4

Modulational Instability in Photorefractive Crystals in the Presence of Wave Mixing

4.1 Introduction

Modulational instability (MI) in a photorefractive medium is studied in the presence of two wave mixing. We then propose and derive a model for forward four wave mixing in the photorefractive medium and investigate the modulational instability induced by four wave mixing effects. By using the standard linear stability analysis the instability gain is obtained. This chapter is organized as follows. Section 4.2 gives a general introduction to Holographic solitons. The basic propagation equation for the two wave mixing (TWM) geometry is presented in Section 4.3 and the modulational instability in this geometry is studied. In section 4.4 the governing equations for the forward four wave mixing geometry is presented. The system is studied without using the undepleted pump beam approximation. The standard linear stability analysis for the coupled equations is carried out and the gain spectrum is obtained. Section 4.5 concludes the chapter giving the main observations.

Modulational instability is a universal process in which tiny phase and amplitude perturbations that are always present in a wide input beam grow exponentially during propagation under the interplay between diffraction (in spatial domain) or dispersion (in temporal domain) and nonlinearity. Instabilities and chaos can occur in many types of nonlinear physical systems. Optical instabilities can be classified as temporal and spatial instabilities depending on whether the electromagnetic wave is modulated temporally or spatially after it passes through the medium. Temporal instability has been studied by various authors\textsuperscript{1-3} and the first experimental
observation of MI in a dielectric material was in 1986. The temporal MI occurs as an interplay between self-phase modulation and group velocity dispersion. In the spatial domain, diffraction plays the role of dispersion. When both diffraction and dispersion are present simultaneously, it results in spatio-temporal MI. Recently, Wen et al. investigated MI in negative refractive index materials. In a rather loose context, MI can be considered as a precursor of self-trapped beam formation.

In the spatial domain, MI manifests itself as filamentation of a broad optical beam through the spontaneous growth of spatial-frequency sidebands. The MI is a destabilization mechanism for plane waves. It leads to delocalization in momentum space and, in turn, to localization in position space and the formation of self-trapped structures. During MI, small amplitude and phase perturbations tend to grow exponentially as a result of the combined effects of nonlinearity and diffraction. As a result of MI in the spatial domain, a large-diameter optical beam tends to disintegrate during propagation. Castillo et al. have provided experimental evidence of such induced MI in a photorefractive bismuth titanium oxide crystal. Saffman et al. study theoretically and experimentally the modulational instability of broad optical beams in photorefractive media. The MI phenomena has been previously observed in various media like Kerr media, electrical circuits, plasmas, parametric band gap systems, quasi-phase-matching gratings and discrete dissipative systems. The transverse instability of counterpropagating waves in PR media is studied by Saffman et al. From the above investigations, it is clear that the study of MI in a medium is both of fundamental as well as of technological importance.

To introduce the concept of MI, we consider the (1 + 1)D nonlinear Schrödinger (NLS) equation and show that this equation exhibits an instability and leads to spatial or temporal modulation of a constant-intensity plane wave. The NLS equation has the form

$$iu_z + \frac{1}{2}u_{xx} \pm |u|^2u = 0,$$  

where $u$ is the amplitude of the beam. The sign of third term is positive for a focusing type of nonlinearity and negative for defocusing nonlinearity.

Equation 4.1 admits plane wave solutions of the form

$$u(z, x) = u_0 \exp[i\phi(z)],$$  

where $u_0$ is a constant and corresponds to the input intensity. Substituting Eq. (4.2) in Eq. (4.1), we get

$$\phi(z) = \pm u_0^2 z.$$  

Hence the plane wave solution can be written as

$$u(z, x) = u_0 \exp[\pm i u_0^2 z].$$  \hspace{1cm} (4.4)

Such a solution shows that the plane wave of amplitude $u_0$ propagates through the nonlinear medium without any change except for acquiring an intensity dependent phase shift. Now, we need to study the stability of these plane waves against small perturbations. For this we use the standard linear stability analysis method, which is an important technique to study the stability of solutions. The method proceeds by first perturbing the plane wave solution in the form

$$u(z, x) = (u_0 + a) \exp[\pm i u_0^2 z],$$  \hspace{1cm} (4.5)

where $a$ is a small complex perturbation. Substituting this in Eq. (4.1) and linearizing in $a$, we get

$$ia_z + \frac{1}{2} a_{xx} \pm (a + a^*) u_0^2 = 0.$$  \hspace{1cm} (4.6)

