Chapter 1

Theoretical Methods and their applications to understand chemistry of Cyclodextrin and cucurbit[n]uril hosts
1.1 Preamble

Understanding of physical and chemical phenomena at the molecular levels has been crucial in modelling of different phenomena in science and engineering. Admitably Chemistry is empirical in nature computational chemistry has emerged as a complementary tool to experiments which can be explored to interpret and rationalize experimental data and further simulate the molecular properties when experiments cannot be carried out. Computational quantum chemistry hinges upon solving the second order, first degree differential equation known as the Schrodinger's equation; the solutions of which yield the molecular electronic structure in ground and excited states, charge distribution, vibrational spectra, thermodynamic parameters, potential energy surfaces, rate constants and dynamics for elementary reactions with varying degree of accuracy depending on size (number of atoms or degrees of freedom) of molecular system. Until the end of nineteenth century the classical mechanics, constituting the Newtons laws governing the motion of material bodies and Maxwells theory of electromagnetism was regarded as fundamental laws of Physics with no limitation whatsoever for their applicability. In 1890s a series of experiments set in motion with radiation emitted from black body, photoelectric effect yielded results which could not be reconciled with the classical mechanics. The resolution of apparent paradoxes posed by these observations and certain other experimental facts was made possible only after introducing the concepts like uncertainty principle and wave particle duality. It was these revolutionary ideas which led to the mathematical formulation of quantum mechanics (QM) that had an immediate and spectacular success in rationalization of these observations. The objective of QM is to solve the Schrödinger equation without employing any empirical data except the universal constants, to obtain the wave function that represents the state of a system. It was soon realized that exact solution of the Schrodinger equation is possible only in a few cases. For systems containing more than one electron only approximate solutions to Schrodinger equation can be obtained. These solutions are associated with certain errors due to incorporation of approximations to varying degrees. Hartree-Fock (HF) theory represent first such successful attempt to derive the wave function of many-electron systems. HF theory referred to be as ab initio (from the beginning or from the first principles) theory i.e., free from empirical parameters, was little used until late 1970’s due to the limitations of computer memory, hard disk storage and the CPU speed. Recent advances in desktop
computers, workstations and supercomputers and availability of different software programs incorporating newer methodologies, have made it possible to derive routinely molecular structure and properties of systems containing up to hundred atoms or more. To improve upon the HF theory the post HF methods such as configuration interaction or Møller-Plesset perturbation theory, coupled cluster theory, etc. were devised. These methods accounting the electron correlation at least partly, however, are computationally demanding and are restricted to small or medium-sized molecules. Unlike in the above theories where the wavefunction has been endowed with the basic variable; the approaches treating the electron density as a fundamental entity have become popular in the recent years. The origin of the latter can be traced to historical Thomas Fermi theory. Prior to the HF method in 1927, Thomas and Fermi independently proposed a semi-classical model to describe an atomic system having large number of electrons. Accuracy of this method was limited due to kinetic energy expressed in terms of drastically approximate electron density and completely neglected electron correlations, albeit, this was a precursor for the rigorous density-based theory. The electron exchange energy incorporated later in 1928 by Dirac, however, lead to atomic energy estimates which were largely inaccurate.

A path-breaking formalism by Hohenberg and Kohn (HK) put forth an attractive alternative to the many-electron wave function based theories. After a year, Kohn and Sham (KS) proposed a theory that essentially connects HK density-based theory to Schrodinger description of quantum mechanics. The HK and KS formalisms, formally have shown that the physical observables of a system in its ground state can be expressed as functionals of the electron density (ED) and led to the density functional theory (DFT) which has proved quite successful. The results of DFT depend on the quality of exchange-correlation potential employed. DFT-based methods promise to provide reliable predictions of the structure and properties of chemical systems with lesser computational effort than that required for post-HF methods. The computational cost of the HF and DFT level calculations on medium/large-sized molecules, practically scales as $O(N^3)$, $N$ being the number of basis functions used in the calculations. DFT is one of the well-established assets of QM to study atoms, molecules, nanoscale structures, biomolecules, polymer chains, and also the interactions therein. Due to ever-increasing computational power, DFT calculations are now feasible for systems containing a few hundred atoms. An outline of electronic structure methods used in the present thesis is given below.
1.2 Many-Electron Systems

The time-independent or time-dependent Schrödinger equation can be used to describe any microscopic many-body system (atom or molecule) depending upon the nature of interaction of its constituent particles. Finding solutions to the Schrödinger equation is essential to elucidation of the electronic structure of atoms, molecules and solids. Hartree-Fock\textsuperscript{2,3} approximation made significant development in solving the Schrödinger equation for a many-electron system, which constitutes an \textit{ab initio} theory based on the fundamental principles of QM. It include exclusively a quantum mechanical effects of electron-correlation beyond the parallel-spin exchange interaction (i.e., incorporates the non-trivial exchange energy) and serves as a first step towards more accurate calculations. A detailed description of time independent Hartree-Fock theory has been discussed.

1.2.1 The Molecular Hamiltonian

Consider a molecule having $N$ electrons bound to $M$ nuclei. The state of such a microscopic system is described by the solution of the Schrödinger equation

$$\hat{H}_\text{mol} |\Psi\rangle = E |\Psi\rangle, \quad (1.1)$$

where, a state ket $|\Psi\rangle$, a wave function representing the state of the molecule with $N$ electrons positioned at space-spin coordinates $\{x_i\} \equiv \{(\vec{r}_i, \omega_i)\}$ and the $M$ nuclei at $\{\vec{R}_A\}$, is represented in coordinate space by $\langle \{x_i\}, \{\vec{R}_A\} |\Psi\rangle = \Psi(\{x_i\}, \{\vec{R}_A\})$. The coordinates $x_i$ for the $i$th electron comprise of space coordinates $\vec{r}_i$ and spin coordinates $\omega_i$. The eigenvalue $E$ in Eq. 1.1 is the total energy of such a system. If one employ atomic units, then $\hbar$ (Planck’s constant/2 $\pi$), $m_e$ (mass of electron), and $|e|$ (electronic charge) are numerically set to unity.

