2.1 Introduction

A spatial soliton results from the balance between linear diffraction and nonlinear self-focusing, usually with a Kerr-type ultrafast nonlinearity. This effect was discovered in 1964. It was Askaryan in 1962 who first suggested the idea that an optical beam can induce a waveguide and guide itself in it. A light beam traveling either in vacuum or in a medium always broadens because of the light's natural wave property of diffraction. But if the beam is incident into a bulk nonlinear material, such as silica glass, it changes the refractive index of the material. The consequent variation of the velocity of light across the wavefront of the beam focuses the beam as if it were passing through a lens. The earliest experimental observation of the self-focusing of optical beams was made in 1964. If the diffraction of the beam can be compensated by the self-focusing of the beam, we get the so called spatial solitons. The perfect balance between diffraction and self-focusing which results in spatial solitons has been found to occur in one and two transverse dimensions, and the solitons are named $(1 + 1)D$ or $(2 + 1)D$ accordingly. When time is also included in the evolution equation, we get the so called spatio-temporal solitons. These spatial solitons have been found to occur in a variety of materials like Kerr materials, photorefractive materials, liquid crystals etc. Peccianti et al. set up an experiment to demonstrate all-optical switching and logic gate blocks using spatial solitons in liquid crystals.

In this chapter, we study the propagation of an optical high-power cylindrically symmetric beam and a pulsed optical high-power beam in a material characterized by cubic-quintic nonlinearity using both analytical and numerical methods taking into consideration of the self-defocusing
effect caused by the presence of free electrons produced due to plasma formation. In section 2.2, a detailed study of two dimensional spatial solitons in a bulk cubic-quintic medium is presented. Spatio-temporal solitons in such a medium stabilized by multiphoton ionization is studied in section 2.3. Both kinds of solitons are studied analytically and numerically. We use the variational method\textsuperscript{35} with a Gaussian ansatz for the analytical analysis. Approximate solutions can be found using this method. The Finite Difference Beam Propagation Method (FD-BPM) is used for the numerical analysis.\textsuperscript{36}

The \((1 + 1)D\) spatial solitons, a continuous wave beam linearly confined in one transverse dimension and self-guided in the other transverse dimension, are stable to perturbations and have been observed experimentally. A \((2 + 1)D\) soliton that is self-guided in both transverse dimensions is not stable with a Kerr-type nonlinearity. Additional mechanisms such as index saturation or inclusion of higher-order nonlinearity helps to stabilize the propagation of such beams. A simple model for the stable propagation of \((2 + 1)D\) solitons may be based on the cubic-quintic nonlinearity.\textsuperscript{5,6} This model has attracted considerable attention. It has been shown that the cubic-quintic nonlinearity correctly describes the dielectric response of the polydiacetylene p-toluene sulfonate (PTS) crystal.\textsuperscript{7} This type of nonlinear response is known for semiconductor-doped glasses\textsuperscript{8} and various \(\pi\) conjugated polymers.\textsuperscript{9–11} In these types of materials the refractive index, \(n\), is of the form

\[
n = n_0 + n_2 I + n_4 I^2, \tag{2.1}
\]

where \(I\) is the beam intensity \(n_0\) is the linear refractive index and \(n_2\) and \(n_4\) are nonlinear coefficients with \(n_2 > 0\) and \(n_4 < 0\), i.e., the higher order nonlinearity is of the saturating kind. The propagation of spatial solitons in a PTS like medium has been studied by various groups.\textsuperscript{12–14}

Now, if we are using a high power laser beam we have to consider the phenomenon of plasma generation through multiphoton absorption. For an extreme high power laser pulse of the order of 10 TW relativistic self-channeling in an underdense plasma has been predicted and experimentally observed over a plasma length of 3 mm. In this regime nearly all molecules of the medium are ionized and relativistic self-focusing develops from an increase of electron inertia under the influence of the intense electromagnetic wave. When such pulses are propagated through air, they can propagate over considerable distances because of the formation of filaments after the plasma has been generated through multiphoton ionization of air.\textsuperscript{15}
dispersive effects are less important in this case. The main mechanism behind the filament formation is related to a dynamic balance between the Kerr self-focusing and the defocusing induced by the plasma.\textsuperscript{16}