Considering that the perturbation consists of two side bands, we can write

$$a(x, z) = u_1(z) \exp[i K x] + v_1(z) \exp[-i K x].$$  \hspace{1cm} (4.7)

This gives a system of two coupled equations in $u_1$ and $v_2$ as

$$u_{1z} = i \left(- \frac{1}{2} K^2 \pm u_0^2\right) u_1 \pm i u_0^2 v_1^*,$$  \hspace{1cm} (4.8)

$$v_{1z}^* = \mp i u_0^2 u_1 + i \left(\frac{1}{2} K^2 \mp u_0^2\right) v_1^*.$$  \hspace{1cm} (4.9)

The above coupled equations can be written in the compact matrix form as

$$\partial_z X = MX,$$  \hspace{1cm} (4.10)

where $M$ is a 2x2 matrix given by

$$M = \begin{pmatrix}
a_{11} & a_{22} \\
a_{33} & a_{44}
\end{pmatrix},$$  \hspace{1cm} (4.11)

and

$$a_{11} = i \left(- \frac{1}{2} K^2 \pm u_0^2\right),$$  \hspace{1cm} (4.12)

$$a_{22} = \pm i u_0^2,$$  \hspace{1cm} (4.13)

$$a_{33} = \mp i u_0^2,$$  \hspace{1cm} (4.14)

$$a_{44} = i \left(\frac{1}{2} K^2 \mp u_0^2\right).$$  \hspace{1cm} (4.15)
This equation has a nontrivial solution only if the determinant of the matrix vanishes. The real part of the eigenvalues of the stability matrix in Eq. (4.11) gives the gain associated with the system.\textsuperscript{15} The eigenvalues are given by the following relation

\begin{equation}
\Lambda_{\pm} = \pm \frac{1}{2} (-K^2 \pm 4K^2u_0^2)^{1/2}.
\end{equation}

For $\Lambda > 0$, the perturbation grows exponentially during propagation with the growth rate or gain given by $Re[\Lambda]$, indicating MI. Equation 4.16 shows that the continuous wave solution is absolutely stable only in the case of a self-defocusing nonlinearity. The solution is unstable in the case of self-focusing nonlinearity. The gain spectrum of modulational instability in the case of focusing nonlinearity is shown in Fig. 4.1 for $u_0 = 1$.

Four wave mixing in the phase conjugate geometry is widely used in photorefractive materials for various applications. It can be used to implement several different computing functions,\textsuperscript{16} optical interconnects,\textsuperscript{17} matrix addition,\textsuperscript{18} and optical correlator.\textsuperscript{19} Nonlinear solutions for photorefractive vectorial two-beam coupling and for forward phase conjugation in photorefractive crystals has been found in.\textsuperscript{20} Recently Jia et al.\textsuperscript{21} experimentally demonstrated degenerate, forward four-wave mixing effects in a self-defocusing PR medium, in both one and two transverse dimensions. They observed the nonlinear evolution of new modes as a function
of propagation distance, in both the near-field and far-field (Fourier space) regions.

4.2 Holographic solitons

Recently, a new kind of spatial solitons, holographic (HL) solitons was proposed by Cohen et al.\textsuperscript{22} They are formed when the broadening tendency of diffraction is balanced by phase modulation that is due to Bragg diffraction from the induced grating. Holographic solitons are solely supported by cross-phase modulation arising from the induced grating, not involving self-phase modulation at all. In 2006,\textsuperscript{23} they showed that the nonlinearity in periodically poled photovoltaic photorefractives can be solely of the cross phase modulation type. The effects of self-phase modulation and asymmetric energy exchange, which exist in homogeneously poled photovoltaic photorefractives, can be considerably suppressed by the periodic poling. They demonstrated numerically that periodically poled photovoltaic photorefractives can support Thirring-type (solitons which exist only by virtue of cross phase modulation) (holographic) solitons. HL solitons in PR dissipative medium was studied by Liu.\textsuperscript{24} Existence of HL solitons in a grating mediated waveguide was studied by Freedman et al.\textsuperscript{25}