The molecular Hamiltonian in Eq. 1.1 in atomic units has the constituents:

$$\hat{H}_\text{mol} = \sum_{i=1}^{N} \frac{1}{2} \nabla_i^2 + \sum_{A=1}^{M} \frac{1}{2M_A} \nabla_A^2 - \sum_{i=1}^{N} \sum_{A=1}^{M} \frac{Z_A}{r_{iA}},$$

where $E$ represents the electronic kinetic energy, $\nabla_i$ and $\nabla_A$ the spatial derivatives with respect to $x_i$ and $\vec{R}_A$, respectively, and $Z_A$ the nuclear charge. The interaction terms include:

- Nuclear-Nuclear repulsion
- Electron-Nuclear attraction
- Electron-Electron repulsion
- Electron-Nuclear repulsion

$$+ \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{r_{ij}}, \quad + \sum_{A=1}^{M} \sum_{B=A}^{M} \frac{Z_A Z_B}{R_{AB}}, \quad (1.2)$$
The first term represents the kinetic energy of the electrons, the second term is the kinetic energy of the nuclei, and the remaining three terms are the Coulomb interaction energies as indicated. The potential energy of $i^{th}$ electron positioned at $\vec{r}_i$ in the field of $M$ nuclei is

$$v(\vec{r}_i) = -\sum_{A=1}^{M} \frac{Z_A}{r_{iA}}$$

(1.3)

with $\vec{r}_{iA} = \vec{r}_i - \vec{R}_A$. The Hamiltonian in Eq. 1.2 can be written as

$$\hat{H}_{mol} = \hat{T}_e + \hat{T}_{\text{Nuc}} + \hat{V}_{e-Nuc} + \hat{V}_{e-e} + \hat{V}_{\text{Nuc-Nuc}}$$

(1.4)

in which the potential energy operator

$$\hat{V}_{e-Nuc} = \sum_{i=1}^{N} v(\vec{r}_i) = -\sum_{i=1}^{N} \sum_{A=1}^{M} \frac{Z_A}{r_{iA}}$$

(1.5)

represents the attractive Coulomb interaction between the electrons with unit negative charge and positively charged nuclei. The $\hat{T}_e$ and $\hat{T}_{\text{Nuc}}$ are the many-electron kinetic energy and the nuclear kinetic energy operators, respectively. The last two operators in Eq. 1.4 represent the Coulomb repulsion among the electrons ($\hat{V}_{e-e}$) and among the nuclei ($\hat{V}_{\text{Nuc-Nuc}}$). In quantum chemistry, the computation of the energy and wave function of an average-size molecule is a formidable task that is alleviated by the Born-Oppenheimer approximation that facilitates decoupling the electronic motion from the nuclear motion.

### 1.2.2 The Born-Oppenheimer Approximation

The Born-Oppenheimer approximation used in simplifying the Schrödinger equation by separating the electronic degrees of freedom from those of the nuclei and gives two sets of Schrödinger equation, one for the electrons and other for the nuclei. This approximation is based on the comparatively slower motions of nuclei than the electrons; $\langle \hat{T}_N \rangle / \langle \hat{T}_e \rangle \sim m_e / M_A \sim 10^{-3}$. Hence, an electron in motion sees relatively static nuclei, while a moving nucleus feels averaged electronic motion. The kinetic energy of the nuclei can be neglected for the electronic motion, thereby the repulsion energy between the nuclei regarded to be fixed. The electronic Hamiltonian describing the motion of $N$ electrons in the field of fixed nuclei can be written as
\[ \hat{H}_e = -\sum_{i=1}^{N} \frac{1}{2} \nabla_i^2 - \sum_{i=1}^{N} \sum_{A=1}^{M} \frac{Z_A}{r_{iA}} + \sum_{i=1}^{N} \sum_{j>i}^{N} \frac{1}{r_{ij}}. \] (1.6a)

or

\[ \hat{H}_e = \sum_i^n \hat{h}(i) + \sum_i^n \sum_j^n \hat{V}(\vec{r}_{ij}), \] (1.6b)

in which

\[ \hat{h}(i) = -\frac{1}{2} \nabla_i^2 - \sum_{A=1}^{M} \frac{Z_A}{r_{iA}} \] (1.7)

is the single-particle operator representing the kinetic energy and potential energy of the \( i^{th} \) electron, and \( \hat{V}(\vec{r}_{ij}) = 1/r_{ij} \), represent the electron-electron Coulomb repulsion, is the two-particle potential energy operator. The Schrödinger equation for the electronic motion is

\[ \hat{H}_e |\Psi_e\rangle = E_e |\Psi_e\rangle, \] (1.8)

where the eigenvalue, \( E_e \) is the electronic energy that depends on the nuclear positions. The electronic wave function \( \langle \{x_i\}, \{\vec{R}_A\}; |\Psi_e\rangle = \Psi_e(\{x_i\}, \{\vec{R}_A\}) \) (the coordinate-space representation of the state \( |\Psi_e\rangle \)) depends explicitly on the electronic coordinates \( \{x_i\} \) but only parametrically on the nuclear coordinates \( \{\vec{R}_A\} \). First keep nuclei fixed and solve Schrödinger equation (Eq. 1.7) for many-electron wave function \( \Psi_e \) to get corresponding electronic energy \( E_e \), and then the Schrödinger equation for the nuclear motion in the average electronic field until self-consistency is reached. A complete self-consistent solution provide the total energy of molecule as a sum of electronic energy \( (E_e) \) and the internuclear repulsion energy \( (V_{Nuc-Nuc}) \) i.e.,

\[ E_{tot}(\{\vec{R}_A\}) = E_e + \sum_{A=1}^{M} \sum_{B=1}^{M} \frac{Z_A Z_B}{R_{AB}}. \] (1.9)

which serves as effective 'potential energy' for the nuclear Hamiltonian:

\[ \hat{H}_{Nuc} = -\sum_{A=1}^{M} \frac{1}{2M_A} \nabla_A^2 + E_{tot}(\{\vec{R}_A\}). \] (1.10)
Thus the nuclei in here move in the potential defined by electronic motion and the fixed inter-nuclear interaction for a given nuclear configuration. The vibrational, rotational and translational motion of the molecule are described by the nuclear Schrödinger equation

$$\hat{H}_{\text{Nuc}} |\Psi_{\text{Nuc}}\rangle = E_{\text{Nuc}} |\Psi_{\text{Nuc}}\rangle$$

(1.11).

The eigenvalue $E_{\text{Nuc}}$ is the contribution within the Born-Oppenheimer approximation to the total energy $E$ of Eq. 1.1. The total wave function of Eq. 1.1 takes form of

$$\Psi(x_i, \{\vec{R}_i\}) = \Psi_e(x_i, \{\vec{R}_i\}) \cdot \Psi_{\text{nuc}}(\{\vec{R}_i\}).$$

(1.12)

within this approximation. Since, we are interested in the electronic wave function at present the subscript 'e' in the electronic Hamiltonian $\hat{H}_e$ and in the electronic wave functions $\{\Psi_e\}$ can be neglected hereafter.

