Chapter 4

Rotation Invariant Moments and Encrypts for
Robust Image Watermarking

Moments and transforms are the scalar quantities that characterise a function and capture its significant properties. More specifically, these are the descriptors that correspond to the projection of the function on a specific basis function, where the type or characteristic of the basis function gives the name to the moment or transform [82]. Normally, moments and transforms are treated separately in many image processing applications, but we cluster rotation invariant moments and transforms in the same group owing to the fact that mathematical formulation of the two is similar except that radial part of the basis functions in case of moments are polynomial functions whereas in case of transforms they are sinusoidal functions. Further, a moment or transform can be categorised depending on its domain of computation (discrete vs. continuous), resilience to particular deformation (invariant vs. non-invariant) and orthogonality (orthogonal vs. non-orthogonal). These categorizations give rise to many moments and transform families, such as discrete non-orthogonal, discrete orthogonal, continuous orthogonal rotation invariants, etc.

Among various invariant moment families, Zernike and pseudo-Zernike moments have been widely used for watermarking. Recently, polar harmonic transforms have also been proposed for watermarking. In this chapter, we introduce other invariant moments and transforms such as Fourier-Mellin moments, orthogonal Fourier-Mellin moments, radial harmonic Fourier moments and angular radial transform for robust image watermarking. We also perform the
detailed analysis of various properties possessed by invariant moments and transforms and their impact on robust watermarking.

4.1. Invariance in Moments and Transforms

Invariant moments and transforms describe the image function by set of measurable quantities called ‘invariants’ that are insensitive to any particular deformation. Mathematically, an invariant, I, is a functional mapping defined on the space of all admissible image functions which does not change its value under degradation operator, D. Thus, for any image function, f, the property of invariance holds if \( I(f) = I(D(f)) \). However, in practice, to accommodate the errors in computation, a weaker constraint is followed, that is, \( I(f) \) should not be significantly different from \( I(D(f)) \) or \( I(f) \approx I(D(f)) \).

The concept of invariance is derived from the theory of algebraic invariants developed by German mathematician Hilbert [111] in the nineteenth century. The modern formulation of geometric invariant theory owes to Gurevich [112]. Specific to an image function, there are two kinds of invariants, namely, geometric invariants and radiometric invariants. A geometric invariant is one which provides invariance to translation, scaling, rotation, affine transformation or any other geometric distortion while radiometric invariants exist with respect to linear contrast stretching, non-linear intensity transforms and convolution. Two popular forms of geometric invariants are moment invariants and transform invariants. Transform invariants are calculated from a certain transform of the image, for example, Fourier transform, Radon transform or wavelet transform while moment invariants are special functions of image moments.

Moment invariants were first introduced by Hu [58] for pattern recognition and image processing community based on the methods of algebraic invariants. Using nonlinear combinations of geometric moments, Hu derived a set of seven invariant moments which have the desirable properties of being invariant under image translation, scaling and rotation. Since then, many researchers have contributed towards the improvements, extensions and generalisations of moment
invariants and their applications in image processing. However, the moment invariants are often faced with the problem of computational difficulty at high orders. Recently, some transforms have been proposed which can be used to generate rotation invariant features. These transforms are much less complex in terms of basis function and have tractable higher order coefficients.

4.2. Mathematical Description

Rotation invariant moments and transforms (RIMTs) is a class of geometric invariants defined on square image functions which does not change its value under rotation. For a given image function, \( f(x, y) \), of size \( N \times N \) pixels in a two-dimensional Cartesian coordinate system, its rotation invariant moment or transform coefficient, \( F_{pq} \), of order \( p \) and repetition \( q \) is defined over a unit disk in the continuous polar domain as follows.

\[
F_{pq} = \lambda \int_{0}^{2\pi} \int_{0}^{\infty} f(r, \theta) V_{pq}^*(r, \theta) r \, dr \, d\theta
\]  

(4.1)

such that \( r = \sqrt{x^2 + y^2} \), \( \theta = \arctan(y/x) \), \( \lambda \) is the normalisation constant and \( V_{pq}^*(r, \theta) \) is the complex conjugate of the basis function, \( V_{pq}(r, \theta) \), which is separable along the radial and angular directions, i.e.

\[
V_{pq}(r, \theta) = R_{pq}(r) A_q(\theta)
\]  

(4.2)

Each moment or transform is uniquely defined by its radial basis function, \( R_{pq}(r) \), and in order to provide rotational invariance, the angular basis function, \( A_q(\theta) \), is given by:

\[
A_q(\theta) = e^{jq\theta} \quad j = \sqrt{-1}
\]  

(4.3)

The magnitude of RIMTs, \( F_{pq} \), is rotation invariant and with appropriate transformations, translation and scale invariance can be achieved, making them more suitable for geometrically invariant watermarking. The total number of
moment or transform coefficients that can be computed up to the given maximum order of moment or transform \( p_{\text{max}} \) and \( q_{\text{max}} \) depends upon constraints set on \( p \) and \( q \) defined for its basis functions.

### 4.3. Properties of Rotation Invariant Moments and Transforms

This section describes some of the common properties of rotation invariant moments (RIMTs) and transforms which make them more suitable for robust image watermarking applications.

#### 4.3.1. Rotation Invariance

The magnitude of a RIMTs, \( |F_{pq}| \), is rotation invariant. If the image is rotated by an angle of \( \phi \) in counter clockwise direction then by replacing the angle \( \theta \) by \((\theta + \phi)\) in Eq. (4.1) we get the RIMTs coefficient of the rotated image as

\[
F_{pq}^\phi = \lambda \int_0^{2\pi} \int_0^1 f(r, \theta + \phi) V^*_p(r, \theta + \phi) r \, dr \, d\theta
\]

\[
= \lambda \int_0^{2\pi} \int_0^1 f(r, \theta + \phi) R_{pq}(r, \theta + \phi) \cdot e^{-jq(\theta + \phi)} r \, dr \, d\theta
\]

\[
= e^{-jq\phi} \times \lambda \int_0^{2\pi} \int_0^1 f(r, \theta) R_{pq}(r, \theta) \cdot e^{-jq\theta} r \, dr \, d\theta
\]

\[
= e^{-jq\phi} \times F_{pq}
\]

Equation (4.6) follows because \( f(r, \theta + \phi) \) is the rotated function value of \( f(r, \theta) \) and \( e^{-jq\phi} \) remains constant in the integral range. Thus the RIMTs coefficients of the rotated and original image are related by Eq. (4.7) imply that

\[
|F_{pq}^\phi| = |e^{jq\phi}| \times |F_{pq}| = |F_{pq}|
\]

This property holds perfectly in case of analog images, whereas for the discrete versions of moments, the property is compromised limiting their applicability for data hiding [71]. Furthermore, the magnitudes of RIMTs

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**References:**

[71] By the author of the text.
coefficients are both horizontal and vertical flip invariant. If $F^h_{pq}$ and $F^v_{pq}$ and are the moments obtained from the horizontally and vertically flipped images of $f(x, y)$, then $F^h_{pq} = (-1)^q F_{pq}$ and $F^v_{pq} = F_{pq}$.

4.3.2. Translation Invariance

By definition, RIMTs are neither translation nor scale invariant. Translation invariance can be achieved by centering the image before computing RIMTs coefficients by using simple 2-dimensional geometric translation transformation on the image function, $f(x, y)$, as follows.

$$
\begin{align*}
x' &= \bar{x} - x ; \quad y' = \bar{y} - y \\
\end{align*}
$$

where

$$
\bar{x} = m_{01}/m_{00} , \quad \bar{y} = m_{01}/m_{00}
$$

and $m_{00}$, $m_{01}$ and $m_{10}$ are the geometric moments of the image given by:

$$
m_{pq} = \sum_{x=0}^{N-1} \sum_{y=0}^{N-1} x^p y^q f(x, y)
$$

The RIMTs coefficients of the translated image, $f'(x', y')$ obtained after transformation (4.9) will be translation invariant.

4.3.3. Scale Invariance

Similarly, scale invariance can be achieved by applying normalisation transformation to get a new image, $f''(x'', y'')$, before RIMTs coefficient computation using following equations:

$$
\begin{align*}
x'' &= (x + 1 - \bar{x})/R ; \quad y'' = (y + 1 - \bar{y})/R , \quad x, y = 0, 1, ..., N-1
\end{align*}
$$

where $(\bar{x}, \bar{y})$ is the centroid of the image computed using Eq. (4.10). The location $(\bar{x}, \bar{y})$ is taken as the centre of the disk and the radius of the disk, $R$, is given by:
\[ R = \sqrt{x_{\text{max}}^2 + y_{\text{max}}^2} \]  

(4.13)

such that

\[ x_{\text{max}} = \max(\bar{x}, N-1-\bar{x}), \quad y_{\text{max}} = \max(\bar{y}, N-1-\bar{y}) \]  

(4.14)

Since the mapping given in Eq. (4.12) suffers from problem associated with the relocation of the centroid of image during scaling, we take the centre of the image as the centre of the disk \((\bar{x} = \bar{y} = N/2)\) and \(R = N/2\) for the inscribed disk or \(R = N/\sqrt{2}\) for outer disk covering the whole image. It is worth mentioning here that this normalisation scheme works only for small scaling factors. When the images are scaled to larger sizes, errors introduced during the interpolation process may restrict the applicability of this property.