A spatial soliton is formed in a nonlinear medium when the optical beam modifies the refractive index of the medium in such a way that it induces a positive lens. As a result, the change in refractive index at the center of the beam is greater than that at its margins. The refractive index structure resembles that of a graded index waveguide. The beam then gets self-trapped and thus forms a spatial soliton. The mechanism responsible for this is the self-focusing effect. Another mechanism that can support spatial solitons arises from the nonlinear phase coupling that results from symmetric energy exchange between two or more mutually coherent beams.

One such example is the quadratic solitons. Quadratic solitons form by the mutual trapping and locking of multiple-frequency waves. The light does not change the refractive index in the case of quadratic nonlinearity of a noncentrosymmetric media. The nonlinear effects are observed when the fundamental frequency becomes phase matched with one of its harmonics. The simplest case corresponds to the process of second harmonic generation (SHG) or optical parametric generation (OPG), where a fundamental frequency wave and its second-harmonic generate each other.
The resulting soliton contains both the fundamental and harmonic fields, which in the simplest case exhibit a classical bell-shape. Solitons form when the material and light propagation conditions inside the quadratic nonlinear crystal are set so that the diffraction and dispersion lengths that measure the spreading of the beams and pulses, and the nonlinear length that measures the strength of the frequency conversion process, are comparable. Each beam continuously loses some of its energy and regains the same amount, such that the net power in each beam is conserved. In this interaction, the field that constitutes the acquired energy (to each of the beams) is phase retarded relative to the primary field of each beam. Thus, as the acquired field is added to the primary field, it effectively slows the phase velocity of the beam. Hence, if the interaction occurs in such a way that the effect is more intense at the center of the beam (or for the lowest spatial frequencies of the beam), then it reduces or eliminates the broadening effects of diffraction.

Such a phase coupling between two mutually coherent beams can be induced through a grating in the refractive index produced by the interference of the two beams. This is the case in HL solitons. Fig. 4.2 shows the schematic representation of the configuration for the existence of HL solitons. Such a soliton is created in the absence of the self-action effect, solely due to the beam cross-coupling through the induced Bragg reflection. Salgueiro et al. studied the composite spatial solitons supported by
mutual beam focusing in a Kerr-like nonlinear medium in the absence of the self-action effects. They predicted the existence of continuous families of single and two-hump composite solitons, and studied their stability and interaction. In 2007, two-dimensional holographic photovoltaic bright spatial solitons were observed in a Cu:K0.25Na0.75Sr1.5Ba0.54Nb5O15 crystal in which two coherent laser beams, a signal beam, as well as a strong and uniform pump beam at 532 nm are coupled to each other via two-wave mixing.27

In the case of a HL dissipative soliton, one beam acts as a pump beam to supply energy for the other beam, which behaves as the signal beam. In such a system the two beam coupling results in an asymmetric energy exchange between the two beams such that energy is transferred from the pump beam into the signal beam. There is a double balance occurring here: diffraction is balanced by nonlinearity and loss is balanced by the gain due to the asymmetric energy transfer. But only the signal beam can propagate as a soliton. Both the bright and dark HL dissipative solitons28 have been studied.

4.3 Two wave mixing geometry

The periodic variation of intensity due to interference of two beams of coherent electromagnetic radiation inside a PR material results in the formation of a volume index grating. The presence of this grating affects the propagation of these two beams. Due to perfectly phase-matched Bragg scatterings, the two beams are strongly diffracted by the index grating. Fig. 4.3 shows the Bragg scattering of the two beams due to the volume index grating. Beam 1 is scattered by the grating and it propagates along the direction of beam 2. Similarly beam 2 is scattered and it propagates along the direction of beam 1.