### 1.2.3 The Antisymmetry Principle of Pauli

There is a stringent requirement on the many-electron wave functions, whether exact or approximate, beside the basic requirements of normalization, square integrability etc. The many-electron wave functions representing a system of $N$ identical electrons must be totally antisymmetric with respect to the interchange of the coordinates $\{x_i\} = \{(\vec{r}_i, \omega_i)\}$ of any two electrons in the system. That is

$$P_{ij} \Psi(x_1...x_i...x_j...x_N) = \Psi(x_1...x_j...x_i...x_N)$$

$$= -\Psi(x_1...x_j...x_i...x_N)$$

(1.13)

where $P_{ij}$ is the permutation operator that interchanges the coordinates of $i$th and $j$th electrons. This is the requirement on the many-electron wave functions from antisymmetry principle or the Pauli principle.

### 1.2.4 The Variational Principle

With the understanding of crucial property of antisymmetry, now we will review an important variational principle used in search of even better approximate wave functions. For any normalized, well-behaved trial wave function $\tilde{\Psi}$ satisfying the same boundary conditions as the exact ground state wave function $\Psi_0$, the expectation value of the Hamiltonian corresponding to $\tilde{\Psi}$ will always be greater than or equal to the ground state energy ($E_0$) corresponding to the exact ground state wave function $\Psi_0$. That is
\[
\langle \widetilde{\Psi} | \hat{H} | \widetilde{\Psi} \rangle \geq \langle \Psi_0 | \hat{H} | \Psi_0 \rangle = E_0, \quad \text{if} \quad \langle \widetilde{\Psi} | \widetilde{\Psi} \rangle = 1 \tag{1.14}
\]

where the equality holds iff \( \widetilde{\Psi} \) is identical to \( \Psi_0 \). This rigorous inequality is the variational principle in quantum mechanics. The ground state wave function \( \Psi_0 \) can be estimated by varying the parameters incorporated in the \( \widetilde{\Psi} \), until the expectation value \( \langle \widetilde{\Psi} | \hat{H} | \widetilde{\Psi} \rangle \) reaches a minimum. The minimum so obtained is the variationally determined estimate of the ground state energy. The energy estimate equals the exact ground state energy if the variationally determined trial function coincides with the ground state wave function.

### 1.2.5 The Hartree-Fock Approximation

Here we introduce the ‘wave function’ for a single electron (i.e. an orbital) comprising an \( N \) electron wave function for an \( N \) electron system. Since an electron is a spin-\( \frac{1}{2} \) fermion, a complete description of its state requires a spatial orbital \( \Psi(\vec{r}) \) and spin function \( \alpha(\omega) \) or \( \beta(\omega) \). The resultant wave functions of a single electron \( \chi(x) = \Psi(\vec{r}) \alpha(\omega) \) or \( \chi(x) = \Psi(\vec{r}) \beta(\omega) \) are called the ‘spin orbitals’. Similar to the spatial orbitals these two spin functions are orthonormal, i.e.,

\[
\langle \alpha | \alpha \rangle = \langle \beta | \beta \rangle = 1. \tag{1.15}
\]

Or,

\[
\begin{align*}
\sum_\omega \langle \alpha | \omega \rangle \langle \omega | \alpha \rangle &= \sum_\omega \langle \beta | \omega \rangle \langle \omega | \beta \rangle = 1 \\
\sum_\omega \alpha^*(\omega) \alpha(\omega) &= \sum_\omega \beta^*(\omega) \beta(\omega) = 1
\end{align*}
\tag{1.16}
\]

and

\[
\langle \alpha | \beta \rangle = \langle \beta | \alpha \rangle = 0. \tag{1.17}
\]

i.e.,

\[
\begin{align*}
\sum_\omega \langle \alpha | \omega \rangle \langle \omega | \beta \rangle &= \sum_\omega \langle \beta | \omega \rangle \langle \omega | \alpha \rangle = 0 \\
\sum_\omega \alpha^*(\omega) \beta(\omega) &= \sum_\omega \beta^*(\omega) \alpha(\omega) = 0
\end{align*}
\tag{1.18}
\]

A point to note herein is that the asterisk (\( ^* \)) in the spin functions \( \alpha(\omega) \) and \( \beta(\omega) \), and in spin orbitals \( \chi_j(x) \) could imply the hermitian adjoint (\( ^\dagger \)) for multi-component spin
orbitals. The spin orbitals \( \chi_j(x) \) are the eigenfunctions of single-particle Hamiltonian (Eq. 1.8) and yields energy \( \epsilon_j \) associated with corresponding orbital as an eigenvalue i.e.,

\[
\hat{h}(i) \chi_j(x_i) = \epsilon_j \chi_j(x_i), \quad (1.19)
\]

The orthonormality of spin orbitals is expressed as

\[
\int dx \chi^*_j(x) \chi_j(x) = \langle \chi_j | \chi_j \rangle = \delta_{ij}, \quad (1.20)
\]

in which \( \int dx \equiv \sum_\omega \int d^3r \) denotes the summation over the spin variables and integration over the spatial ones simultaneously. A “Hartree-Product” wave function \( \Psi^{\text{HP}} \) can be constructed from such spin orbitals for a system of \( N \) non-interacting electrons, that is

\[
\hat{H} \Psi^{\text{HP}} = \sum_j \hat{h}(i) \Psi^{\text{HP}} = \left( \sum_i (\epsilon_i + \epsilon_j + \cdots + \epsilon_k) \right) \Psi^{\text{HP}} \quad (1.21)
\]

where

\[
\Psi^{\text{HP}}(x_1, x_2, \ldots, x_N) = \chi_i(x_1) \chi_j(x_2) \cdots \chi_k(x_N). \quad (1.22)
\]

The many-electron wave function from Hartree theory\(^7\) as a simple product of spin orbitals is not antisymmetric; but Pauli principle necessitates antisymmetry. By placing the spin orbitals in a determinant form (Slater determinant),\(^8\) one can construct a many-electron wave function that satisfy Pauli’s antisymmetry principle. Thus,

\[
\Psi(x_1, x_2, \ldots, x_N) = -\frac{1}{\sqrt{N!}} \begin{vmatrix}
\chi_j(x_1) & \chi_j(x_2) & \cdots & \chi_j(x_N) \\
\chi_k(x_1) & \chi_k(x_2) & \cdots & \chi_k(x_N) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_i(x_1) & \chi_i(x_2) & \cdots & \chi_i(x_N)
\end{vmatrix} \quad (1.23a)
\]