4.3.4. Orthogonality

The basis polynomial of a moments or transform, \(V_{pq}(r, \theta)\), given in Eq. (4.2) is said to be orthogonal over the interior of the unit disk if it satisfies the following condition.

\[
\int_0^{\pi/2} \int_0^{\pi/2} V_{pq}^*(r, \theta) V_{p'q'}(r, \theta) r \, dr \, d\theta = \frac{1}{\lambda} \delta_{pp'} \delta_{qq'}
\]  

(4.15)

where \(\delta\) is known as the Kronecker delta defined as:

\[
\delta_{ij} = \begin{cases} 
1 & \text{for } i = j \\
0 & \text{otherwise}
\end{cases}
\]  

(4.16)

The orthogonal property of the basis function eliminates the information redundancy, making each moment or transform coefficient uniquely usable for embedding a distinct watermark symbol.

4.3.5. Robustness against Noise

Due to orthogonal property and the proven superiority of RIMTs over other moments and transforms in terms of their feature extraction capabilities, they
provide robustness against noise and independence from uniform change in intensity, illumination change, blurring and contrast.

4.4. Rotation Invariant Moments and Transforms for Watermarking

This section describes some popularly used rotation invariant moments and transforms for a variety of image processing applications that require invariant image representation. As mentioned earlier, the radial function in case of moments are polynomial functions containing exponential and factorial terms whereas in case of transforms they are sinusoidal functions making the former more computation intensive. Table 4.1 summarises the major characteristics of these moments and transforms such as constraints on the orders and repetition for their radial and angular basis functions, normalisation factor, number of moments or transform coefficients that can be computed, invariance to various geometric operations and orthogonality.

4.4.1. Zernike Moments (ZMs)

Zernike moments (ZMs) were first introduced by Teague [104] in 1980 on the basis of a class of Zernike polynomials. The ZMs, \( Z_{pq} \), of order \( p \geq 0 \) and repetition \( q \geq 0 \) for a continuous image function, \( f(r, \theta) \), in the polar domain are defined over a unit disk as follows:

\[
Z_{pq} = \frac{p+1}{\pi} \int_{0}^{2\pi} \int_{0}^{1} f(r, \theta) V_{pq}^{*}(r, \theta) r \, dr \, d\theta
\]

(4.17)

where \( V_{pq}^{*}(r, \theta) \) is the complex conjugate of the complex Zernike polynomial \( V_{pq}(r, \theta) \) given by:

\[
V_{pq}(r, \theta) = R_{pq}^{ZM}(r) e^{i\eta \theta}
\]

(4.18)
Table 4.1. Characteristics of rotation invariant moments and transforms.

<table>
<thead>
<tr>
<th>Moment/Transform</th>
<th>Constraint on (p) and (q)</th>
<th>Normalisation factor ((\lambda))</th>
<th>Maximum no. of moments/transforms up to (p_{\text{max}}) and (q_{\text{max}})</th>
<th>Invariance to</th>
<th>Orthogonality</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(p = 0, 1, 2, \ldots)</td>
<td>(p + 1) (\pi)</td>
<td>((1 + p_{\text{max}})(2 + p_{\text{max}}))/2</td>
<td>D D D H</td>
<td>✓</td>
</tr>
<tr>
<td>ZMs</td>
<td>(</td>
<td>q</td>
<td>\leq p)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(p -</td>
<td>q</td>
<td>= \text{even})</td>
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<td></td>
</tr>
<tr>
<td>PZMs</td>
<td>(p = 0, 1, 2, \ldots)</td>
<td>(p + 1) (\pi)</td>
<td>((1 + p_{\text{max}})^2)</td>
<td>D D I H</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>q</td>
<td>= 0, 1, 2, \ldots)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FMMs</td>
<td>(p = 0, 1, 2, \ldots)</td>
<td>(p + 1) (\pi)</td>
<td>((1 + p_{\text{max}})(1 + 2q_{\text{max}}))</td>
<td>D D I H</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>q</td>
<td>= 0, 1, 2, \ldots)</td>
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</tr>
<tr>
<td>OFMMs</td>
<td>(p = 0, 1, 2, \ldots)</td>
<td>(p + 1) (\pi)</td>
<td>((1 + p_{\text{max}})(1 + 2q_{\text{max}}))</td>
<td>D D I H</td>
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<td></td>
<td>(</td>
<td>q</td>
<td>= 0, 1, 2, \ldots)</td>
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<tr>
<td>RHFMs</td>
<td>(p = 0, 1, 2, \ldots)</td>
<td>(1/2\pi)</td>
<td>((1 + p_{\text{max}})(1 + 2q_{\text{max}}))</td>
<td>D D D H</td>
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<td></td>
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<tr>
<td>ARTs</td>
<td>(p = 0, 1, 2, \ldots)</td>
<td>(1/2\pi)</td>
<td>((1 + 2p_{\text{max}})(1 + 2q_{\text{max}}))</td>
<td>D D I H</td>
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<td></td>
<td>(</td>
<td>q</td>
<td>= 0, 1, 2, \ldots)</td>
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<tr>
<td>PCETs</td>
<td>(</td>
<td>p</td>
<td>= 0, 1, 2, \ldots)</td>
<td>(1/\pi)</td>
<td>((1 + p_{\text{max}})(1 + 2q_{\text{max}}))</td>
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<td>(</td>
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<td>PCTs</td>
<td>(</td>
<td>p</td>
<td>= 0, 1, 2, \ldots)</td>
<td>(2/\pi)</td>
<td>((1 + p_{\text{max}})(1 + 2q_{\text{max}}))</td>
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<td></td>
<td>(</td>
<td>q</td>
<td>= 0, 1, 2, \ldots)</td>
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</tr>
<tr>
<td>PSTs</td>
<td>(</td>
<td>p</td>
<td>= 1, 2, \ldots)</td>
<td>(1/\pi)</td>
<td>((1 + p_{\text{max}})(1 + 2q_{\text{max}}))</td>
</tr>
<tr>
<td></td>
<td>(</td>
<td>q</td>
<td>= 0, 1, 2, \ldots)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

R: Rotation;  
S: Scaling;  
T: Translation;  
N: Noise;  
D: Direct;  
I: Indirect;  
H: High;  
L: Low;
The Zernike radial polynomial, \( R_{pq}^{\text{ZM}}(r) \), is defined as:

\[
R_{pq}^{\text{ZM}}(r) = \sum_{s=0}^{(p-|q|)} \frac{(-1)^s(p-s)!|q|^{p-2s}}{s!(p+|q|)!} \frac{(p+|q|)!}{2^s} \frac{(p-|q|)!}{2^{p-s}} \frac{1}{2^s} \quad p-|q| = \text{even} 
\]  

The Zernike polynomial, \( V_{pq}(r, \theta) \), given in Eq. (4.18) is orthogonal over the unit disk as given follows:

\[
\pi \int_{0}^{2\pi} \int_{0}^{1} V_{pq}(r, \theta) V_{p'q'}(r, \theta) r \, dr \, d\theta = \frac{\pi}{p+1} \delta_{pp'} \delta_{qq'} 
\]  

ZMs invariants possess excellent image reconstruction in terms of finite number of moments. ZMs are defined over the continuous unit disk and its basis functions are orthogonal that make each moment coefficient to the image unique and no redundancy occur between moment features. In addition, features like minimum information redundancy, magnitude invariance to rotation and flipping, better immunity to noise and separable radial and angular components make ZMs superior to other invariant moments. By virtue of all these features, ZMs are the most popular among the invariant moment and transform families and have been found useful in almost all application areas that require invariant feature representation such as image watermarking [69-72, 75], face and character recognition [113], edge detection [114], image segmentation [115], surface modeling [116], palmprint verification [117]. Some low order ZMs have also been proposed for robust audio and video watermarking [118-119].

**4.4.2. Pseudo-Zernike Moments (PZMs)**

Another class of orthogonal rotation invariant moments based on Zernike polynomials are pseudo-Zernike moments (PZMs) given by Bhatia and Wolf [120]. These moments differ from ZMs only in terms of their real-valued radial polynomial function which is defined as follows.
These moments have behavior similar to that of ZMs and provide approximately double number of coefficients for the given order of moments. Therefore, for a given maximum order, PZMs have more low order moments. This is an important aspect for image description as it is known that the low order moments represent the global characteristics of an image more precisely while the high order moments represent the fine details [121]. However, high computational complexity and early numerical instability limit their use for watermarking applications requiring high capacity information embedding.