In this section, we present the derivation of the model equation for the TWM geometry29,30 and study the MI in it. Consider the interaction of two laser beams inside a photorefractive medium (see Fig. 4.3). A stationary interference pattern is formed, if the two beams are of the same frequency.
Figure 4.3: Bragg scattering from the gratings formed in PR media. Grating induced by the pair of beams (top). Beam $A_2$ is diffracted by the grating producing beam $A_1$ (middle). Beam $A_1$ is diffracted by the grating producing beam $A_2$ (bottom).
Two wave mixing geometry

Let the electric field of the two beams be written as

\[ E_j = A_j \exp[i(\omega t - k_j \cdot r)], \text{ for } j = 1, 2. \] (4.17)

Here, \( A_1 \) and \( A_2 \) are the amplitudes, \( \omega \) is the angular frequency, and \( k_1 \) and \( k_2 \) are the wave vectors.

The medium is assumed to be isotropic and both beams are polarized perpendicular to the plane of incidence. The total intensity of the beams is

\[ I = |E|^2 = |E_1 + E_2|^2, \] (4.18)

which can be expressed as

\[ I = |A_1|^2 + |A_2|^2 + A_1 A_2^* \exp[iK \cdot r] + A_2 A_1^* \exp[-iK \cdot r], \] (4.19)

where

\[ K = k_2 - k_1. \]

The magnitude of the vector \( K \) is \( 2\pi/\Lambda \) where \( \Lambda \) is the period of the fringe pattern.

Eq. 4.19 represents a spatial variation of optical energy in the photorefractive medium. Such an intensity pattern will generate and redistribute charge carriers. As a result, a space charge field is created in the medium. This field induces an index volume grating via the Pockels effect. In general, the index grating will have a spatial phase shift relative to the interface pattern.

The index of refraction including the fundamental component of the intensity-induced gratings can be written as

\[ n = n_0 + \left[ \frac{n_1 A_1^* A_2}{2I_0} \exp[i\phi] \exp[-iK \cdot r] + cc \right], \] (4.20)

where \( n_0 \) is the index of refraction when no light is present, \( \phi \) is real and \( n_1 \) is a real and positive number. The wave equation reduces to the Helmholtz equation for highly monochromatic waves like laser beams which can be viewed as a superposition of many monochromatic plane waves with almost identical wave vectors

\[ \nabla^2 E + \omega^2 n^2 E/c^2 = 0, \] (4.21)

where \( E = E_1 + E_2. \)

Now let

\[ E_j = A_j(x, y, z) \exp[i\omega t - i\beta_j z]. \]
Modulational Instability in Photorefractive Crystals in the Presence of Wave Mixing

Here $A(x, y, z)$ is the complex amplitude that depends on position, $\beta_1$ and $\beta_2$ are the $z$ component of the wave vectors $k_1$ and $k_2$ inside the medium respectively and $z$ is measured along the central direction of propagation. We solve for the steady state so that $A_j$ is taken to be time independent.

$$
\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \omega^2 n^2/c^2 \right) E = 0. \tag{4.22}
$$

Substituting Eq. (4.20) and solving by neglecting the second order term $n_1^2$ and using the fact that for highly directional monochromatic waves with $a >> \lambda$, $\partial^2 A/\partial z^2$ can be neglected and we get

$$
2i\beta_1 \frac{\partial A_1}{\partial z} = \nabla^2_\perp A_1 + \frac{\omega^2 n_0 n_1}{c^2 I_0} e^{-i\phi} A_2^* A_2 A_1, \tag{4.23}
$$

and

$$
2i\beta_2 \frac{\partial A_2}{\partial z} = \nabla^2_\perp A_2 + \frac{\omega^2 n_0 n_1}{c^2 I_0} e^{-i\phi} A_1^* A_1 A_2. \tag{4.24}
$$