The factor, \((N!)^{-1/2}\), on the right is the normalization constant. The Slater determinant describes \( N \) electrons occupying \( N \) spin orbitals \( (\chi_1, \chi_2, \ldots, \chi_i, \ldots, \chi_j, \ldots, \chi_N) \) without specifying the occupancies. Interchange of coordinates of two electrons lead to interchange two rows, which changes the sign of the determinant. Thus the \( N \) electron Slater determinantal wave function fulfills the requirement of the antisymmetry principle and it may conveniently be written as

\[
\Psi(x_1, x_2, \ldots, x_N) = \begin{vmatrix}
\chi_1(x_1) & \chi_2(x_2) & \cdots & \chi_N(x_N) \\
\chi_2(x_1) & \chi_1(x_2) & \cdots & \chi_N(x_N) \\
\vdots & \vdots & \ddots & \vdots \\
\chi_N(x_1) & \chi_N(x_2) & \cdots & \chi_1(x_N)
\end{vmatrix}, \quad (1.23b)
\]

where only the principal diagonal spin orbitals are retained, and the electron coordinates are labelled strictly in the order \( x_1, x_2, \ldots, x_N \). Although strictly wave function
corresponding to the state \( \{ \chi_i x_j \cdots \chi_j \cdots \chi_N \} \) defined on a position-spin coordinate basis \( \{(x_j, x_2, \ldots, x_N)\} \) must be written as
\[
\Psi(x_1, x_2, \ldots, x_N) = \langle x_1, x_2, \ldots, x_N \mid \chi_i x_j \cdots \chi_j \cdots \chi_N \rangle.
\] (1.23c)

A very important effect of the electron-electron exchange correlation associated with the correlated motion of two electrons with parallel spins is brought by antisymmetrizing a Hartree product; however the motion of electrons with opposite spins remains uncorrelated. Such a single determinant is used as an approximate ground state many-electron wave function with spin orbitals obtained from HF equations, Hartree-Fock (HF) approximation as described below.

Within variational principle and the orthonormality conditions imposed on the spin orbitals, HF equations can then be derived by varying the spin orbitals in a way that energy corresponding to a single determinant they form is minimized. If we start with a 'trial' Slater determinant wavefunction,
\[
\left| \tilde{\Psi} \right\rangle = \left| \chi_i x_j \cdots \chi_k \right\rangle
\]
\[
= (N!)^{-1/2} \sum_{n=1}^{N^3} \left\{ \sum_{n=1}^{P_n} \left\{ \chi_i(x_1) \chi_j(x_2) \cdots \chi_k(x_N) \right\} \right\},
\] (1.24)

where \( P \) is an operator generating \( n \)th permutation of the \( N \) electrons and \( P_n \) is number of interchanges required to obtain required permutation. The energy corresponding to this wave function is
\[
\langle \tilde{\Psi} | \hat{H} | \tilde{\Psi} \rangle = \langle \tilde{\Psi} | \sum_{i} \left[ \hat{h}(i) + \sum_{j>i} \hat{V}(i,j) \right] | \tilde{\Psi} \rangle
\] (1.25)

With some algebraic manipulations and orthonormality of the spin orbitals one arrives at
\[
\langle \tilde{\Psi} | \hat{H} | \tilde{\Psi} \rangle = \sum_{i} \langle i | \hat{h} | i \rangle + \sum_{j>i} \langle i | \hat{V}(i,j) | j \rangle - \langle i | \hat{V}(j,i) | j \rangle - \langle j | \hat{V}(i,j) | i \rangle,
\] (1.26)
in which the abbreviations imply,
\[
\langle i | \hat{h} | i \rangle = \int dx_1 \chi_i^*(x_1) \hat{h} \chi_i(x_1);
\] (1.27)
\[
\langle j | \hat{V}(i,j) | j \rangle = \int dx_1 dx_2 \chi_i^*(x_1) \chi_j^*(x_2) r_{ij}^{-1} \chi_i(x_1) \chi_j(x_2);
\] (1.28)
and
\[
\langle j | \hat{V}(i,j) | i \rangle = \int dx_1 dx_2 \chi_i^*(x_1) \chi_j^*(x_2) r_{ij}^{-1} \chi_j(x_1) \chi_i(x_2).
\] (1.29)
By applying variational principle for true ground state ket \( |\Psi_0\rangle \) one can write \( E_0 = \langle \Psi_0 | \hat{H} | \Psi_0 \rangle \leq \langle \Psi | \hat{H} | \Psi \rangle \). Now if \( |\Psi\rangle = |\Psi_0\rangle \) then HF ground state energy in terms of the spin orbitals is

\[
E_{0}^{\text{HF}} = \sum_{i} \langle i | \hat{h} | i \rangle + \sum_{i} \sum_{j \neq i} \left( \langle ij | j \rangle - \langle ji | j \rangle \right).
\]

(1.30)

To minimize \( E_{0}^{\text{HF}}[\{\chi_i\}] \) by varying the spin orbitals provided \( \int dx \chi^*_i(x)\chi_j(x) = \langle i | j \rangle = \delta_{ij} \)

i.e. that the orbitals remain othonormal. To this end, we consider a functional \( L_0[\{\chi_i\}] \) of spin orbitals

\[
L_0[\{\chi_i\}] = E_0[\{\chi_i\}] - \sum_{i} \sum_{j} e_{ij} \langle i | j \rangle - \delta_{ij},
\]

(1.31)

where \( e_{ij} \) are the Lagrange multipliers. Extremum of \( E_{0}^{\text{HF}}[\{\chi_i\}] \) is obtained by setting the first variation in \( L_0[\{\chi_i\}] \) to zero for an arbitrary infinitesimal variation in spin orbitals, i.e.,

\[
\delta L_0 = \delta \left( E_{0}^{\text{HF}}[\{\chi_i\}] - \sum_{i} \sum_{j} e_{ij} \langle i | j \rangle - \delta_{ij} \right) = 0 \quad \text{for} \quad (\chi_i \rightarrow \chi_i + \delta \chi_i).
\]

It turns out that the above equation holds when the orbitals satisfy:

\[
\hat{f}(x)\chi_i(x) = \sum_{j} e_{ij} \chi_j(x) \quad (i = 1, 2, ..., N),
\]

(1.32)

where

\[
\hat{f}(x) = \hat{h} + \hat{V}^{\text{HF}}(x)
\]