4.4.3. Fourier-Mellin Moments (FMMs)

Also known as rotational moments (RMs), Fourier-Mellin moments are given by Sheng and Duvernoy [122]. In the most general form, the FMMs, \( M_{pq} \), of order \( p \geq 0 \) and repetition \( |q| \geq 0 \) for a two-dimensional image function, \( f(r, \theta) \), over a unit disk in the polar domain is given by:

\[
M_{pq} = \iint_{0}^{2\pi} f(r, \theta) V_{pq} (r, \theta) r \, dr \, d\theta
\]  

(4.22)

The radial basis function of FMMs, \( R_{pq,lm}^{FMM} (r) \), is expressed in terms of monomials with exponential power to the radius as follows.

\[
R_{pq,lm}^{FMM} (r) = r^p \cos^l \theta \sin^m \theta , \quad \text{for } p \geq 0 \text{ and } |q| \geq 0
\]  

(4.23)

While the parameter \( l,m = 0,1,2,\ldots,\infty \) can take any value, most of the image processing applications use \( l = m = 0 \), for which the FMMs are defined as follows:

\[
M_{pq} = \iint_{0}^{2\pi} f(r, \theta) \, r^p \, e^{-jq\theta} \, r \, dr \, d\theta
\]  

(4.24)
Despite the ease of computation and invariance to rotation which can be extended to image translation, FMMs are not widely used as these moments are not generally independent [121] and do not satisfy the orthogonality condition.

### 4.4.4. Orthogonal Fourier-Mellin Moments (OFMMs)

Orthogonal Fourier-Mellin moments (OFMMs) were improved by Sheng and Shen [123] and belong to a class of circularly orthogonal invariant moments. The OFMMs, $O_{pq}$, of order $p \geq 0$ and repetition $|q| \geq 0$ for a two-dimensional image function, $f(r, \theta)$, over a unit disk in the polar domain are given by:

$$O_{pq} = \frac{p + 1}{\pi} \int_{0}^{2\pi} \int_{0}^{1} f(r, \theta) V_{pq}^* (r, \theta) r \, dr \, d\theta$$  \hspace{1cm} (4.25)

The radial polynomial, $R_p^{OFMM}(r)$, of OFMMs is expressed as:

$$R_p^{OFMM}(r) = \sum_{s=0}^{p} (-1)^{p+s} \frac{(p+s+1)! \, r^s}{s! \, (p-s)! \, (s+1)!}, \quad p \geq 0 \text{ and } |q| \geq 0$$  \hspace{1cm} (4.26)

These moments differ significantly from FFMs in that the radial polynomials of these polynomials satisfy the orthogonality condition given in the Eq. (4.15). OFMMs are found more useful for small images as they provide more number of moments for a given order of moments [123]. Due to these features, the OFMMs are used in a variety of pattern recognition applications such as optical character recognition [124], fingerprint verification [125], object recognition [126] and face recognition [127].

### 4.4.5. Radial Harmonic Fourier Moments

Recently, Ren et al. [128] introduced radial harmonic Fourier moments (RHFMs) belonging to the class of orthogonal rotational invariant moments (ORIMs). The RHFMs, $H_{pq}$, of order $p \geq 0$ and repetition $|q| \geq 0$ for a continuous image function, $f(r, \theta)$, in the polar domain are defined over a unit disk as follows:
\[ H_{pq} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r f(r, \theta) V_{pq}^* (r, \theta) r \, dr \, d\theta \quad (4.27) \]

The radial basis function of RHFMs is a sinusoidal function in \( r \) defined as:

\[
R^\text{RHFMs}_p (r) = \begin{cases} 
\frac{1}{\sqrt{r}} & p = 0 \\
\sqrt{\frac{2}{r}} \cos(\pi pr), & p = \text{even} \\
\sqrt{\frac{2}{r}} \sin(\pi(p + 1)r), & p = \text{odd}
\end{cases} \quad (4.28)
\]

The RHFMs basis function is orthogonal in the interior of the unit circle and satisfies the condition given in Eq. (4.15) for \( \lambda = 1/2\pi \). RHFMs support inherent features like magnitude invariance, minimum information redundancy and resilience to noise that are essential for developing invariant watermarking schemes. Their sinusoidal radial basis function is less computation intensive and exhibits better numerical stability compared to other orthogonal rotation invariant moments [129]. The major problem associated with RHFMs is that they become singular at \( r = 0 \). Thus special care is taken when \( r \) becomes zero. In discrete images, if the size of image is even, which is the case in our analysis, this condition does not arise. Due to these attractive characteristics, RHFMs are found useful for a variety of applications like in tumor cell recognition [130, 131], character recognition [132] and are introduced for robust image watermarking in this thesis.

4.4.6. Angular Radial Transforms (ARTs)

ARTs is a rotation-invariant transform adopted in MPEG-7 as a region-based shape descriptors [133]. The ARTs coefficient, \( A_{pq} \), of order \( p \geq 0 \) and repetition \( |q| \geq 0 \) for a continuous image function, \( f(r, \theta) \), in the polar domain are defined over a unit disk as follows:

\[ A_{pq} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r f(r, \theta) V_{pq}^* (r, \theta) r \, dr \, d\theta \quad (4.29) \]
The radial basis function of ARTs, \( R_p^{ART}(r) \), is a harmonic function in \( r \) defined as:

\[
R_p^{ART}(r) = \begin{cases} 
1, & p = 0 \\
2 \cos(\pi r), & \text{otherwise}
\end{cases}
\]  

(4.30)

Similar to rotation invariant moments, ARTs offer magnitude invariance to rotation and excellent resilience to noise while exhibiting low computation complexity and better numerical stability. ARTs have been found useful for applications like video security system [134], shape retrieval [135] and logo recognition systems [136] and are introduced image watermarking in this thesis.

### 4.4.7. Polar Harmonic Transforms

Recently, Yap et al. [137] introduced a set of two dimensional rotation invariant transforms based on orthogonal projection basis, collectively known as polar harmonic transforms (PHTs) to generate rotation invariant features. PHTs consist of three transforms, namely, polar complex Exponential transform (PCETs), polar cosine Transform (PCTs) and polar sine transform (PSTs). The PCETs coefficient, \( F_{pq}^{PCET} \), of order \( |p| \geq 0 \) and repetition \( |q| \geq 0 \) is given by:

\[
F_{pq}^{PCET} = \frac{1}{\pi} \int_{0}^{2\pi} \int_{0}^{1} f(r, \theta) [V_{pq}^{PCET}(r, \theta)]^* r \, dr \, d\theta
\]  

(4.31)

where \( V_{pq}^{PCET}(r, \theta) \) is the PCETs basis function having radial basis function, \( R_p^{PCET}(r) \), as a complex exponential in the radial direction:

\[
R_p^{PCET}(r) = e^{i2\pi pr^2}
\]  

(4.32)

The basis function of PCETs is also orthogonal as given by:

\[
\int_{0}^{2\pi} \int_{0}^{1} V_{pq}^{PCET}(r, \theta) [V_{p'q'}^{PCET}(r, \theta)]^* f(r, \theta) r \, dr \, d\theta = \pi \delta_{pp'} \delta_{qq'}
\]  

(4.33)
Similar to the form of PCETs, the other two sets of harmonic transforms, PCTs and PSTs, are defined as:

\[
F_{pq}^{PCT} = \lambda \int_0^{2\pi} \int_0^1 f(r, \theta) \left[ V_{pq}^{PCT} (r, \theta) \right] r \, dr \, d\theta, \quad p \geq 0, |q| \geq 0
\]  
(4.34)

\[
F_{pq}^{PST} = \lambda \int_0^{2\pi} \int_0^1 f(r, \theta) \left[ V_{pq}^{PST} (r, \theta) \right] r \, dr \, d\theta, \quad p \geq 1, |q| \geq 0
\]  
(4.35)

such that

\[
V_{pq}^{PCT} (r, \theta) = R^p_{p} \left( r e^{jq\theta} \right) = \cos(\pi \, p \, r^2) e^{jq\theta}
\]  
(4.36)

\[
V_{pq}^{PST} (r, \theta) = R^p_{p} \left( r e^{jq\theta} \right) = \sin(\pi \, p \, r^2) e^{jq\theta}
\]  
(4.37)

\[
\lambda = \begin{cases} 
1/\pi & p = 0 \\
2/\pi & p \neq 0
\end{cases}
\]  
(4.38)

All the three transforms belonging to PHTs are characterised by invariance to geometric transformations, excellent resilience to noise, minimum information redundancy and low computation complexity. These transforms are found useful for invariant image watermarking [55], fingerprint classification [138] and invariant image description [139].