Now for the case when the two laser beams enter the medium from the same side at $z = 0$

$$
\beta_1 = \beta_2 = k \cos \theta = \frac{2\pi}{\lambda} n_0 \cos \theta.
$$

Here, $2\theta$ is the angle between the beams inside the medium. Simplifying we obtain

$$
i \frac{\partial A_1}{\partial z} = \frac{1}{2k \cos \theta} \nabla^2_\perp A_1 + \frac{\Gamma}{2I_0} A_1 |A_2|^2, \tag{4.25a}
$$

$$
i \frac{\partial A_2}{\partial z} = \frac{1}{2k \cos \theta} \nabla^2_\perp A_2 + \frac{\Gamma}{2I_0} A_2 |A_1|^2, \tag{4.25b}
$$

where

$$
\Gamma = \frac{2\pi n_1}{\lambda \cos \theta} e^{-i\phi}, \tag{4.26}
$$

is the complex coupling coefficient. A similar model was used to study the existence of Holographic solitons.\textsuperscript{22}

A prerequisite for obtaining Holographic focusing is that the induced grating be in phase with the intensity grating. If it is shifted by $\tau$, then the grating leads to holographic defocusing. If it is $\pm\pi/2$ phase shifted with respect to the intensity grating, then the interaction will yield an asymmetric energy exchange between the two beams because of the two
beam coupling property of the photorefractive material. Therefore, $\phi = 0$ for the existence of bright solitons. This gives the coupling constant as

$$\Gamma = \frac{2\pi n_1}{\lambda \cos \theta}. \quad (4.27)$$

The next step is to study the propagation of a broad optical beam through the PR medium. We study MI of a one dimensional broad optical beam. Hence the $y$ dependent term in the transverse Laplacian in Eq. (4.25) can be neglected. For a broad optical beam, the diffraction term in Eq. (4.25) can be set to zero giving the steady state solutions as

$$A_j = \sqrt{P} \exp \left[ -i \frac{\Gamma P}{2I_0} z \right], \quad (4.28)$$

for $j = 1, 2$.

To study MI, we consider small perturbations of the steady state solutions as

$$A_j = (\sqrt{P} + a_j) \exp \left[ -i \frac{\Gamma P}{2I_0} z \right]. \quad (4.29)$$

Substituting in Eq. (4.25) and neglecting the quadratic and higher order terms in $a_j$, the perturbations $a_1$ and $a_2$ are found to satisfy the following linearized set of two coupled equations:

$$i \frac{\partial a_1}{\partial z} = \frac{1}{2k \cos \theta} \frac{\partial^2 a_1}{\partial x^2} + \frac{\Gamma P}{2I_0} (a_2 + a_2^*), \quad (4.30)$$

$$i \frac{\partial a_2}{\partial z} = \frac{1}{2k \cos \theta} \frac{\partial^2 a_2}{\partial x^2} + \frac{\Gamma P}{2I_0} (a_1 + a_1^*). \quad (4.31)$$

It is important to note that the evolution of the perturbations depend solely on the cross phase modulation. To solve Eq. (4.30) and Eq. (4.31), we assume that the perturbations be composed of two side bands:

$$a_j(x, z) = U_j(z) \exp[i \kappa x] + V_j(z) \exp[-i \kappa x]. \quad (4.32)$$

The substitution of Eq. (4.32) in Eqs. (4.30) and Eq. (4.31) results in a set of four homogeneous equations in $U_1, U_2, V_1$ and $V_2$ as

$$\frac{\partial U_1}{\partial z} = i \frac{\kappa^2}{2k \cos \theta} U_1 - i \frac{\Gamma P}{2I_0} (U_2 + V_2^*), \quad (4.33)$$

$$\frac{\partial V_1^*}{\partial z} = -i \frac{\kappa^2}{2k \cos \theta} V_1^* + i \frac{\Gamma P}{2I_0} (U_2 + V_2^*), \quad (4.34)$$

$$\frac{\partial U_2}{\partial z} = i \frac{\kappa^2}{2k \cos \theta} U_2 - i \frac{\Gamma P}{2I_0} (U_1 + V_1^*), \quad (4.35)$$

$$\frac{\partial V_2^*}{\partial z} = -i \frac{\kappa^2}{2k \cos \theta} U_1 + i \frac{\Gamma P}{2I_0} (U_1 + V_1^*). \quad (4.36)$$
This set has a nontrivial solution only if the determinant of the coefficient matrix vanishes. The eigenvalues of the system are obtained as

\[
\Lambda_{1\pm} = \pm \frac{1}{2A} \left( \frac{-I_0 \kappa^4 - 2A \Gamma P \kappa^2}{I_0} \right)^{1/2},
\]