(1.33)

is the “Fock operator”. The first term (cf. Eq. 1.7) and second term (HF potential operator) in the Fock operator are single-particle operator. HF potential operator represents electron-electron interaction in an average way. Thus the complex many-electron problem is replaced by a set of one-electron problems in which \( V^{\text{HF}} \) represents one-electron effective potential. Each electron only sees the average field generated by all other electrons is the basis of the HF theory and hence be regarded as a mean field theory. Explicitly,

\[
\hat{V}^{\text{HF}}(x_i) = \sum_{j} (\hat{J}_j(x_i) - \hat{K}_j(x_i)),
\]

(1.34)
in which \( \hat{J}_j(x_1) = \int \! dx_2 \, \chi_j^*(x_2) r_{12}^{-1} \chi_j(x_2) \) is a Coulomb operator that represents the average local potential at the position \( x_1 \) due to an electron in \( \chi_j(x_2) \) and acts on orbitals as
\[
\hat{J}_j(x_1) \chi_i(x_1) = \left[ \int \! dx_2 \, \chi_j^*(x_2) r_{12}^{-1} \chi_j(x_2) \right] \chi_i(x_1).
\] (1.35)

The nonlocal operator – \( \hat{K}_j(x_1) \), called the exchange operator, represents the parallel-spin exchange type electron-electron correlation introduced by the antisymmetric for the many-electron wave function, is defined by its action on the spin orbital \( \chi_i(x_1) \) through
\[
- \hat{K}_j(x_1) \chi_i(x_1) = - \left[ \int \! dx_2 \, \chi_j^*(x_2) r_{12}^{-1} \chi_j(x_2) \right] \chi_i(x_1).
\] (1.36)

Operating \( \hat{K}_j(x_1) \) on \( \chi_i(x_1) \) involves an exchange of electrons at \( x_1 \) and \( x_2 \) to the right of \( r_{12}^{-1} \) in Eq. 1.36 relative to that in Eq. 1.35. Thus, the action of \( \hat{K}_j(x_1) \) on the orbital \( \chi_i(x_1) \) obviously cannot be viewed in terms of a local potential for electron in \( \chi_j(x_1) \). In fact, it is this exchange operator which renders the Fock operator manifestly nonlocal in space.

The HF set of equations (Eq. 1.32) so obtained is not in the canonical (standard) eigenvalue form, however, one can make it so by performing a unitary transformation of the spin orbitals to obtain a unique set of orbitals \( \{ \chi_i' \} \) in which the non-diagonal \( \varepsilon_{ij} \) vanish. Thus,
\[
\hat{f}(x) \chi_i'(x) = \varepsilon_i + \varepsilon_i' \chi_i'(x) \quad (i = 1, 2, \ldots, N).
\] (1.37)

The spin orbitals \( \{ \chi_i' \} \) obtained as the solutions of these eigenvalue equations are called the canonical spin orbitals. If the primes are dropped the equations turn out to be
\[
\hat{f}(x) \chi_i(x) = \varepsilon_i \chi_i(x) \quad (i = 1, 2, \ldots, N),
\] (1.38)

and represent a canonical set, yielding the standard form of the HF equations. Since the HF operator \( \hat{f}(x) \) has a dependence on the spin orbitals \( \chi_i \) (eigenfunctions) through the Hatree potential \( V^{HF}(x) \) (Eq. 1.34), the HF equations actually are pseudo eigenvalue equations, and must be solved iteratively. Iteration is completely equivalent to finding the energy minimum that manifests as a 'fixed point' in the parlance of nonlinear dynamics. Starting from some trial orbitals one first constructs the Fock operator and then uses it to obtain improved orbitals which are the input for a new Fock operator. The iterative
procedure is continued until a self consistency is reached so that the effective field \( V^{\text{HF}}(x) \) no longer changes (i.e. remain within some specified criterion of convergence). The spin orbitals obtained as the solutions of the HF equations (Eq. 1.38) are the set of orthonormal orbitals \( \{ \chi_i \} \) with orbital energies \( \{ \epsilon_i \} \). The \( N \) orbitals \( \{ \chi_a \} \) with lowest energies \( \{ \epsilon_a \} \) form the occupied orbitals and the Slater determinant \( \left| \Psi^\text{HF}_0 \right\rangle = \left| \chi_1 \chi_2 \cdots \chi_a \cdots \chi_{N} \right\rangle \) formed from these orbitals is the best ground state single determinantal HF wave function in a variational sense. The remaining orbitals \( \{ \chi_r \} \) in the set \( \{ \chi_i \} \) with orbital energies \( \{ \epsilon_r \} \) are called the unoccupied or virtual orbitals.

In order solve HF equation explicit form to the spin orbitals viz. restricted and unrestricted spin orbitals have been given. Restricted spin orbitals constrained to have the same spatial wave function for \( \alpha \) (spin up) and \( \beta \) (spin down) spin functions, this formalism is restricted Hartree-Fock (RHF) theory. While unrestricted Hartree-Fock (UHF) theory based on unrestricted spin orbitals with different spatial wave functions for \( \alpha \) and \( \beta \) spin functions. The RHF can adequately describe the ground state of closed-shell systems, whereas UHF is crucial for open-shell systems including the excited states. We confine ourselves to the RHF formalism since we are concerned to optimizing ground state geometries of host, guest and their complexes. The restricted set of spin orbitals, therefore, has the form

\[
\chi_j(x) = \begin{cases} 
\psi_j(x) & \text{if } \alpha \text{ spin function} \\
\psi_j(x) & \text{if } \beta \text{ spin function}
\end{cases}
\]

Thus, RHF wave function \( \left| \psi^\text{HF}_0 \right\rangle = \left| \psi_1 \psi_2 \cdots \psi_a \psi_a \cdots \psi_{N/2} \psi_{N/2} \right\rangle \) for a closed-shell system of \( N \) electrons represents the ground state. Substituting the orbitals \( \psi_j(x) \) or \( \overline{\psi}_j(x) \) in the HF equations (Eq. 1.38) and summing over the spin functions lead to identical spatial equations, in terms of the spatial orbitals \( \{ \psi_j(x) \} \), called 'closed-shell' HF equations

\[
\hat{f}^{(\alpha)} \psi_j(\vec{r}) = \epsilon_j \psi_j(\vec{r}). \tag{1.40}
\]

The HF operator in Eq. (1.40) has a form

\[
\hat{f}^{(\alpha)} = \hat{h}(\vec{r}) + \sum_p^{N_S} \left( 2\hat{J}_p(\vec{r}) - \hat{K}_p(\vec{r}) \right), \tag{1.41}
\]
in which the operators \( \hat{J}_p(\vec{r}) \) and \( \hat{K}_p(\vec{r}) \) are defined as in Eqs. 1.35 and 1.36 with the spin orbitals replaced by their spatial parts. Here, the exchange operator \( \hat{K}_p(\vec{r}) \) is only for parallel spin orientations; the exchange energy obtained thereby is always \( \leq 0 \).