We study the propagation of an optical high power beam through a PTS like medium. In this high energy regime, we have to consider the phenomenon of plasma generation through multiphoton absorption.\textsuperscript{17} Multiphoton absorption is a nonlinear process, in contrast with the one-photon absorption process. It has a self defocusing effect on the material. The counteracting self-defocusing effect of both photoionized free electrons and the quintic nonlinearity restricts the unbounded growth of the Kerr nonlinearity. The study of spatial solitons in a bulk Kerr medium with multiphoton ionization has been carried out by Herrmann et al.\textsuperscript{18} Couairon\textsuperscript{19} has studied the dynamics of light filaments formed when a femtosecond laser pulse propagates in air taking into consideration the plasma generated using photoionization and has given an extensive review\textsuperscript{20} on the main aspects of ultrashort laser pulse filamentation in various transparent media such as air (gases), transparent solids and liquids. The refractive index now takes the form

\begin{equation}
    n = n_0 + n_2 I + n_4 I^2 - \frac{N_e}{2n_0 N_{cr}},
\end{equation}

where $N_e$ is the number density of free electrons and $N_{cr}$ is the critical plasma density. The beam evolution is studied using the cubic-quintic nonlinear Schrödinger equation with the effect of multiphoton ionization.

Now, when a pulsed optical beam propagates through a bulk nonlinear medium, it is affected by diffraction and dispersion simultaneously and at the same time the two effects become coupled through the nonlinearity of the medium. Such a space-time coupling leads to various nonlinear effects like the possibility of spatio-temporal collapse or pulse splitting and the formation of light bullets. The idea of "light bullets" was first proposed by Silberberg.\textsuperscript{21} Optical spatiotemporal solitons, or the so called "light bullets", have attracted growing interest because of their potential applications in ultra fast all-optical switching in a bulk medium. The binary information once brought by temporal solitons has to be retreated in all-optical systems. Therefore, there is a growing interest in spatiotemporal solitons.\textsuperscript{22} In a medium with purely cubic nonlinear response light bullets (LB) formally exist. But in the higher dimensions (2D and 3D) they are unstable against the spatiotemporal collapse induced by the combined effects of nonlinearity and anomalous dispersion. They are not stable in the uniform self-focusing Kerr medium\textsuperscript{23} but stability can be
achieved in saturable,\textsuperscript{16,24} quadratically nonlinear,\textsuperscript{25-27} and graded-index Kerr media.\textsuperscript{28} Spatio-temporal solitons can also be found in self-induced-transparency media,\textsuperscript{29} in off-resonance two-level systems,\textsuperscript{30} as well as in engineered tandem structures made with quadratically nonlinear pieces.\textsuperscript{31} A fully localized "light bullet" in all the three dimensions of space and time has not yet been found in an experiment but two-dimensional (2D) spatio-temporal solitons in bulk $\chi^2$ media, such as lithium iodate ($\text{LiIO}_3$) and barium metaborate (BBO) has been observed.\textsuperscript{32} This was the first experimental observation of a 2D spatio-temporal solitons in a second harmonic generation setting. This experiment was performed using tilted-pulse technique using highly elliptical beams. As a result diffraction is minimal in the remaining spatial transverse dimensions. Stable spinning solitons have also been predicted to exist in $\chi^2$ media combined with the self-defocusing Kerr nonlinearity.\textsuperscript{33} Stable spinning solitons of the same kind have also been found in an optical model based on the cubic-quintic nonlinear Schrödinger equation.\textsuperscript{34}

2.2 \textbf{Spatial solitons stabilized by multiphoton ionization}

2.2.1 Model equation and analysis

An electric field $E(r, t)$ in a dielectric medium satisfies Maxwell’s equation in the form

$$\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 D}{\partial t^2} = \nabla(\nabla.E), \quad (2.3)$$

where $D = \varepsilon E$ is the displacement vector in the dielectric medium, with $\varepsilon$ being the dielectric constant relative to vacuum and it is approximately equal to $n^2$; $n$ being the refractive index. For a medium characterized by cubic nonlinearity, $D$ can be written as, $D = (n_0(\omega) + n_2(\omega)|E|^2)E$ where, $E = \frac{1}{2}[A(r, t) \exp i(\omega t - k z) + c.c]$. Using this, we can reduce the Maxwell’s equation to

$$2ik \frac{\partial A}{\partial z} + \nabla^2 A + 2kk_0n_2 |A|^2 A = 0, \quad (2.4)$$

for a cubic medium where, $k = \omega/c$, $k_0 = n_0 k$, and $A$ is the amplitude of the beam. This is the cubic nonlinear Schrödinger equation.