\[
\Lambda_{2\pm} = \pm \frac{1}{2A} \left( \frac{-I_0 \kappa^4 + 2A \Gamma P \kappa^2}{I_0} \right)^{1/2},
\]

where \( A = \kappa \cos \theta \). The plane wave solution is stable if perturbations at any wave number \( \kappa \) do not grow with propagation. This is the case as long as \( \kappa \) is imaginary. MI gain will exist only when \( \text{Re}[\Lambda] > 0 \). This condition is satisfied only by the second set of eigenvalues. The gain associated with the system is given by

\[
G = |\mathcal{R}(\Lambda_{\pm})|.
\]

A typical plot of the gain spectrum is given in Fig. 4.4. We consider the case of BaTiO\(_3\) crystal with a space charge field of \( 10^4 \text{V/m} \), electro-optic coefficient \( r_{42} = 1640 \times 10^{-12} \text{m/V} \) and refractive index \( n = 2.4 \) which gives a coupling constant \( \Gamma = 20 \text{ cm}^{-1} \). Fig. 4.5 gives the variation of gain with the angle \( \theta \). The gain increases with decrease in angle.
Forward four wave mixing geometry

Figure 4.5: Variation of gain with respect to the angle between the two beams. The lowermost curve is for $\theta = \pi/4$, the dotted curve is for $\pi/8$ and the topmost curve is for $\pi/16$.

Figure 4.6: Forward four wave mixing in photorefractive media in the transmission geometry.
4.4 Forward four wave mixing geometry

In the two wave mixing case, two coherent beams interfere inside a photorefractive medium and produce a volume index grating. In the case of optical phase conjugation, using four-wave mixing, a third beam is incident at the Bragg angle from the opposite side and a fourth beam is generated. Now we consider the situation in which the third beam is incident from the front at the Bragg angle and a diffracted beam is generated. That is we consider the interaction of four beams in a PR medium in the forward geometry. We assume that all the beams have the same frequency $\omega$. We propose a model for the observation of MI in a PR medium induced by four wave mixing. The method proceeds by first writing the four coupled equations for the present geometry. Of the four beams, let beams $A_2$ and $A_3$ be the pump beam, $A_1$ be the signal beam and $A_4$ be the generated beam. Beam $A_1$ is coherent with beam $A_2$ and beam $A_3$ is coherent with beam $A_4$. Thus the index grating consists of two contributions: $A_1^* A_2$ and $A_3^* A_4$. The index of refraction including the fundamental component of the intensity-induced gratings can thus be written as

$$n = n_0 + \left[ \frac{n_1 (A_1^* A_2 + A_3^* A_4)}{2I_0} \exp[i\phi] \exp[-iK.r] + cc \right], \quad (4.40)$$

where $k_2 - k_1 = k_4 - k_3 = K$.

We have

$$E = \sum_{j=1}^{4} E_j, \quad (4.41)$$

and

$$n^2 = \frac{n_0 n_1 (A_1^* A_2 + A_3^* A_4)}{I_0} \exp[i\phi] \exp[-iK.r] + cc. \quad (4.42)$$

This gives

$$\nabla^2 E = \sum_{j=1}^{4} \left( \frac{\partial^2 A_j}{\partial z^2} - 2i\beta_j \frac{\partial A_j}{\partial z} - A_j \beta_j^2 + \frac{\partial^2 A_j}{\partial x^2} \right) \exp[i(\omega t - \beta_j r)]. \quad (4.43)$$
Substituting the above equations in Eq. (4.21) gives
\[
\frac{\partial^2 A_1}{\partial x^2} \exp[-i\beta_1 z] + \frac{\partial^2 A_2}{\partial x^2} \exp[-i\beta_2 z] + \frac{\partial^2 A_3}{\partial x^2} \exp[-i\beta_3 z]
+ \frac{\partial^2 A_4}{\partial x^2} \exp[-i\beta_4 z] - 2i\beta_1 \frac{\partial A_1}{\partial z} \exp[-i\beta_1 z] - 2i\beta_2 \frac{\partial A_2}{\partial z} \exp[-i\beta_2 z]
- 2i\beta_3 \frac{\partial A_3}{\partial z} \exp[-i\beta_3 z] - 2i\beta_4 \frac{\partial A_4}{\partial z} \exp[-i\beta_4 z]
= -\frac{\omega^2 n_0 n_1}{c^2 I_0} \{(A_1^* A_2 + A_3^* A_4) \exp[i\phi] \exp[-iK.r] + c.c\}. \quad (4.44)
\]