The HF equations (Eq. 1.40) are solved by introducing a finite set of spatial basis functions, \( \{ \varphi_{\mu} \} \), in terms of which the spatial molecular orbitals \( \{ \psi_i(\vec{r}) \} \) are expressed, i.e.,

\[
\psi_i(\vec{r}) = \sum_{\mu} C_{\mu i} \varphi_{\mu}(\vec{r}) \quad (i = 1, 2, \ldots, K) .
\] (1.42)

In the limit \( K \to \infty \), this would be an exact expansion for molecular orbitals giving the basis set \( \{ \varphi_{\mu} \} \) with infinite dimensional; however practical simplicity it must be restricted to a finite basis set. The spatial functions, \( \{ \varphi_{\mu}(\vec{r}) \} \), represent a fixed set of atomic orbitals, and frequently the symmetry governs which functions could be used. The problem of determining the HF molecular orbitals is reduced to the problem of calculating the set of expansion coefficients \( C_{\mu i} \). This scheme of obtaining molecular orbitals leads to the linear combination of atomic orbitals (LCAO) approximation; a more detailed discussion of this method will be presented later in Section 1.5. Substituting Eq. 1.42 in HF equations Eq. 1.40 yields a set of algebraic equations

\[
\sum_{\nu} F_{\mu \nu} C_{\nu i} = \epsilon_i \sum_{\nu} S_{\mu \nu} C_{\nu i} \quad (i = 1, 2, \ldots, K),
\] (1.43)

which in matrix notation incarnates as

\[
\mathbf{F} \mathbf{C} = \mathbf{S} \mathbf{C} \mathbf{\epsilon}
\] (1.44)

This method of transforming the HF equations (which actually are integro-differential equations) to a set of linear algebraic equations (matrix equations) was introduced by both Roothaan and Hall9 independently, and the matrix equation (Eq. 1.44) was thereafter termed the Roothaan-Hall equation. In Eq. 1.44, \( \mathbf{\epsilon} \) is a diagonal matrix of the orbital energies \( \epsilon_i \), \( \mathbf{C} \) is a matrix of expansion coefficients \( C_{\mu i} \), \( \mathbf{S} \) is an overlap matrix with elements

\[
S_{\mu \nu} = \int d^3 r \varphi_{\mu}^*(\vec{r}) \varphi_{\nu}(\vec{r}),
\]

and \( \mathbf{F} \) is the Fock matrix with elements

\[
F_{\mu \nu} = H_{\mu \nu}^{1\text{-body}} + G_{\mu \nu},
\] (1.45)

in which the hermitian 1-body Hamiltonian matrix is given in terms of its matrix elements by

\[
H_{\mu \nu}^{1\text{-body}} = \int d^3 r \varphi_{\mu}^*(\vec{r}) \hat{h}(\vec{r}) \varphi_{\nu}(\vec{r}).
\] (1.46)
This 1-body Hamiltonian matrix is required to be calculated only once, as it remains fixed throughout the iterative procedure for obtaining the orbitals. The elements in the second term in Eq. 1.45 contain two-electron integrals, and their evaluation is a major time-consuming part during the iteration. Explicitly

\[ G_{\mu\nu} = \sum_{ij} P_{ij}[(\mu\nu | \sigma\lambda) - \frac{1}{2} (\mu\lambda | \sigma\nu)] \]  

(1.47)

where \( P_{ij} \) are the expansion coefficients in the atomic orbital expansion of the charge density given by

\[ \rho(\vec{r}) = 2 \sum_{i}^{N/2} \Psi_i^*(\vec{r}) \Psi_i(\vec{r}) = \sum_{ij} P_{ij} \phi_i^*(\vec{r}) \phi_j(\vec{r}), \]  

(1.48)

in which the matrix \( P \) defined through

\[ P_{ij} = 2 \sum_{k}^{N/2} C_{ik} C_{kj}^* \]  

(1.49)

is the density matrix. The terms in the square bracket in Eq. 1.47 are the two-electron integrals over spatial orbitals defined by

\[ (\mu\nu | \sigma\lambda) = \int d^3r_1 \, d^3r_2 \, \phi_i^*(\vec{r}_1) \phi_j(\vec{r}_2) \, \phi_k^*(\vec{r}_1) \phi_l(\vec{r}_2). \]  

(1.50)

Several quantities of physical interest can be expressed in terms of these matrices. For instance, the HF ground state energy is expressible as

\[ E_0 = \frac{1}{2} \sum_{\mu} \sum_{\nu} P_{\mu\nu} (H_{1\text{-body}} + F_{\mu\nu}). \]  

(1.51)

Also the electric dipole moment,

\[ \vec{\mu} = - \sum_{\mu} \sum_{\nu} P_{\mu\nu} (\nu | \mu) + \sum_{A} Z_A \vec{R}_A, \]  

(1.52)

and the net effective charge on an atom,

\[ q_A = Z_A - \sum_{\mu \in A} (PS)_{\mu\mu}, \]  

(1.53)

in which \( (PS)_{\mu\mu} \) is interpreted as the “number of electrons” in \( \phi_\mu \). This scheme of associating charge on atoms is called the Mulliken population analysis.\(^{10}\) The Roothaan-Hall equations (Eq. 1.44) can further be simplified by transforming \( \{\phi_\mu\} \) to an orthonormal basis set \( \{\phi'_\mu\} \) in which matrix \( S \) becomes a unit matrix. The transformation is facilitated through
\[ \varphi'_\mu = \sum_v X_{\mu v} \varphi_v, \]  
(1.54)

where the transformation matrix \( X \) is obtained as \( X = S^{-1/2} \). In the orthonormal basis set \( \{ \varphi'_\mu \} \), the Roothaan-Hall equations (Eq. 1.44) become

\[ F' C' = C' \varepsilon \]  
(1.55)

Where \( F' = X^\dagger F X \) and \( C' = X^{-1} C \) are respectively the transformed Fock and the coefficient matrices. Diagonalizing \( F' \) one obtains \( \varepsilon \) and \( C' \). The HF approximation in practice makes use of these transformed Roothaan-Hall equations.