### 4.5. Generalised Computational Framework

In digital image processing, the image function, \( f(x, y) \), is discrete and often defined in a discrete domain \([0, N-1] \times [0, N-1]\) with the pixel values \( f(i, k) \), \( i, k = 0, 1, \ldots, N-1 \), whereas moment and transform invariants are defined over the unit disk in continuous polar domain. Though a 2-dimensional image function, \( f(i, k) \), can easily be converted into the polar domain using a suitable interpolation process [140], it has been observed that this mapping introduces new type of error known as interpolation error which adversely affects the accuracy in computation [141]. In order to avoid interpolation error, the traditional method computes \( F_{pq} \) directly in the Cartesian domain using zeroth-order approximation (ZOA) of the double integration involved in Eq. (4.1) as follows:
\[ F_{pq} = \lambda \sum_{i=0}^{N-1} \sum_{k=0}^{N-1} f(i, k) R_{pq}(r_{ik}) e^{-jq_{ik}} \Delta x \Delta y \] (4.39)

such that

\[ \Delta x = \Delta y = \frac{2}{D}, \quad r_{ik}^2 = x_i^2 + y_k^2 \leq 1 \text{ and } \theta_{ik} = \arctan \left( \frac{y_k}{x_i} \right) \] (4.40)

and the coordinates \( (x_i, y_k) \) are given by

\[ x_i = \frac{2i - N + 1}{D}, \quad y_k = \frac{2k - N + 1}{D}, \quad i, k = 0, 1, \ldots, N - 1 \] (4.41)

where

\[ D = \begin{cases} N & \text{for inscribed circular disk contained in the square image} \\ N\sqrt{2} & \text{for outer circular disk containing the whole square image} \end{cases} \] (4.42)

The coordinate \( (x_i, y_k) \) is the center of the square pixel grid \( (i, k) \). The mapping converts the square domain into an approximated unit disk. As an example, an \( 8 \times 8 \) pixel grid shown in Fig. 4.1(a) is approximated by an inscribed unit disk: \( x_i^2 + y_k^2 \leq 1 \) (with \( D = 8 \)), shown in Fig. 4.1(b) and with outer circular disk containing the entire image (with \( D = 8\sqrt{2} \)), shown in Fig. 4.1(c).

Fig. 4.1. Approximation of square image onto unit disk (a) An \( 8 \times 8 \) image grid, (b) inscribed circle approximated by square grids (c) outer circle containing whole image.
In order to embed the watermark completely inside the image, all moment and transform-based watermarking applications use inscribed disk for computing the RIMTs coefficients. Therefore, in this thesis, we have taken $D = N$ and
\[ \Delta x = \Delta y = \frac{2}{N}. \]

Further, if all coefficients $F_{pq}$ of an image function $f(x, y)$ up to maximum order $p_{\text{max}}$ and $q_{\text{max}}$ are known, then image function can be reconstructed using following inverse transformation equation.

\[ \hat{f}(i, k) = \sum_{p=p_{\text{min}}}^{p_{\text{max}}} \sum_{q=q_{\text{min}}}^{q_{\text{max}}} F_{pq} V_{pq}(x_i, y_k) \]  

(4.43)

where $p_{\text{min}}, p_{\text{max}}, q_{\text{min}}$ and $q_{\text{max}}$ are the respective minimum or maximum values of a moment or transform according to its definition. The reconstructed image function, \( \hat{f}(i, k) \), bears more resemblance to the original image function, \( f(i, k) \), for higher orders. The quality of reconstructed image can be measured using mean square reconstruction error (MSRE), denoted by $\varepsilon$, defined as:

\[ \varepsilon = \frac{\sum_{i=0}^{N-1} \sum_{k=0}^{N-1} [f(i, k) - \hat{f}(i, k)]^2}{\sum_{i=0}^{N-1} \sum_{k=0}^{N-1} f^2(i, k)} \]  

(4.44)

4.6. Existing Invariant Watermarking Scheme

Among various invariant moment and transform-based watermarking schemes, we observe that the method given by Xin et al. [71] using ZMs/PZMs is one of the best methods. A variation of this scheme is presented by Li et al. [55] using PHTs. We conducted large number of experiments and found that the original approach given by Xin et al. [71] is slightly better than [55] in terms of watermark robustness. This scheme [71] is also extensively used by other researchers for designing robust watermarking schemes in the invariant domain as well as feature-based watermarking schemes [55, 66, 68, 72, 74-77]. Therefore, in order
to have a common platform for the comparative performance analysis of
watermark schemes in terms of critical parameters like robustness, visual
imperceptibility and embedding capacity, the method of [71] is preferred. Using
their scheme, each bit of the watermark bit sequence, \( B = \{b_i, i = 1, 2, \ldots, L\} \)
consisting of \( L \) number of information bits such that \( b_i \in \{0, 1\} \), is embedded into
host image by modulating some selected moment or transform coefficient through
quantization index modulation [144]. This section summarises some of the
important steps required for the implementation of watermarking algorithm while
the details can be found in [71].

4.6.1. Selection of Moments and Transform Coefficients

While analysing the invariance properties of ZMs/PZMs, Xin et al. [71] observed
that all moments with repetition \( q = 4m, m \) integer, cannot be computed
accurately due to geometric approximation errors which arise as the area covered
by digitised pixels does not equal the exact area of the unit disk. Like ZMs/PZMs,
PHTs are also defined over the unit disk in the continuous domain, hence
analogous results are observed for PHTs by Li et al. [55]. These results can be
easily verified for other moments and transforms under investigation and are also
presented later in this chapter (Section 4.7.1.1). Furthermore, due to the conjugate
symmetry of moments, only \( F_{pq} \) with \( p \geq 0 \) and \( q \geq 0 \) have independent
magnitudes. Therefore, we can define \( S = \{F_{pq}, 0 \leq p \leq p_{\text{max}}, 0 \leq q \leq q_{\text{max}}, q \neq 4m\} \)
as the set of all \( F_{pq} \) that can be used for embedding and depending on the length
of watermark bit sequence, \( L \leq |S| \), a number of moments and transform
coefficients are selected from set \( S \) to form the feature vector
\( F = \{F_{p_1, q_1}, F_{p_2, q_2}, \ldots, F_{p_L, q_L}\} \).

4.6.2. Modulation of Moment and Transform Coefficients

Each bit \( b_i \) from \( B \) is embedded into the corresponding element \( F_{p_i, q_i} \) of \( F \) to
obtain modified feature vector \( \tilde{F} = \{\tilde{F}_{p_1, q_1}, \tilde{F}_{p_2, q_2}, \ldots, \tilde{F}_{p_L, q_L}\} \) where \( \tilde{F}_{p_i, q_i} \) is the
quantized version of \( F_{p_i, q_i} \) computed as follows.
\[
\left\lfloor \tilde{F}_{p,q_i} \right\rfloor = \left[ \frac{F_{p,q_i} - d_i(b_i)}{\Delta} \right] \Delta + d_i(b_i), \quad i = 1,2,\ldots,L
\]  

(4.45)

where \( \left\lfloor \cdot \right\rfloor \) denotes the rounding operation, \( \Delta \) is the quantization step size and \( d_i(.) \) is the dither value for the \( i \)th quantizer with constraint \( d_i(1) = d_i(0) + \Delta/2 \). The elements of dither vector \( \langle d_i(0), d_2(0), \ldots, d_L(0) \rangle \) are pseudo randomly generated with uniform distribution over the interval \([0, \Delta/2]\).

### 4.6.3. Formation of Watermarked Image

After modifying the selected moments and transform coefficients, the spatial watermark signal, \( w(x, y) \), is reconstructed using Eq. (4.43) as follows.

\[
w(x, y) = \sum_{i=1}^{L} \left[ \varepsilon_{p,q_i} V_{p,q_i} (x, y) + \varepsilon_{p,-q_i} V_{p,-q_i} (x, y) \right]
\]  

(4.46)

where \( \varepsilon_{p,q_i} = \tilde{F}_{p,q_i} - F_{p,q_i} \) and \( \varepsilon_{p,-q_i} = \tilde{F}_{p,-q_i} - F_{p,-q_i} \), \( i = 1,2,\ldots,L \) and the final watermarked image \( f_{\text{wat}}(x, y) \) is formed as \( f_{\text{wat}}(x, y) = f(x, y) + w(x, y) \).