There are many materials which show quintic nonlinear effect in addition to the cubic one. In this case, the Maxwell’s equation can be sim-
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plified to obtain the cubic-quintic nonlinear Schrödinger equation. Hence the dynamics of the amplitude $A$ of a laser beam in a PTS like medium is governed by the cubic-quintic nonlinear Schrödinger equation of the form

$$2ik \frac{\partial A}{\partial z} + \nabla^2 A + 2k k_0 n_2 |A|^2 A + 2k k_0 n_4 |A|^4 A = 0. \quad (2.5)$$

Since we are using a very high power laser beam we have to consider the effect of multiphoton ionization as well. The reduced Maxwell’s equation for the slowly varying amplitude $A$ now takes the form

$$2ik \frac{\partial A}{\partial z} + \nabla^2 A + 2k k_0 n_2 |A|^2 A + 2k k_0 n_4 |A|^4 A$$

$$- \rho a A \int_{-\infty}^{\eta} |A(t')|^{2N} dt' = 0, \quad (2.6)$$

where $N$ is the number of quanta necessary to ionize the molecules, $\eta = t - z/v$ is the time of the moving frame of the pulse maximum and $\rho$ and $a$ are constants.

Here, we are considering the propagation of the beam along the $z$ direction and variation along the radial direction. So we will use cylindrical coordinates for our analysis. Hence Eq. (2.6) takes the form

$$2ik \frac{\partial A}{\partial z} = - \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A}{\partial r} \right) + 2k \lambda_1 |A|^2 A$$

$$+ 2k \lambda_2 |A|^4 A + \rho a A \int_{-\infty}^{\eta} |A(t')|^{2N} dt', \quad (2.7)$$

where $\lambda_1 = -k_0 n_2$ and $\lambda_2 = -k_0 n_4$.

The time dependence of the beam is taken into account by the ansatz, $A(z, r, \eta) = B(z, r)T(\eta)$. $T(\eta)$ is the normalized input shape.

Eq. (2.7) can be obtained from the Lagrangian

$$L = \frac{i}{2} \left( B \frac{\partial B^*}{\partial z} - B^* \frac{\partial B}{\partial z} \right) T + \frac{r}{2k} \frac{\partial B}{\partial r} \frac{\partial B^*}{\partial r} T$$

$$+ r \frac{\lambda_1}{2} |B|^4 T^3 + r \frac{\lambda_2}{3} |B|^6 T^4 + r \frac{\rho a |B|^{2N+2}}{2k} \frac{1}{N+1} Tg(\eta), \quad (2.8)$$

where

$$g(\eta) = \int_{-\infty}^{\eta} T^{2N} dt.$$
For a solution of this problem, let us assume a trial solution of the form

$$B(z, r) = C(z) \exp\left[ -\frac{r^2}{2w(z)^2} + ib(z)r^2 \right],$$  \hspace{1cm} (2.9)

where $C(z)$ is the maximum amplitude, $b(z)$ is the curvature parameter, $w(z)$ is the beam radius. Ideally, the trial function should include a possibility to model the dynamically varying radial shape function of the beam. But that will make the variational analysis more complicated.

The reduced Lagrangian is then given by

$$\langle L \rangle = \int_0^\infty Lr \, dr,$$  \hspace{1cm} (2.10)

$$\langle L \rangle = \frac{i}{2} T \left( C \frac{\partial C^*}{\partial z} - C^* \frac{\partial C}{\partial z} \right) \frac{w^3 \sqrt{\pi}}{4} + b_z |C|^2 T w^5 \frac{\sqrt{\pi}}{8}$$

$$+ \frac{|C|^2 T}{2k} \left\{ \frac{1}{w^4} + 4b^2 \right\} w^5 \frac{\sqrt{\pi}}{8} + \frac{\lambda_1}{2} |C|^4 T^3 w^3 \frac{\sqrt{\pi}}{8\sqrt{2}}$$

$$+ \frac{\lambda_2}{3} |C|^6 T^4 w^3 \frac{\sqrt{\pi}}{12\sqrt{3}} + \frac{\rho a}{2k (N + 1)^{5/2}} T g(\eta) w^3 \frac{\sqrt{\pi}}{4}. \hspace{1cm} (2.11)$$

Now we can find the variation of $\langle L \rangle$ with respect to the various Gaussian parameters $C(z)$, $C(z)^*$, $w(z)$ and $b(z)$. We have

$$\frac{\partial}{\partial z} \left( \frac{\partial \langle L \rangle}{\partial C} \right) - \frac{\partial \langle L \rangle}{\partial C} = 0.$$  \hspace{1cm} (2.12)