The HF approximation provides remarkable results in many cases, but has its limitations associated with its inadequacy in incorporating the intricate electron correlation beyond parallel-spin exchange. Many methods have been proposed to improve upon the HF approximation, e.g., Configuration Interaction (CI), Coupled cluster theory, Many-body perturbation theory etc., which are collectively termed the post-HF methods. We define the 'missing' correlation energy as the difference between the exact nonrelativistic ground state energy \( E_{\text{exact}} \) of the system and the HF ground state energy \( E_0^{\text{HF}} \),

\[ E_{\text{corr}} = E_{\text{exact}} - E_0^{\text{HF}}. \]  
(1.56)

In order to account for correlation energy one ought to go beyond the single Slater determinant description of the many-electron wave function.

After reviewing aspects of the wave function approach to handle the \( N \)-electron system, we now turn to the density functional theory (DFT) that treats charge density as a basic variable for the ground state description. This approach allows one to replace the complex \( N \)-electron wave function \( \Psi(x_1, x_2, \ldots, x_N) \) and the Schrödinger equation by the electron density \( \rho(\vec{r}) \), and the associated calculation scheme. To begin with, the historical Thomas-Fermi theory, a basis for the modern DFT of Hohenberg and Kohn, have been discussed.

**1.3 Density Functional Theory**

The idea of using electron density distribution as a basic variable to deal with many-electron system in ground state was proposed independently by Thomas and Fermi. They imagined the electrons in an atom as a Fermi gas, obeying Fermi-Dirac statistics and the Pauli exclusion principle. This statistical model was used in approximating the electron
distribution from which the electrostatic potential due to nucleus and electrons and the energy of atom were estimated by assuming the phase space distribution of electrons with maximally two electrons in every $h^3$ ($h=\text{Planck’s constant}$) of its volume and that there is an effective potential field $V_{\text{eff}}(\vec{r})$, practically unchanged over a volume $\Delta V$ in coordinate space, is determined from the electron charge distribution. In the uniform electron gas regime, the electrons move independent of each other, and their translational momenta can be taken to be directed isotropically in the momentum space. It is further assumed that electrons occupy the phase space as densely as possible consistent with the exclusion principle, that is, their kinetic energy is a minimum, which is consistent with working at the absolute zero temperature.

The total kinetic energy in atomic units is therefore given by

$$T_{\text{TF}}[\rho] = (3/10) \left(3\pi^2\right)^{2/3} \int \rho^{5/3}(\vec{r}) d^3r = C_F \int \rho^{5/3}(\vec{r}) d^3r$$

This method, called the local density approximation (LDA), first introduced for the kinetic energy functional is in fact one of the most significant ideas in modern DFT as a starting point. The total ground state energy is constructed by adding the classical electrostatic energies of electron-nuclear attraction and classical electron-electron repulsion (Hartree potential) in Eq. 1.66, as:

$$E_{\text{TF}}[\rho(\vec{r})] = C_F \int \rho^{5/3}(\vec{r}) d^3r - Z \int \frac{\rho(\vec{r})}{r} d^3r + \frac{1}{2} \int \int \frac{\rho(\vec{r}_1)\rho(\vec{r}_2)}{|\vec{r}_1 - \vec{r}_2|} d^3r_1 d^3r_2,$$

which is the energy functional in TF theory for atoms. The ground state electron density is then varied to minimize the energy functional, thereby facilitating the determination of electron density and the ground state energy by minimizing the energy functional.

Minimization of energy functional $E_{\text{TF}}[\rho(\vec{r})]$ by variational method such that:

$$\delta \left[ E_{\text{TF}}[\rho(\vec{r})] - \mu_{\text{TF}} \left( \int \rho(\vec{r}) d^3r - N \right) \right] = 0,$$

where the chemical potential,

$$\mu_{\text{TF}} = \frac{\delta E_{\text{TF}}[\rho(\vec{r})]}{\delta \rho(\vec{r})} = \frac{5}{3} C_F \rho(\vec{r})^{2/3} + V_{\text{eff}}(\vec{r}),$$

is introduced as a Lagrange multiplier with the constant particle number. $V_{\text{eff}}(\vec{r})$ is the effective electrostatic potential perceived by an electron at point $\vec{r}$ due to the nucleus (‘external potential’) and the entire electron distribution, i.e.,
TF theory provides a qualitatively good account of the total energies of atoms. The most serious defect in the TF theory is possibly it does not lead to bonding between atoms.\textsuperscript{15} Moreover its serious shortcomings (a) $\rho_{\text{TF}}$ has an infinite value at $r = 0$ and (b) it falls off as $\sim r^{-3}$ for neutral atoms; for positive ions $\rho(\vec{r})$ have a finite cut off for.

The basic problem that still remained unsolved was to overcome the difficulties in estimating the kinetic energy, and the nonclassical part (exchange and correlation) of the electron-electron interactions. The pioneering work of Hohenberg and Kohn (1964) as well as Kohn and Sham (1965) eliminated these discrepancies. Hohenberg and Kohn provided the fundamental theorems of existence of the exact energy functional $E[\rho(\vec{r})]$ and the exact variational principle so that the TF theory may be regarded as a crude approximation of the exact density functional theory (DFT). Kohn and Sham introduced the concept of a fictitious non-interacting system corresponding to the interacting system, with kinetic energy that can be computed accurately and is remarkably close to the kinetic energy of the interacting system.

DFT\textsuperscript{12,16-20} successfully provides qualitatively precise electronic structure of atoms, molecules, and solids. The quantum mechanical problems for given $N$ and $V_{\text{ext}}(\vec{r})$ have the wave function $\Psi$ as a basic variable. On the other hand, in DFT the electron density $\rho(\vec{r})$ can be used as basic variable for a ground state description. It is evident that the external potential $V_{\text{ext}}(\vec{r})$ and the total number of electrons $N$ both uniquely fix the Hamiltonian of the system, which then gives the ground state wave function $\Psi$, and the ground state energy via the variational principle, as is proved from the Hohenberg-Kohn theorems.