Multiplying Eq. (4.44) with \(\exp[i\beta_3 z]\) and equating the coefficients, we get the following four coupled equations for the present geometry:

\[
i \frac{\partial A_1}{dz} = \frac{1}{2k \cos \theta} \nabla^2 A_1 + \frac{\Gamma}{2I_0} (A_1 A_2^* + A_3 A_4^*) A_2, \quad (4.45a)
\]
\[
i \frac{\partial A_2}{dz} = \frac{1}{2k \cos \theta} \nabla^2 A_2 + \frac{\Gamma}{2I_0} (A_1^* A_2 + A_3^* A_4) A_1, \quad (4.45b)
\]
\[
i \frac{\partial A_3}{dz} = \frac{1}{2k \cos \theta} \nabla^2 A_3 + \frac{\Gamma}{2I_0} (A_1 A_2^* + A_3 A_4^*) A_4, \quad (4.45c)
\]
\[
i \frac{\partial A_4}{dz} = \frac{1}{2k \cos \theta} \nabla^2 A_4 + \frac{\Gamma}{2I_0} (A_1^* A_2 + A_3^* A_4) A_3, \quad (4.45d)
\]

where \(I_0 = I_1 + I_2 + I_3 + I_4\) and \(\beta_1 = \beta_2 = \beta_3 = \beta_4 = k \cos \theta\).

The model permits plane wave solutions of the form \(A_j(x, z) = \sqrt{P} \exp[-i\frac{\Gamma P}{2I_0} z]\). The next step is to carry out a linear stability analysis of the plane wave solutions. For this the plane wave solution is perturbed as

\[
A_j = (\sqrt{P} + a_j(x, z)) \exp[-i\frac{\Gamma P}{2I_0} z], \quad (4.46)
\]

where \(a_j\) is a small complex perturbation. Inserting this into the coupled Eq. (4.45) and linearizing around the solution yields the equations for the perturbations:

\[
i \frac{\partial a_1}{\partial z} = -\frac{\Gamma P}{2I_0} a_1 + \frac{1}{2k \cos \theta} \frac{\partial^2 a_1}{\partial x^2} + \frac{\Gamma P}{2I_0} (2a_2 + a_2^* + a_3 + a_4), \quad (4.47)
\]
\[
i \frac{\partial a_2}{\partial z} = -\frac{\Gamma P}{2I_0} a_2 + \frac{1}{2k \cos \theta} \frac{\partial^2 a_2}{\partial x^2} + \frac{\Gamma P}{2I_0} (2a_1 + a_1^* + a_3^* + a_4), \quad (4.48)
\]
\[
i \frac{\partial a_3}{\partial z} = -\frac{\Gamma P}{2I_0} a_3 + \frac{1}{2k \cos \theta} \frac{\partial^2 a_3}{\partial x^2} + \frac{\Gamma P}{2I_0} (a_1 + a_2^* + 2a_4 + a_4^*), \quad (4.49)
\]
Modulational Instability in Photorefractive Crystals in the Presence of Wave Mixing

Now, we assume that the spatial perturbation $a(x, z)$ is composed of two side band plane waves, i.e.