### 4.6.4. Watermark Extraction

The watermark extraction process tries to get an estimate of embedded sequence \( \hat{b} = (\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_L) \) from the received image, \( f_{\text{wat}}(x, y) \), which may be a distorted version of \( f_{\text{wat}}(x, y) \) with possibly low error rate. The process of extraction is similar to the embedding process. The feature vector \( F' = \{F'_{p,q_i}, F'_{p',q_2}, \ldots, F'_{p',q_L}\} \) is recomputed from possibly attacked watermarked image. Each \( F'_{p,q_i} \) is requantized using Eq. (4.45) with both dither functions \( d_i(0) \) and \( d_i(1) \) to compute \( \left| F'_{p,q_i,0} \right| \) and \( \left| F'_{p,q_i,1} \right| \), respectively. A minimum distance decoder is then used to estimate the bit embedded in \( F'_{p,q_i} \) as follows.

\[
\hat{b}_i = \arg \min_{j \in \{0,1\}} \left[ \left| F'_{p,q_i,0,j} \right| - \left| F'_{p,q_i,1,j} \right| \right], \quad i = 1,2,\ldots,L
\]  

(4.47)
4.7. Experimental Analysis

We perform numerous experiments to comparatively analyse rotation invariant moments and transforms for various properties possessed by them and their effectiveness for robust watermarking. The empirical results in this section are organised into two parts. The first part presents an analysis of moments and transforms on the basis of various properties possessed by them whereas the comparative performance evaluation of invariant image watermarking schemes is given in the second part. All algorithms are implemented in Visual C++ 6.0 under Windows 7 environment on Intel 2.50 GHz processor with 2 GB RAM. We perform entire experimentation with twelve standard 256-level gray scale images of size $256 \times 256$ pixels normally used in image processing applications [145]. These images are given in Appendix B. While plotting the graphs in this section we have used dotted lines for moments and solid lines for transforms while marker styles are used to distinguish various moments and transforms.

4.7.1. Analysis of Properties of RIMTs

In this section, we elaborate on the significant properties such as computational accuracy, quality of reconstructed images, numerical instability, rotation and scale invariance and computational complexity of various RIMTs.

4.7.1.1. Computational Accuracy of RIMTs

The simplified form of double integration for the computation of $F_{pq}$ in Eq. (4.39) and the constraint $x_i^2 + y_i^2 \leq 1$ in Eq. (4.40) introduce numerical integration error and geometric error, respectively. These errors were first analysed extensively by Liao and Pawlak [142] and Pawlak [143] for ZMs/PZMs and later by Xin et al. [71]. Li et al. [55] also pointed out the presence of these errors for PHTs which can be generalised analogously for all RIMTs defined in the continuous unit disk. The inaccuracy in the computation of RIMTs coefficients of order $p$ and repetition $q = 4m, m$ integer, can be demonstrated through a constant image analysis. The theoretical values of various rotation invariant moments and
transform coefficients for constant image function \( f(r,\theta) = k \), can be derived directly through analytical integration and must evaluate to

\[
F_{pq} = \begin{cases} 
  k, & p = q = 0 \\
  0, & \text{otherwise}
\end{cases}
\]  
\text{for ZMs/PZMs, OFMMs, PCETs, PCTs}

\[
F_{pq} = \begin{cases} 
  4k/p\pi, & p = 1,3,5,\ldots, q = 0 \\
  0, & \text{otherwise}
\end{cases}
\]  
\text{for PSTs}

\[
F_{pq} = \begin{cases} 
  2k\pi/(p+2), & q = 0 \\
  0, & \text{otherwise}
\end{cases}
\]  
\text{for FMMs}

\[
F_{pq} = \begin{cases} 
  k/2, & p = q = 0 \\
  2k(-1)^{p-1}/(p\pi)^2, & p \neq 0 \text{ and } q = 0 \\
  0, & \text{otherwise}
\end{cases}
\]  
\text{for ARTs}

and for RHFM

\[
F_{pq} = \begin{cases} 
  2k/3, & p = q = 0 \\
  k(\sqrt{2}p\sin(p\pi) - S(\sqrt{2}p))/p\pi\sqrt{p}, & p = \text{even and } q = 0 \\
  k(\sqrt{2}p\cos(p\pi) + C(\sqrt{2}(-p-1))/(\sqrt{(-p-1)}))/\pi(p+1), & p = \text{odd and } q = 0 \\
  0, & \text{otherwise}
\end{cases}
\]

where, \( C(\cdot) \) and \( S(\cdot) \) are Fresnel \( C \) and \( S \) integrals, respectively.

The theoretical values of various RIMTs for constant image function \( f(r,\theta) = 1.0 \) with \( 256 \times 256 \) pixels grid up to order \( p_{\text{max}} \leq 10 \) and selected repetitions \( q = 4m \) for \( m = 0,1,2 \) are given in Table 4.2 while Table 4.3 displays the corresponding values computed using zeroth-order approximation rounded up to four decimal positions. The values computed using traditional zeroth-order approximation method differs significantly from the accurate values due to the presence of geometric approximation and numerical integration errors. As a result of these errors, all \( F_{pq} \) for which \( q = 4m \) becomes unusable for watermark embedding [55, 71], resulting in lower embedding capacity.
Table 4.2. Theoretical values of various RIMTs with order $p_{\text{max}} \leq 10$ and repetition $q = 0, 4, 8$ for constant image function $f(r, \theta) = 1.0$ of size $256 \times 256$ pixels.

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Table 4.3. Computed values of RIMTs using ZOA method with order \( p_{\text{max}} \leq 10 \) and repetition \( q = 0, 4, 8 \) for constant image function \( f(r, \theta) = 1.0 \) of size 256×256 pixels.

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4.7.1.2. Image Reconstruction Quality

The quality of reconstructed image can be analysed qualitatively by visual inspection or quantitatively by computing the mean square reconstruction error (MSRE), \( \varepsilon \), between the original and the reconstructed images. Table 4.4 displays the reconstructed image of Lena using various moments and transforms at varying order of moments and transforms. Total number of moments or transforms coefficients used for reconstruction and the values of \( \varepsilon \) are given below the images. As observed from the visual analysis of the reconstructed images, RHFMs provide the best quality of reconstructed images at all orders of moments followed by ZMs and PZMs. It is also observed that orthogonal moments provide better quality of reconstructed images than transforms up to order of moments \( p_{\text{max}} \leq 20 \). In particular, PZMs, FMMs, OFMMs and PCETs failed to reconstruct the image at \( p_{\text{max}} > 20 \). However, the given order of moments and transforms generates different number of moments or transform coefficients depending upon the constraints set on the radial basis function. Therefore, we reanalyse the MSRE (\( \varepsilon \)) as a function of number of moments or transforms coefficients used for image reconstruction \( (L) \) and observed that RHFMs provide the best reconstruction quality followed by PCTs, PZMs, ZMs, PSTs, ARTs, PCETs, OFMMs and FMMs in that order. It is also observed that, in general, image quality deteriorates at the vicinity of \( r = 0 \) indicated by a bright or dark spot at the center of image.

4.7.1.3. Numerical Stability

Another major issue associated with the computation of RIMTs is that they become numerically instable at high orders. This is due to the underflow error (also referred to as overflow) and finite precision error [147]. As observed by [55, 71, 147], moments and transforms higher than certain order cannot be computed accurately even with double precision. We observe that different moments and transforms exhibit different levels of numerical stability depending on their radial basis function. A simple and efficient method of measuring numerical stability is to analyse the behavior of average MSRE (\( \varepsilon \)) with respect to the order of moments.
Table 4.4. Reconstructed images of Lena of size $256 \times 256$ pixels

<table>
<thead>
<tr>
<th>RIMTs</th>
<th>Order of moments and transforms $P_{max}$</th>
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<tr>
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<td>ZMs</td>
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<tr>
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<td>$L=66, \epsilon=0.06191$</td>
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<tr>
<td>PZMs</td>
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<tr>
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</tr>
<tr>
<td></td>
<td>$L=231, \epsilon=0.05473$</td>
</tr>
<tr>
<td>OFMMs</td>
<td><img src="image16.png" alt="Image" /> <img src="image17.png" alt="Image" /> <img src="image18.png" alt="Image" /> <img src="image19.png" alt="Image" /> <img src="image20.png" alt="Image" /></td>
</tr>
<tr>
<td></td>
<td>$L=231, \epsilon=0.03423$</td>
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<tr>
<td>RHFMs</td>
<td><img src="image21.png" alt="Image" /> <img src="image22.png" alt="Image" /> <img src="image23.png" alt="Image" /> <img src="image24.png" alt="Image" /> <img src="image25.png" alt="Image" /></td>
</tr>
<tr>
<td></td>
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<tr>
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<tr>
<td>PCETs</td>
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<tr>
<td></td>
<td>$L=441, \epsilon=0.14594$</td>
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<tr>
<td>PCTs</td>
<td><img src="image36.png" alt="Image" /> <img src="image37.png" alt="Image" /> <img src="image38.png" alt="Image" /> <img src="image39.png" alt="Image" /> <img src="image40.png" alt="Image" /></td>
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<tr>
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<td>$L=231, \epsilon=0.06648$</td>
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<tr>
<td>PSTs</td>
<td><img src="image41.png" alt="Image" /> <img src="image42.png" alt="Image" /> <img src="image43.png" alt="Image" /> <img src="image44.png" alt="Image" /> <img src="image45.png" alt="Image" /></td>
</tr>
<tr>
<td></td>
<td>$L=210, \epsilon=0.08407$</td>
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</table>
We reconstruct all twelve sample images given in Appendix B at varying order of moments and transforms $1 \leq p_{\text{max}} = q_{\text{max}} \leq 50$ using traditional ZOA computational framework. The average MSRE, $\varepsilon$, as a function of $p_{\text{max}}$ for moments and transforms are plotted in Fig. 4.2(a) and (b), respectively. Although ARTs and FMMs are non-orthogonal, they are included here for the sake of comparison. Theoretically, an increase in the order of moments or transforms should improve the quality of reconstructed image and provide decrease in MSRE. As observed from the figure, this holds only up to certain value of $p_{\text{max}}$, beyond which MSRE starts rising quickly due to numerical instability in the computation of moments or transform coefficient at higher orders.