Thus the variation with $C(z)$ and $C(z)^*$ gives

$$C \frac{\partial \langle L \rangle}{\partial C} = \frac{i}{2} T C C^* w^3 + b_z |C|^2 T \frac{w^5}{2}$$

$$+ \frac{|C|^2 T}{2k} \left\{ \frac{1}{w^4} + 4b^2 \right\} \frac{w^5}{2} + \frac{\lambda_1}{2\sqrt{2}} |C|^4 T^3 w^3$$

$$+ \frac{\lambda_2}{3\sqrt{3}} |C|^6 T^4 w^3 + \frac{\rho a}{2k (N + 1)^{5/2}} T g(\eta) w^3,$$  \hspace{1cm} (2.13)

and
Model equation and analysis

\[ C^* \frac{\partial \langle L \rangle}{\partial C^*} = -i \frac{T}{2} C^* C_z w^3 + b_z |C|^2 T \frac{w^5}{2} + |C|^2 T \left\{ \frac{1}{w^4} + 4b^2 \right\} \frac{w^5}{2} + \frac{\lambda_1}{2 \sqrt{2}} |C|^4 T^3 w^3 \]
\[ + \frac{\lambda_2}{3 \sqrt{3}} |C|^6 T^4 w^3 + \frac{\rho a}{2k (N + 1)^{5/2}} T g(\eta) w^3. \]  

(2.14)

Subtracting Eq. (2.14) from Eq. (2.13) and using Eq. (2.12), we get

\[ \frac{\partial}{\partial z} (w^2 C C^*) = 0. \]

\[ \Rightarrow w(0)^2 |C(0)|^2 = w(z)^2 |C(z)|^2 = E_0. \]  

(2.15)

Adding Eq. (2.13) and Eq. (2.14) we obtain

\[ i(C C_*^* - C^* C_z) = -2b_z |C|^2 w^2 - \frac{2 |C|^2}{2k} \left\{ \frac{1}{w^4} + 4b^2 \right\} w^2 \]
\[ - \frac{2 \lambda_1}{\sqrt{2}} |C|^4 T^2 + \frac{4 \lambda_2}{3 \sqrt{3}} |C|^6 T^3 + \frac{4 \rho a g(\eta) |C|^{2N+2}}{2k (N + 1)^{5/2}}. \]  

(2.16)

Now, the variation of \( \langle L \rangle \) with respect to \( w(z) \) and \( b \) gives

\[ \frac{\partial \langle L \rangle}{\partial w} = \frac{3}{2} iT (CC_*^* - C^* C_z) + 5b_z |C|^2 T \frac{w^4}{2} + \frac{|C|^2 T}{4k} \]
\[ + 10 |C|^2 T b^2 \frac{w^4}{2k} + \frac{3 \lambda_1}{4 \sqrt{2}} |C|^4 T^3 w^2 + \frac{\lambda_2}{3 \sqrt{3}} |C|^6 T^4 w^2 \]
\[ + \frac{3 \rho a}{2k (N + 1)^{5/2}} T g(\eta) w^2, \]  

(2.17)

and

\[ \frac{\partial \langle L \rangle}{\partial b} = 0 \Rightarrow \frac{d}{dz} \left( w^5 |C|^2 \right) = \frac{6b}{k} |C|^2 w^5. \]  

(2.18)

From this and considering the fact that \( w^2 |C|^2 \) is a constant, we can write

\[ \frac{dw}{dz} = \frac{4bw}{2k}. \]  

(2.19)
This implies

\[ b(z) = \frac{k d(\ln w)}{2} \frac{d}{dz} \]  

(2.20)

Comparing Eq. (2.16) and Eq. (2.17) we obtain

\[ b_z w^2 + \frac{5}{2k w^2} + \frac{4b^2 w^2}{2k} + \frac{9\lambda_1}{2\sqrt{2}} |C|^2 T^2 \]
\[ + \frac{10\lambda_2}{3\sqrt{3}} |C|^4 T^3 + \frac{6(2N + 1)\rho a}{2k(N + 1)} \frac{|C|^{2N}}{(N + 1)^{5/2}} g(\eta) = 0. \]  

(2.21)

Now, combining Eq. (2.21) with the derivative form of Eq. (2.19), we obtain

\[ \frac{d^2 w}{dz^2} = -\frac{20(2N + 1)\rho a g(\eta) E_0^N}{(2k)^2(N + 1)^{5/2} w^{2N + 1}} \]
\[ - \frac{36\lambda_1 T^2 E_0}{4k\sqrt{2} w^3} - \frac{40\lambda_2 T^3 E_0^2}{6k\sqrt{3} w^5} - \frac{24(2N + 1)\rho a g(\eta) E_0^N}{(2k)^2(N + 1)^{5/2} w^{2N + 1}}. \]  

(2.22)

Here \( |C|^2 \) has been eliminated by using the fact that \( w^2 |C|^2 = E_0 \).