$$a_j(x, z) = U_j(z) \exp[i\kappa x] + V_j(z) \exp[-i\kappa x].$$

Substituting we get eight homogeneous equations as

$$\frac{\partial U_1}{\partial z} = -i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)U_1 - i\frac{\Gamma P}{2I_0}(2U_2 + V_1^* + U_3 + V_4^*),$$
$$\frac{\partial U_2}{\partial z} = -i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)U_2 - i\frac{\Gamma P}{2I_0}(2U_1 + V_1^* + V_3^* + U_4),$$
$$\frac{\partial U_3}{\partial z} = -i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)U_3 - i\frac{\Gamma P}{2I_0}(U_1 + V_2^* + 2U_4 + V_3^*),$$
$$\frac{\partial U_4}{\partial z} = -i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)U_4 - i\frac{\Gamma P}{2I_0}(V_1^* + U_2 + 2U_3 + V_3^*),$$
$$\frac{\partial V_1^*}{\partial z} = i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)V_1^* + i\frac{\Gamma P}{2I_0}(2V_2^* + U_2 + V_3^* + U_4),$$
$$\frac{\partial V_2^*}{\partial z} = i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)V_2^* + i\frac{\Gamma P}{2I_0}(2V_1^* + U_1 + U_3 + V_4^*),$$
$$\frac{\partial V_3^*}{\partial z} = i\left(-\frac{\Gamma P}{2I_0} + \frac{\kappa^2}{2k \cos \theta}\right)V_3^* + i\frac{\Gamma P}{2I_0}(V_1^* + U_2 + 2V_4^* + U_4),$$
$$\frac{\partial V_4^*}{\partial z} = i\left(-\frac{\Gamma P}{2I_0} - \frac{\kappa^2}{2k \cos \theta}\right)V_4^* + i\frac{\Gamma P}{2I_0}(U_1 + V_2^* + 2V_3^* + U_3).$$

The eight coupled equations obtained above can be written in a compact matrix form as, $\partial_t X = MX$, where $M$ is an 8x8 matrix with $X = [U_1 \ U_2 \ U_3 \ U_4 \ V_1^* \ V_2^* \ V_3^* \ V_4^*]^T$.

This system has a nontrivial solution only if the determinant of the matrix vanishes. The real part of the eigenvalues of the stability matrix $\alpha$: Eq. (4.52) gives the gain associated with the system. Out of the eight roots of the system, only the root with maximum positive value contributes to the MI gain. Here only the eigenvalue given by

$$\Lambda^2 = \pm \frac{1}{2A} \left( -I_0 \kappa^4 + 4 \Delta \Gamma P \kappa^2 \right)^{1/2},$$

(4.53)
concludes to the MI of the system. We note that the gain vanishes for all values of \( \kappa \) greater than \( \kappa_{\text{max}}^2 = 4A\Gamma P/I_0 \). Defining \( \gamma = \Gamma P/I_0 \), we can rewrite the gain as

\[
\frac{G}{\gamma} = \sqrt{4\left(\frac{\kappa}{\kappa_{\text{max}}}\right)^2 \left(1 - \left(\frac{\kappa}{\kappa_{\text{max}}}\right)^2\right)}.
\] (4.54)

Figure 4.7: A typical plot showing the gain spectrum of the system in the forward four wave mixing process with respect to the spatial perturbation \( \kappa \).

The variation of the gain coefficient in the forward four wave mixing process with respect to the spatial perturbation is plotted in Fig. 4.7. Such instabilities are useful for pattern formation. A transverse modulation instability of a single beam or counterpropagating beams is a general mechanism that leads to pattern formation in nonlinear optics.\(^{31,32}\) We expect that similar results will be obtained using the present geometry.

4.5 Conclusions

We first studied MI occurring in a PR medium in the two wave mixing geometry and further modeled the forward four wave mixing occurring in a PR medium and studied MI in this geometry. In both the cases, the geometry is such that the nonlinear effect is manifested through holographic focusing. MI does not rely on self-phase-modulation self-focusing.
but results only by virtue of the competition between induced periodic modulation of the refractive index and diffraction of the beam. The MI gain spectrum is obtained for both two wave mixing and forward four wave mixing geometry. Such instabilities will be useful for pattern formation. Photorefractive materials are attractive for the studies of pattern formation as their slow time constant gives the possibility of observing the spatiotemporal dynamics of the system in real time. This also reduces the demands on experimental equipment where speed is often a crucial parameter. Another advantage of photorefractive pattern formation is that patterns can be observed using optical powers of tens of mW. In contrast, pattern formation using other nonlinearities require optical powers in the order of 1W.

References


