations like fusion, duplication, barycentric subdivision, shadow graph etc. Moreover, $\lambda$-number can also be discussed in the context of line graph, middle graph, total graph, square graph of arbitrary graph.

4.12 Concluding Remarks

It is observed that all the close transmitters become very close due to expansion of network. Then obviously more number of channels are required. We found that the graphs studied in this chapter will fulfill this requirement. Thus, our results are very much applicable to serve the purpose of expansion of transmitters network.

The next chapter is targeted to discuss radio labeling of graphs.