On integrating the above equation, we get an equation for the variation of \( w(z) \) as

\[ \frac{1}{2} \left( \frac{dw}{dz} \right)^2 + \Pi(w) = 0. \]  

(2.23)

This is analogous to the equation of a particle moving in a potential well. The potential \( \Pi(w) \) is given by

\[ \Pi(w) = -\frac{10}{(2k)^2 w^2} - \frac{18\lambda_1 T^2 E_0}{4k\sqrt{2} w^2} - \frac{10\lambda_2 T^3 E_0^2}{6k\sqrt{3} w^4} \]
\[ - \frac{24(2N + 1)\rho a g(\eta) E_0^N}{2N(2k)^2(N + 1)^{5/2} w^{2N}} + c. \]  

(2.24)

The phase \( \phi(z) \) of \( C(z) \) \((C(z) = |C(z)| \exp[i\phi(z)]\) is obtained from Eq. (2.16) and also using Eq. (2.21) as

\[ \frac{d\phi}{dz} = \frac{4}{2k w^2} + \frac{7\lambda_1}{2\sqrt{2}} |C|^2 T^2 + \frac{8\lambda_2}{3\sqrt{3}} |C|^4 T^3 \]
\[ + \frac{2(7N + 4)\rho a g(\eta)|C|^{2N}}{(N + 1)2k(N + 1)^{5/2}}. \]  

(2.25)
Introducing $w(z)/w_0 = y(z)$, Eq. (2.23) becomes

$$\frac{1}{2} \left( \frac{dy}{dz} \right)^2 + \Pi(y) = 0,$$

(2.26)

where

$$\Pi(y) = \frac{\mu}{y^2} + \frac{\nu}{y^4} + \frac{\xi}{y^{2N}} + K,$$

(2.27)

with

$$\mu = -\frac{10}{4k^2w_0^4} - \frac{18\lambda_1T^2E_0}{4k\sqrt{2}w_0^4},$$

$$\nu = -\frac{10\lambda_2T^3E_0^2}{6k\sqrt{3}w_0^6},$$

$$\xi = \frac{24(2N + 1)\rho ag(\eta)E_0^N}{2N(2k)^2(N + 1)^{5/2}w_0^{2N+2}},$$

and

$$K = \frac{c}{w_0^2}.$$

Now, let us assume that the beam at $z = 0$ has $w(0) = w_0$ and $[dw(z)/dz]_{z=0} = 0$. This gives $K = -(\mu + \nu + \xi)$.

Depending on the values of $\mu, \nu$ and $\xi$ we can identify four different regimes for the propagation of the beam.

Figure 2.1: Qualitative plot of the potential function $\Pi(y)$ when all the nonlinearities are of defocusing nature ($\frac{\mu + \nu}{\xi} > 0$). Dotted line represents the linear case.
1) $\frac{\mu + \nu}{\xi} > 0$. This condition implies defocusing due to both third and fifth order nonlinearity as well as the nonlinearity due to the multiphoton effect. We can clearly see from Fig. 2.1 that the beam diffracts faster than in the purely linear case.

2) $-1 < \frac{\mu + \nu}{\xi} < 0$. This condition implies focusing due to the third order nonlinearity and defocusing due to a weak fifth order nonlinearity. The multiphoton effect is also of defocusing nature. We can see (Fig. 2.2) that the effect of nonlinearity is to focus the beam.

![Figure 2.2: Qualitative plot of the potential function $\Pi(y)$ when third order nonlinearity is of focusing nature and all other nonlinearities are of defocusing nature (weak fifth order) ($-1 < \frac{\mu + \nu}{\xi} < 0$). Dotted line represents the linear case.](image)

3) $-2.5 < \frac{\mu + \nu}{\xi} < -1$. In this case the third order nonlinearity is of focusing type and there is a strong fifth order nonlinearity. A potential well has been created. The spreading of the beam is stopped at the zeros of the potential function (Fig. 2.3).

4) $\frac{\mu + \nu}{\xi} = -2.5$. This is the limit case. The potential well has degenerated into a single point. The diffraction of the beam is exactly compensated by the focusing effect of the nonlinearity and beam propagates without any change in its shape (Fig. 2.4). The collapse of the beam has been arrested and we get a stable $(2 + 1)D$ spatial soliton which propagates through the medium without any shape change.
Figure 2.3: Qualitative plot of the potential function $\Pi(y)$ when third order nonlinearity is of focusing nature and all other nonlinearities are of defocusing nature (strong fifth order) ($-2.5 < \frac{\mu + \nu}{\xi} < -1$). Dotted line represents the linear case.