We also observe that the average value of $\varepsilon$ is the smallest for RHFMs at low order of moments as compared to average MSRE values of other moments and transforms. This fact is also reported in [137]. However, RHFMs become numerically more unstable as moment order becomes higher because of rapid changes in their basis functions at higher order of moments, especially in the vicinity of $r = 0$. In our experiments, we observe that RHFMs are numerically stable up to orders $p_{\text{max}} \leq 30$, ZMs up to $p_{\text{max}} \leq 44$, PZMs up to $p_{\text{max}} \leq 23$, OFMMs up to $p_{\text{max}} \leq 22$ and FMMs up to $p_{\text{max}} \leq 10$. Furthermore, among transforms, PCETs exhibit numerical instability at $p_{\text{max}} = 32$, followed by PSTs at $p_{\text{max}} = 30$, PCTs at $p_{\text{max}} = 27$ and ARTs at $p_{\text{max}} = 26$. In general, we find that all transforms under investigation are numerically stable up to $p_{\text{max}} \leq 25$.

4.7.1.4. Magnitude Invariance Properties

By definition, the magnitudes of the rotation invariant moments and transform coefficients of the original and the rotated image must be exactly same for analog images while the scale invariance holds due to mapping given in Eq. (4.41). However, when implemented in the discrete domain, these invariance properties are severely compromised due to errors in computation and vary with the types of moments or transforms. We analyse the magnitude invariance properties of various RIMTs under investigation for rotation and scaling using mean square error (MSE) [142] defined as follows.
Fig. 4.2. Analysis of numerical stability with respect to maximum order

\[ 1 \leq p_{\text{max}} \leq 50 \] for (a) moments and (b) transforms
where $|F_{pq}|$ and $|F'_{pq}|$ are magnitudes of moments or transforms of original and geometrically manipulated images respectively and $L$ is the total number of moments or transform coefficients participating in the computation. We analyse the invariance properties of various RIMTs for order $p_{\text{max}} = q_{\text{max}} = 10$.

We first analyse the rotation invariance performance of moments and transforms after rotating the internal disk of all sample images of Appendix B at different angles from $0^\circ$ to $90^\circ$ with an interval of $10^\circ$. As evident from Fig. 4.3, transforms, in general, possess superior rotation invariance properties than moments. PCTs and ARTs provide better rotation invariance compared to PCETs and PSTs. Among moments, RHFMs provide the best results for almost all rotation angles. The behaviors of ZMs and PZMs are similar with ZMs providing slightly better rotation invariance while OFMMs exhibit poor performance for all rotation angles. FMMs exhibit peculiar behavior with MSE increasing for rotation angles from $0^\circ$ to $45^\circ$ and then symmetrically decreasing till $90^\circ$.

Further, we analyse the scale invariance performance of moments and transforms with scaling factors ranging from 0.25 to 2.00 with standard images of size $256 \times 256$ pixels. The curves showing average MSE vs. scaling factors for various RIMTs are shown in Fig. 4.4. The results for scale invariance are similar to that for rotation invariance. PCTs provide better scale invariance compared to ARTs, PCETs and PSTs in the given order of performance. Among the moments, RHFMs perform the best results which are closely followed by ZMs/PZMs for all scaling factors, while FMMs exhibit poor scale invariance. It is also observed that the scale invariance property is severely affected when the scaling factor is less than one. This trend is attributed to the fact that the moments and transforms are computed more inaccurately for low resolution images because they are more affected by geometric error and numerical integration error [146]. The contribution of these errors is low for high resolution images and, therefore, the magnitude of MSE is smaller for higher scaling factors.
Fig. 4.3. Analysis of rotation invariance properties for (a) moments and (b) transforms.
Fig. 4.4. Analysis of scale invariance properties for (a) moments and (b) transforms.
4.7.1.5. Time Complexity

Analytically, the time complexity of computing $F_{pq}$ for $p \leq p_{\text{max}}$ and $q \leq q_{\text{max}}$ is $O(N^2 p_{\text{max}} q_{\text{max}})$ for moments and $O(N^2)$ for transforms. However, for a given order $p_{\text{max}}$ and $q_{\text{max}}$, the actual time taken to compute all coefficients of a particular moment or transform depends on the number of radial polynomials, sinusoidal functions, exponential terms and factorials present in its basis function. We investigate the time taken (in millisecs) for computing all RIMTs for $p_{\text{max}} = q_{\text{max}} = 30$. Figure 4.5 shows the curves depicting the average time taken in terms of CPU elapse time for computing all $F_{pq}$ for sample Lena image of size $256 \times 256$ pixels. It may be noted here that the time taken does not depend on the contents of gray scale images, therefore, only one image is considered for time analysis. Further, some moments and transforms have been grouped because their radial basis functions are similar and involve approximately same number of computations. As seen from the figure, computation of transforms takes much lesser time compared to moments which are more computation intensive because their radial basis functions are polynomials in degree $p$ while that of transforms consist of low-complexity sinusoidal functions.

![Fig. 4.5. Analysis of CPU elapse time for computation of RIMTs.](image-url)
4.7.2. Comparative Performance Analysis for Watermarking

We perform numerous experiments to perform the comparative analysis of RIMTs for robust image watermarking with respect to visual imperceptibility, embedding capacity and watermark robustness against various attacks. Figures 4.6 and 4.7 show the watermarked versions and the corresponding spatial watermark signals embedded using various moment-based and transform-based schemes, respectively. The watermarked versions are obtained after embedding a 128-bit randomly generated watermark signal into the sample Lena image of size $256 \times 256$ pixels and spatial watermark embedded is multiplied by ten for better display after setting negative values to zero.

![Watermarked version and Spatial signal added](image)

**(a) ZMs** (PSNR = 50.22 dB) **(b) PZMs** (PSNR = 52.03 dB) **(c) FMMs** (PSNR = 45.49 dB) **(d) OFMMs** (PSNR = 48.03 dB) **(e) RHFMs** (PSNR = 64.54 dB)

**Fig. 4.6. Examples of invariant moment-based watermarking schemes.**

![Watermarked version and Spatial signal added](image)

**(a) ARTs** (PSNR = 47.16dB) **(b) PCETs** (PSNR = 44.81dB) **(c) PCTs** (PSNR = 54.44dB) **(d) PSTs** (PSNR = 46.49dB)

**Fig. 4.7. Examples of invariant transform-based watermarking schemes.**
4.7.2.1. Visual Imperceptibility of Watermarked Images

The visual imperceptibility of watermarked images depends on the embedding strength determined by quantization step ($\Delta$) and number of bits embedded [71]. For comparative analysis, a pseudo-randomly generated watermark sequence of length 128-bits is embedded in all sample images and average $PSNR$ values as a function of embedding strength $\Delta$ varying from 0.8 to 2.0 are plotted as shown in Fig. 4.8. It is evident from the figure that RHFs provide the highest quality of watermarked images and imperceptibility of ZMs/PZMs-based watermarking scheme is better than PCTs-based schemes followed by PSTs, ARTs, OFMMs, PCETs and FMMs. It can also be observed that ARTs and OFMMs produce almost similar quality of watermarked images while FMMs provide the poorest visual imperceptibility at all embedding strengths. Comparing these results with the performance orders of various moments and transforms for their reconstruction capabilities, we observe that better reconstruction capabilities ensure better quality of watermarked images. Thus, visual imperceptibility directly depends on the reconstruction capabilities of RIMTs. For all our further experiments, we choose quantization step $\Delta=1.0$ to ensure acceptable visual imperceptibility yielding $PSNR \geq 44dB$ for all moments and transforms-based watermarking schemes.