Figure 2.4: Qualitative plot of the potential function $\Pi(y)$ when focusing due to the third order nonlinearity is completely balanced by the defocusing due to the fifth order nonlinearity and multiphoton ionization ($\frac{\mu + \nu}{\xi} = -2.5$). This is the limit case. Dotted line represents the linear case.

A three-dimensional plot of the normalized soliton intensity versus the
time \( \eta \) and the radial variable \( r \) is plotted (Fig. 2.5).

![Figure 2.5: Three-dimensional plot of the normalized soliton intensity versus the time \( \eta \) and the radial variable \( r \).](image)

2.2.2 Numerical analysis

Eq. 2.6 is numerically studied using the Finite Difference Beam Propagation Method (FD-BPM). It is a cylindrical partial differential equation that can be "integrated" forward in \( z \) by a number of standard techniques. In this approach, the field in the transverse plane is represented only at discrete points on a grid, and at discrete planes along the propagation direction \( z \). Given the field at one \( z \) plane, we can find the field at the next \( z \) plane. This is then repeated to determine the field throughout the structure.

Let \( \Psi_{i}^{s+1} \) denote the field at transverse grid point \( i \) and longitudinal plane \( s \), and assume that the grid points and planes are equally spaced by \( \Delta r \) and \( \Delta z \) apart, respectively. The radial and longitudinal dimensions are discretized by the values \( r_{i} \) and \( z_{s} \) according to the relations

\[
\begin{align*}
\text{for radial:} & \quad r_{i} = i\Delta r, \\
\text{for longitudinal:} & \quad z_{s} = s\Delta z.
\end{align*}
\]

(2.28) and (2.29)

Simplifying we get a tridiagonal matrix of the form

\[
-c_{1}\Psi_{i+1}^{s+1} + d\Psi_{i}^{s+1} - c_{3}\Psi_{i-1}^{s+1} = c_{1}\Psi_{i+1}^{s} + c_{2}\Psi_{i}^{s} + c_{3}\Psi_{i-1}^{s}.
\]

(2.30)

This can be easily solved using Thomas Algorithm\(^{37}\) as discussed in section 1.3. Once the field at \( s \) is known, we can determine the field at \( s + 1 \) and so on.
We integrated Eq. (2.6) using the result obtained from the variational analysis as initial condition. The numerical parameters of the simulation has been chosen so as to fit the usual experimental configurations. Here, we have chosen $n_0 = 1.6755$, $n_2 = 2.2 \times 10^{-12} \text{ cm}^2/\text{W}$ and $n_4 = -8 \times 10^{-22} \text{ cm}^4/\text{W}^2$ which are the nonlinear coefficients of PTS at wavelength 1600nm.\(^\text{38}\) Similarly, for AlGaAs, with $n_0 = 3$, $n_2 = 1.5 \times 10^{-13} \text{ cm}^2/\text{W}$, $n_4 = -5 \times 10^{-23} \text{ cm}^4/\text{W}^2$ at wavelength 1550 nm.\(^\text{38}\) The beam intensity is chosen as $1.1 \times 10^9 \text{ W/m}^2$. The outcome of these simulations (see Fig. (2.6)) agrees very well with that obtained from the variational approach. The beam propagates without any change in shape.

Figure 2.6: Numerically computed beam profile after it propagates a distance of 1 mm through the medium. Here Intensity is in $\text{W/m}^2$ and distance and transverse beam profile are in micrometers.

2.3 Spatio-temporal solitons stabilized by multiphoton ionization

2.3.1 Analytical and numerical analysis

We can generalize Eq. (2.5) to include both the temporal and spatial effects such that the new equation describes the propagation of an optical pulse
in a bulk medium whose transverse dimensions remain much larger than
the beam size. In this case we have to include the effects of diffraction,
dispersion and nonlinearity. Thus the general form of the (3+1)D nonlinear
Schrödinger equation governing the evolution of the electromagnetic field
\( A \) in a cubic-quintic isotropic dispersive medium can be written as

\[
2ik \frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial \tau^2} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}
- 2k\lambda_1 |A|^2 A - 2k\lambda_2 |A|^4 A = 0. \tag{2.31}
\]

The change in the refractive index by the effect of Kerr nonlinearity, defocusing quintic nonlinearity and the self-induced ionization can be written as,
\[
n = n_0 + n_2 I + n_4 I^2 - N_e/2n_0 N_{cr},
\]
where \( n_0 \) is the number density of free electrons and \( N_{cr} \) is the critical plasma density. In this case, \( D \) takes the form \( D = (n_0(\omega) + n_2(\omega) |E|^2 + n_4(\omega) |E|^4 - N_e/2n_0 N_{cr})^2 E \), where,
\[
N_e = \int_{-\infty}^{\tau} |E|^{2n} \, d\tau'.
\]
Using a simple integration rule, the electron density can be approximated for an integration up to the peak of the pulse, as \( N_e = g(\tau) N_0 |E|^{2n} \), where \( g(\tau) = 0.5(\tau_{\text{min}} + \tau) \).

Now the evolution equation can be written as

\[
2ik \frac{\partial A}{\partial z} + \frac{\partial^2 A}{\partial \tau^2} + \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2}
- 2k\lambda_1 |A|^2 A - 2k\lambda_2 |A|^4 A - \beta^{(N)} |A|^{2N} A = 0, \tag{2.32}
\]
where \( \beta^{(N)} \) is the \( N \)-photon absorption coefficient.

Here we are considering the propagation of the pulsed beam along
the \( z \) direction. We will take spherical coordinates for our analysis. The
comoving coordinate \( \tau \) can be treated on the same footing as a spatial coordinate. We can introduce the spatiotemporal radius \( r \) as,
\[
r = (x^2 + y^2 + \tau^2)^{1/2}.
\]
Now Eq. (2.32) takes the form

\[
2ik \frac{\partial A}{\partial z} = -\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial A}{\partial r} \right) - 2k\lambda_1 |A|^2 A \\
+ 2k\lambda_2 |A|^4 A + \beta^{(N)} |A|^{2N} A. \tag{2.33}
\]

We follow the standard variational method \(^35\) for the analysis of this
equation. For this we first write the Lagrangian of the above equation as

\[
L = i \frac{r^2}{2} \left( A A^* - A^* \frac{\partial A}{\partial z} \right) + \frac{r^2 \partial A \partial A^*}{2k} \\
+ \frac{r^2}{2} \lambda_1 |A|^4 + \frac{r^2}{3} \lambda_2 |A|^6 + \frac{r^2}{2k} \beta^{(N)} |A|^{2N+2}. \tag{2.34}
\]
We proceed by assuming a trial solution of the form

\[ A(r, z) = C(z) \exp\left[-\frac{r^2}{2w(z)^2} + ib(z)r^2 + i\phi r^2\right], \]  

(2.35)

where \( C(z) \) is the maximum amplitude, \( w(z) \) is the beam width, \( b(z) \) is the curvature parameter and \( \phi \) is the phase.

Now the reduced Lagrangian for the system can be written as

\[
\langle L \rangle = \int_0^\infty Lrdr
\]

\[
= \frac{i}{2}(CC^* - C^*C_z)w^4 + 3b_z |C|^2 w^6
+ \frac{3 |C|^2}{2k} \left( \frac{1}{w^4} + 4b^2 \right) w^6 + \frac{\lambda_1}{8} |C|^4 w^4
+ \frac{\lambda_2}{27} |C|^6 w^4 + \frac{\beta^{(N)}}{kN^3} |C|^{2N+2} w^4.
\]  

(2.36)

Now we can find the variation of \( \langle L \rangle \) with respect to the various Gaussian parameters \( C(z), C(z)^*, w(z) \) and \( b(z) \):

\[
C\frac{\partial \langle L \rangle}{\partial C} - C^*\frac{\partial \langle L \rangle}{\partial C^*} \Rightarrow \frac{i}{2}CC^*_z w^4 + \frac{i}{2}C^*C_z w^4.
\]  

(2.37)

This gives

\[
\frac{\partial}{\partial z}(w^3CC^*) = 0.
\]

That is

\[
w^3 |C|^2 = E_0 = w_0^3 |C_0|^2.
\]  

(2.38)

\[
C\frac{\partial \langle L \rangle}{\partial C} + C^*\frac{\partial \langle L \rangle}{\partial C^*} = \frac{i}{2}w^4(CC^*_z - C^*C_z)
+ 6b_z|C|^2w^6 + \frac{6}{2k}|C|^2 \left( \frac{1}{w^4} + 4b^2 \right) w^6
+ \frac{\lambda_1}{2} |C|^4 w^4 + \frac{2\lambda_2}{9} |C|^6 w^4 + \frac{2\beta^{(N)}}{k^2} |C|^{2N} w^4 = 0.
\]  

(2.39)

The variation of \( \langle L \rangle \) with \( w \) gives
\[
\frac{\partial \langle L \rangle}{\partial w} = i w^3 (C^* C_z - C^* C_z) + 18 b_z |C|^2 w^5 + \frac{3}{k} |C|^2 w + \frac{36}{k} |C|^2 b^2 w^5 + \frac{\lambda_1}{2} |C|^4 w^3 + \frac{4}{27} \lambda_2 |C|^6 w^3 + \frac{4 \beta^{(N)}}{k^2} |C|^{2N} w^3 = 0. \tag{2.40}
\]