4.7.2.2. Embedding Capacity

The maximum embedding capacity of a watermarking scheme using moments or transforms with maximum order $p_{\text{max}}$ and $q_{\text{max}}$ is given by $|S|$, the cardinality of set $S$, which represents the set of all $F_{pq}$ that can be used for watermarking. The algebraic expressions for deriving the cardinality for various moments and transforms for given values of $p_{\text{max}}$ and $q_{\text{max}}$ are given in Table 4.5. The maximum values of $p_{\text{max}}$ and $q_{\text{max}}$ for which the moments and transforms remain numerically stable and the corresponding maximum embedding sizes supported, $L_{\text{max}}$, are also provided for a quick comparison. As evident from the table, RHFs provide maximum data embedding size because they provide more number of moments and have better numerical stability.
Fig. 4.8. Visual imperceptibility relative to embedding strength ($\Delta$) for (a) moments and (b) transforms
Further, we embed randomly generated watermarks of varying lengths $L$, given by different orders $p_{\max} = q_{\max}$ from 1 to 50 in sample images and compute average BER after extracting watermark sequence from un-attacked watermarked image. The curves depicting total number of bits embedded vs. average BER obtained using various RIMTs-based watermarking schemes are plotted as shown in Fig. 4.9. As seen from the figure, acceptable values of BER are obtained up to maximum embedding strength, $L_{\max}$, given by the proposed order, as shown in Table 4.5.

<table>
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<tr>
<th>Moment/transform</th>
<th>Cardinality of set $S$</th>
<th>Proposed order</th>
<th>$L_{\max}$</th>
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</thead>
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<td>$</td>
<td>S</td>
<td>= \begin{cases} 3p_{\max}^2 + 8p_{\max} \ 16 &amp; \text{if } p_{\max} = 4i \ 3p_{\max}^2 + 10p_{\max} + 3 \ 16 &amp; \text{if } p_{\max} = 4i + 1 \ 3p_{\max}^2 + 8p_{\max} + 4 \ 16 &amp; \text{if } p_{\max} = 4i + 2 \ 3p_{\max}^2 + 10p_{\max} + 7 \ 16 &amp; \text{if } p_{\max} = 4i + 3 \end{cases}$</td>
</tr>
<tr>
<td>PZMs[71]</td>
<td>$</td>
<td>S</td>
<td>= \begin{cases} 3p_{\max}^2 + 6p_{\max} \ 8 &amp; \text{if } p_{\max} = 4i \ 3p_{\max}^2 + 6p_{\max} - 1 \ 8 &amp; \text{if } p_{\max} = 4i + 1 \ 3p_{\max}^2 + 6p_{\max} \ 8 &amp; \text{if } p_{\max} = 4i + 2 \ 3p_{\max}^2 + 6p_{\max} + 3 \ 8 &amp; \text{if } p_{\max} = 4i + 3 \end{cases}$</td>
</tr>
<tr>
<td>FMMs</td>
<td>$</td>
<td>S</td>
<td>= \begin{cases} 3q_{\max} (p_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i \ (3q_{\max} + 1)(p_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i + 1 \ (3q_{\max} + 2)(p_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i + 2 \ (3q_{\max} + 3)(p_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i + 3 \end{cases}$</td>
</tr>
<tr>
<td>OFMMs</td>
<td>$</td>
<td>S</td>
<td>= \begin{cases} p_{\max} (3q_{\max}) \ 4 &amp; \text{if } q_{\max} = 4i \ p_{\max} (3q_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i + 1 \ p_{\max} (3q_{\max} + 2) \ 4 &amp; \text{if } q_{\max} = 4i + 2 \ p_{\max} (3q_{\max} + 3) \ 4 &amp; \text{if } q_{\max} = 4i + 3 \end{cases}$</td>
</tr>
<tr>
<td>RHFMs</td>
<td>$</td>
<td>S</td>
<td>= \begin{cases} p_{\max} (3q_{\max}) \ 4 &amp; \text{if } q_{\max} = 4i \ p_{\max} (3q_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i + 1 \ p_{\max} (3q_{\max} + 2) \ 4 &amp; \text{if } q_{\max} = 4i + 2 \ p_{\max} (3q_{\max} + 3) \ 4 &amp; \text{if } q_{\max} = 4i + 3 \end{cases}$</td>
</tr>
</tbody>
</table>

Table 4.5. Embedding capacity of various RIMTs

<table>
<thead>
<tr>
<th>Moment/transform</th>
<th>Cardinality of set $S$</th>
<th>Proposed order</th>
<th>$L_{\max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARTs, PCETs, PCTs, PSTs</td>
<td>$</td>
<td>S</td>
<td>= \begin{cases} p_{\max} (3q_{\max}) \ 4 &amp; \text{if } q_{\max} = 4i \ p_{\max} (3q_{\max} + 1) \ 4 &amp; \text{if } q_{\max} = 4i + 1 \ p_{\max} (3q_{\max} + 2) \ 4 &amp; \text{if } q_{\max} = 4i + 2 \ p_{\max} (3q_{\max} + 3) \ 4 &amp; \text{if } q_{\max} = 4i + 3 \end{cases}$</td>
</tr>
</tbody>
</table>
Fig. 4.9. BER vs. length of watermark sequence for (a) moments and (b) transforms.
It is worth mentioning here that an average BER in the neighborhood of 0.5 is achieved at embedding capacity higher than $L_{\text{max}}$ because the probability of getting an accurate bit value (0 or 1) is 0.5, even though the moments and transforms are numerically unstable. RHFMs provide the highest embedding capacity relative to others while FMMs are found least desirable for high capacity embedding systems. ZMs provide lower embedding capacity because they offer fewer moments for embedding as repetition $q$ depends on the order due to the condition $|q| \leq p$ and must satisfy the additional constraint $p - |q| = \text{even}$. PZMs offer approximately double number of moments than ZMs, but exhibit numerical instability at very low order of moments ($p_{\text{max}} = 22$) thus providing lower embedding capacity.

4.7.2.3. Watermark Robustness

To analyse the robustness of watermark against various attacks, 128-bit watermarks are embedded into images given in Appendix B of size $256 \times 256$ pixels using quantization step $\Delta = 1.0$. All graphs presented in this section are plotted after computing the average BER of all images with at least 50 different pseudo-randomly generated watermark sequences.

i. Robustness to Rotation and Flipping

We perform experiments with nine different rotation angles at an interval of $5^\circ$ in the first octant and plot average BER curves obtained after extracting the watermark using the different RIMTs-based watermarking schemes. Rotations beyond $45^\circ$ are equivalent to similar rotations with $\pm 90^\circ$. The output watermarked images are rotated using inverse rotation transformation for better pixel sampling [147]. As evident from graphs in Fig. 4.10, in general, transforms provide better robustness to rotation than moments and among moments RHFMs outperform in comparison to other moments. On comparing the graphs in Fig. 4.2, showing rotation invariance property of RIMTs, with curves in Fig. 4.10, it can be noticed that robustness to rotation attack is in accordance with the invariance property satisfied by various RIMTs. Further, we obtain $BER = 0$ at angles $\pm 90^\circ$.
and ±180° for all watermarking schemes due to 4-way symmetric property of their basis functions.

Fig. 4.10. Watermark robustness to rotation for (a) moments and (b) transforms.
ii. Robustness to Scaling

The watermarked images of standard size $256 \times 256$ pixels are scaled with various scaling factors ranging from 0.25 to 2.0 using bilinear interpolation. No re-scaling is required prior to watermark extraction as image normalisation due to mapping (Eq. 4.41) provides magnitude invariance against scaling. Figure 4.11 shows the average BER curves obtained after extracting 128-bit watermark with various RIMTs-based watermarking schemes. It is observed that transforms are more robust against scaling attack than moments. In particular, PCTs-based watermarking scheme is able to extract the watermark with average BER < 0.10 even if the watermarked image is scaled down to the smallest size of $64 \times 64$ pixels. As mentioned earlier, the lower is the scaling factor, the higher is the average BER value. This trend is observed due to the reason that the moments and transforms are computed inaccurately because of geometric error and numerical integration error which are more prominent for low resolution images generated by small scaling factors. Furthermore, like rotation, the robustness to scaling attack is also directly linked with the performances of magnitude invariance properties of RIMTs for image resize.

iii. Robustness to JPEG compression

JPEG compression is a popular image compression technique which is commonly used for exchange of images and video over Internet. We examine the behavior of BER for JPEG compression with twelve different quality factors, $Q$, ranging from 15 to 100. The average BER curves obtained after extracting 128-bit watermark sequence are depicted in Fig. 4.12. It is evident from the figure that transforms provide better robustness than moments. The performances of PCTs and ARTs are found almost similar closely followed by PCETs and PSTs. Once again among the moments, RHFMs provide better robustness to compression. In general, both moments and transforms are found robust to higher compression factor, $Q \geq 60$. 