Similarly, after some algebra, we get the following equations:

\[
b = \frac{k}{2w} \frac{dw}{dz}, \tag{2.41}
\]

and

\[
d^2 w \over dz^2 = -\frac{3}{k^2 w^3} - \frac{\lambda_1 |C|^2}{2kw} - \frac{20}{81kw} \lambda_2 |C|^4 - \frac{4 \beta^{(N)}}{3k^2 N^2 w} |C|^{2N+2}. \tag{2.42}
\]

Integration of Eq. (2.42) using Eq. (5.18) and introducing the normalized variables, \[w(z)/w_0 = y(z),\]

\[
\frac{1}{2} \left( \frac{dy}{dz} \right)^2 + \Pi(y) = 0, \tag{2.43}
\]

where

\[
\Pi(y) = \frac{\mu}{y^2} + \frac{\nu}{y^3} + \frac{\xi}{y^6} + \frac{\alpha}{y^{3N-1}} + K. \tag{2.44}
\]

Figure 2.7: Qualitative plot of the potential function for different magnitudes and sign of nonlinearity.
with

\[ \mu = -\frac{3}{(2k^2w_0^4)}, \]

\[ \nu = -\frac{\lambda_1 E_0}{6kw_0^5}, \]

\[ \xi = \frac{10\lambda_2 E_0^2}{243kw_0^8}, \]

\[ \alpha = -\frac{4\beta^{(N)} E_0^{(N)}}{3k^2N^2(3N - 4)w_0^{(3N+1)}}, \]

\[ K = c/w_0^2, \]

and \( c \) is a constant of integration.

This represents a particle in a potential well. Based on the magnitude and sign of the nonlinearity we can identify four different regimes of propagation. Only when the focussing due to third order nonlinearity is compensated by the defocusing due to fifth order nonlinearity and the free electrons produced due to plasma formation, we get a stable light bullet. All the four regimes are plotted in Fig. 2.7. The linear case is also plotted for reference (f5). When all the nonlinearities are of defocusing kind the beam diffracts even faster than the purely linear case (f1).

Figure 2.8: Normalized beam profile at the input face of the medium. Inset: Two dimensional view of the beam profile.
A stable light bullet is formed when the focussing of the beam is completely compensated by the defocusing due to fifth order nonlinearity and the free electrons. This is the limit case $(f4)$. The plots for intermediately high nonlinearity are shown in $(f2)$ and $(f3)$.

In order to verify the results from the variational approach, we studied the system numerically using the finite-difference beam propagation method (FD-BPM). The solution obtained using the variational method was used as the input for the numerical simulation. In our simulations we could observe a stable light bullet which propagated through the medium preserving its spatial and temporal radius. We propagated a 50 micrometer, 20 fs pulse through a medium with cubic-quintic nonlinearity. The nonlinear parameters of the medium are $n_0 = 3$, $n_2 = 1.5 \times 10^{-13} cm^2/W$, $n_4 = -5 \times 10^{-23} cm^4/W^2$ at wavelength 1550 nm. The normalized initial profile of the beam is shown in Fig. 2.8 and the output profile after 5 diffraction lengths of travel through the medium is given in Fig. 2.9. From the figures we can clearly see that the beam propagated through the medium maintaining its spatio-temporal profile. This is the signature of the formation of light bullets.

Figure 2.9: Numerically computed normalized beam profile using the solution of variational method as the input after it propagates 5 diffraction lengths through the medium.
2.4 Conclusions

In this chapter, first we have presented the studies on the propagation of a high energy laser beam through a PTS like medium characterized by both third and fifth order nonlinearities. We have employed both analytical and numerical methods. The energy of the beam considered in the present problem is sufficiently high enough to produce multiphoton ionization. Solutions are obtained using the variational formulation. It is found that multiphoton ionization helps in containing the catastrophic breakdown of the beam and helps in forming stable solitons. We could also show analytically the formation of stable solitons. This solution was taken as the initial condition for the numerical simulation. The soliton is found to propagate without any shape change.

We have further shown the existence of stable light bullets in a medium with cubic-quintic nonlinearity and self-defocusing effect of free electrons due to plasma formation. The system was studied both analytically and numerically and the results were found to be in good agreement. These LBs have potential application in all optical communication. It has been demonstrated numerically that the three-dimensional low energy spinning solitons are unstable in the CQ model. The consideration of multiphoton ionization term in the evolution equation may stabilize the spinning light bullets in the cubic-quintic model. However, high-energy spinning solitons are stable in the cubic-quintic model.

References


REFERENCES