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Fig. 4.11. Watermark robustness to scaling for (a) moments and (b) transforms.
Fig. 4.12. Watermark robustness to JPEG compression for (a) moments and (b) transforms.
(iv) Robustness to Stirmark attacks

Next, we analyse the performances of moments-based and transforms-based watermarking schemes against various image processing distortions such as median filtering, additive noise, cropping, shearing transformation and other random attacks using an automated evaluation Stirmark benchmark. To examine the watermark robustness under these attacks, all watermarked images of size $256 \times 256$ pixels are attacked with Stirmark 4.0 tool. The average $BER$ values for various RIMTs-based watermarking schemes under investigation are presented in Table 4.6. As observed from the table, PCTs provide minimum $BER$ values for shearing and JPEG compression attack while RHFMs perform better against cropping, median filtering and removal attacks.

For quick appraisal, we count the number of complete failures when value of $BER \geq 0.25$ (entries encircled with solid lines in the table) and near failures when $0.20 \leq BER < 0.25$ (entries encircled with dotted lines in the table) and write them at the bottom of each column. Further, we assign relative ranks on the basis of number of failures reported. For equal number of failures, the absolute values of average $BER$s, indicating failure, are used for ordering among different moments and transforms. For example, two failures are reported in case of each ZMs and RHFMs and the values of average $BER$ are higher in case of ZMs, therefore, we rank RHFMs above ZMs. Similarly, between PSTs and PZMs, PSTs are ranked above PZMs.

Clearly, PCTs provide the best watermark robustness against all attacks while non-orthogonal FMMs perform the worst. Among the moments, RHFMs have been found more robust to Stirmark attacks compared to other moments.
Table 4.6. Stirmark test results

<table>
<thead>
<tr>
<th>Attack</th>
<th>Average BER (acceptable value ≤ 0.25)</th>
<th>Moments</th>
<th>Transforms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>ZMs</td>
<td>PZMs</td>
</tr>
<tr>
<td>Shearing</td>
<td>x:0.00 y:0.01</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>x:0.00 y:0.05</td>
<td>0.0076</td>
<td>0.0187</td>
</tr>
<tr>
<td></td>
<td>x:0.01 y:0.00</td>
<td>0.0000</td>
<td>0.0476</td>
</tr>
<tr>
<td></td>
<td>x:0.05 y:0.00</td>
<td>0.0088</td>
<td>0.0329</td>
</tr>
<tr>
<td></td>
<td>x:0.01 y:0.01</td>
<td>0.0001</td>
<td>0.0378</td>
</tr>
<tr>
<td></td>
<td>x:0.05 y:0.05</td>
<td>0.0235</td>
<td>0.0978</td>
</tr>
<tr>
<td>Cropping</td>
<td>&lt;10%</td>
<td>0.0000</td>
<td>0.0054</td>
</tr>
<tr>
<td></td>
<td>10-20%</td>
<td>0.0307</td>
<td>0.0755</td>
</tr>
<tr>
<td>JPEG quality</td>
<td>&lt;25</td>
<td>0.0251</td>
<td>0.0256</td>
</tr>
<tr>
<td></td>
<td>25-50</td>
<td>0.0007</td>
<td>0.0123</td>
</tr>
<tr>
<td></td>
<td>&gt;50</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Median Filter</td>
<td>3x3</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>5x5</td>
<td>0.0070</td>
<td>0.0546</td>
</tr>
<tr>
<td></td>
<td>7x7</td>
<td>0.0545</td>
<td>0.1248</td>
</tr>
<tr>
<td>Rotation</td>
<td>±0.25</td>
<td>0.0385</td>
<td>0.0596</td>
</tr>
<tr>
<td></td>
<td>±0.5</td>
<td>0.0790</td>
<td>0.0687</td>
</tr>
<tr>
<td></td>
<td>with scale ±0.25</td>
<td>0.1268</td>
<td>0.1483</td>
</tr>
<tr>
<td></td>
<td>with crop ±0.5</td>
<td>0.1976</td>
<td>0.2673</td>
</tr>
<tr>
<td>Removal attack</td>
<td>10 rows, 10 cols</td>
<td>0.0526</td>
<td>0.2324</td>
</tr>
<tr>
<td></td>
<td>20 rows, 20 cols</td>
<td>0.1065</td>
<td>0.3243</td>
</tr>
<tr>
<td>Additive Noise</td>
<td>20%</td>
<td>0.0983</td>
<td>0.1083</td>
</tr>
<tr>
<td></td>
<td>30%</td>
<td>0.1396</td>
<td>0.1286</td>
</tr>
<tr>
<td></td>
<td>50%</td>
<td>0.2568</td>
<td>0.2580</td>
</tr>
<tr>
<td></td>
<td>75%</td>
<td>0.4678</td>
<td>0.5109</td>
</tr>
<tr>
<td>No. of Failures</td>
<td></td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Relative Rank</td>
<td></td>
<td>3&lt;sup&gt;rd&lt;/sup&gt;</td>
<td>7&lt;sup&gt;th&lt;/sup&gt;</td>
</tr>
</tbody>
</table>
4.8. Conclusions

In this chapter, we introduce Fourier-Mellin moments (FMMs), Orthogonal Fourier-Mellin moments (OFMMs), radial harmonic Fourier moments (RHFMs) and angular radial transform (ARTs) for robust image watermarking. We perform a detailed comparative evaluation of watermarking systems regarding critical parameters like visual imperceptibility, embedding capacity and watermark robustness against geometric transformations, common signal processing distortions and Stirmark attacks. We also analyse various properties possessed by different rotation invariant moments and transforms in order to investigate relationships between their performance for robust watermarking and associated properties such as accuracy, magnitude invariance, reconstruction capabilities and computational complexity. Following significant conclusions are drawn from experiments performed and results obtained in the chapter.

i. Computational accuracy, numerical stability and watermark embedding capacity: Embedding capacity of a watermarking scheme largely depends on the computational accuracy and numerical stability of the moments or transforms used for watermarking. Among moments, RHFMs provide highest embedding capacity (713 bits). It is observed that ZMs are numerically stable up to $p_{\text{max}} \leq 44$, PZMs up to $p_{\text{max}} \leq 23$, RHFMs up to orders $p_{\text{max}} \leq 30$, OFMMs up to $p_{\text{max}} \leq 22$ and FMMs up to $p_{\text{max}} \leq 10$. Although, ZMs provide better numerical stability and reconstruction capabilities at high order of moments up to $p_{\text{max}} = 44$, they provide fewer coefficients for embedding as repetition $q$ depends on the order ($|q| \leq p$) and must satisfy the condition $p - |q| = \text{even}$. Further, PZMs provide approximately double the number of moments than ZMs but exhibit numerical instability at very low order of moments ($p_{\text{max}} = 23$). We find that all transforms under investigation are numerically stable up to $p_{\text{max}} \leq 25$. At this order of transforms PCTs, PCETs and ARTs provide equivalent embedding capacity of 494 bits while PSTs provide lower embedding capacity (475 bits).
ii. *Image reconstruction capabilities and visual imperceptibility:* It is observed that RHFMs provide the highest reconstruction capabilities with minimum value of MSRE ($\varepsilon=0.03358$) and the best quality of watermarked image with the values of $PSNR$ lying between $63.43\, dB$ to $51.08\, dB$ while FMMs provide the lowest reconstruction capabilities ($\varepsilon=5.38370$) and poor quality watermarked images with the highest value of $PSNR = 48.74\, dB$. The performances of ZMs and PZMs are almost similar. Similarly among transforms, PCTs provide better reconstruction capabilities and quality of reconstructed images compared to other transforms.

iii. *Invariance properties and watermark robustness:* Robustness of a watermarking scheme is directly related with the invariance properties possessed by moments and transforms. We observe that transforms exhibit better magnitude invariance and subsequently transform-based watermark schemes are found more robust to geometric attacks.

iv. *Computational complexity:* The major overhead while embedding or extracting watermark is related with the computation of moments or transform coefficients. The time complexity of computing $F_{pq}$ for $p \leq p_{\text{max}}$ and $q \leq q_{\text{max}}$ is $O(N^2 p_{\text{max}} q_{\text{max}})$ for moments except RHFMs and $O(N^2)$ for RHFMs and transforms. Thus, RHFMs and transforms based watermarking schemes are more suitable for real-time applications that need to perform in limiting computing environment.

The lower performances of moment-based watermarking schemes are attributed to their computational intensive basis function and numerical instability issues. Therefore improving accuracy and speed of computation will enhance the performances of various watermarking schemes. We devote Chapter 5 to the design and implementation of accurate and fast computational framework for RIMTs. Further, we propose the use of RHFMs and PCTs as they outperform in their respective moment and transform families. In addition, ZMs are chosen because they are of special interest to various researchers and one of the best state-of-art watermarking schemes is also based on them [71].