### 1.3.1 The Hohenberg-Kohn Theorems

The Hohenberg-Kohn (HK) theorems\textsuperscript{21} offer a one-to-one map between $V_{\text{ext}}(\vec{r})$ and the electron density $\rho(\vec{r})$ for a nondegenerate many-electron ground state, and assure the existence of an energy functional $E[\rho(\vec{r})]$ that reaches its minimum value for the exact density; establishing thereby a variational principle with $\rho(\vec{r})$ as the variable.
**Theorem I:** The nondegenerate ground state electron density $\rho(\vec{r})$ of a system of interacting electrons determines, up to an additive constant, the external, one-body multiplicative potential $V_{\text{ext}}(\vec{r})$ in which the electrons move and thus the Hamiltonian and hence all physical properties of the system at the nondegenerate ground state. Conversely, the one-body multiplicative external potential determines uniquely the ground state density.

**Theorem II:** Given an external potential $V_{\text{ext}}(\vec{r})$, the correct ground state density $\rho_0(\vec{r})$ minimizes the energy functional $E[\rho]$ i.e.,

$$E_0 = E[\rho_0] \leq E[\tilde{\rho}],$$

where $\tilde{\rho}(\vec{r})$ is the 'trial' density, $\rho_0(\vec{r})$ being the exact ground state one. The equality holds iff $\rho(\vec{r}) = \rho_0(\vec{r}) \forall \vec{r}$.

This theorem is once again founded upon the variational principle. The energy functional $E[\rho]$ in Eq. 1.80 is expressible as

$$E[\rho] = F_{\text{HK}}[\rho] + \int \rho(\vec{r}) V_{\text{ext}}(\vec{r}) d^3r,$$

where the HK functional $F_{\text{HK}}[\rho]$ is defined as the sum of kinetic energy functional $T[\rho]$ and the electron-electron energy functional $V_{\text{ee}}[\rho]$, i.e.,

$$F_{\text{HK}}[\rho] = \langle \Psi[\rho]|(\hat{T} + \hat{V}_{ee})|\Psi[\rho] \rangle = T[\rho] + V_{ee}[\rho].$$

The HK functional $F_{\text{HK}}[\rho]$ is a universal functional of density and has the same form for all electronic (bound-state) structure problems. However, the explicit density dependence is unknown and its determination is a challenging task in DFT.

These original HK theorems concern with nondegenerate $N$-electron ground state and $V$-representable density obtained from an $N$ electron antisymmetric ground state wave function associated with some external potential. Another class of densities can be obtained from antisymmetric wave function (need not be a ground state) associated with any multiplicative, one-body external potential $V_{\text{ext}}(\vec{r})$. These 'N-representable' densities satisfy the weaker requirements viz., (a) $\rho \geq 0$ (b) $\int \rho(\vec{r}) d^3r = N$ , and (c) $\int |\nabla \rho(\vec{r})|^2 d^3r < \infty$.

The conditions of nondegeneracy and $V$-representability have been lifted using an alternative formulation by Levy (modified HK-like theorems).
1.3.2 Levy-Constrained Search Definition of $F[\rho]$ and $E[\rho]$

The HK functional $F_{\text{HK}}[\rho]$ is defined only for densities those are $N$ and $V$-representable. This limitation can be overcome by Levy through constrained search formalism in which universal functional $F_L[\rho]$ (like HK) is defined for all $N$-representable $\rho(\vec{r})$ for the sum of kinetic and repulsion energies:

$$F_L[\rho] = \text{Min}_{\Psi_\rho \Rightarrow \rho} \langle \Psi_\rho | (\hat{T} + \hat{V}_{ee}) | \Psi_\rho \rangle.$$ \hspace{1cm} (1.92)

The functional $F_L[\rho]$ searches all antisymmetric many-electron wave functions $\Psi_\rho$ yielding the given $\rho(\vec{r})$ that need not be $V$-representable. $F_L[\rho]$ then gives minimum expectation value, among the different expectation values. For $F_L[\rho]$ to be a valid universal variational functional, it is necessary that

$$F_L[\rho]+ \int \rho(\vec{r}) V_{ee}(\vec{r}) \, d^3 r = E[\rho] \geq E_0, \text{ and}$$

$$F_L[\rho_0]+ \int \rho_0(\vec{r}) V_{ee}(\vec{r}) \, d^3 r = E[\rho_0] = E_0.$$ \hspace{1cm} (1.93a)

In the Levy constrained-search definition of $F_L[\rho]$, density $\rho$ need not be $V$-representable, since $F_{\text{HK}}[\rho] = F_L[\rho]$ it is $V$-representable. Furthermore, the degeneracy of $\Psi_0$ does not affect the proof.

1.3.3 The Kohn-Sham Formulation of DFT

After the generalization of the original HK theorems, the KS formalism introduced concept of orbitals in DFT such that the kinetic energy can be computed to a sufficiently good accuracy; a small residual correction here can be handled separately. Ultimately the KS idea dominates the present-day density functional calculations.

Although the HK theorem guarantees the existence of energy functional $E[\rho]$ that reaches its minimum for the correct density in accordance with the energy-functional variational principle, the theorem gives no explicit prescription for construction of its crucial ingredient, $F_{\text{HK}}[\rho]$, defined to be

$$F_{\text{HK}}[\rho] = T[\rho] + V_{ee}[\rho]$$

$$= T[\rho] + J[\rho] + (V_{ee}[\rho] - J[\rho]).$$ \hspace{1cm} (1.94)

Nonclassical $V_{ee}$
Here $J[\rho]$ represents the Coulomb repulsion energy among the electrons, and the electron-electron interaction term $(V_{ee}[\rho] - J[\rho])$ is the actual exchange-correlation energy that constitutes a major part of the KS ‘exchange-correlation energy’ $E_{XC}[\rho]$. The relation expressing kinetic energy explicitly in terms of the electron density is not yet established, however, it can be computed exactly from the many-particle wave function. Accordingly Kohn and Sham\textsuperscript{24} proposed a method wherein a imaginary system of $N$ noninteracting electrons with the same ground state density and the energy as the corresponding physical system is considered. A major portion of kinetic energy $T[\rho]$ of the physical system can be extracted as the kinetic energy $T_{S}[\rho]$ of the noninteracting system thereby introducing an auxiliary set of orbitals $\{\phi_{i}\}$. The kinetic energy $T_{S}[\rho]$ is then evaluated accurately as the sum of the orbital kinetic energies, sweeping the extra kinetic energy ingredient $(T[\rho] - T_{S}[\rho])$, into the total exchange-correlation energy of the system that incorporates non classical electron-electron interaction. That is

$$F_{ijk}[\rho] = T_{S}[\rho] + J[\rho] + \left(\frac{T[\rho] - T_{S}[\rho]}{E_{XC}[\rho]}\right)(V_{ee}[\rho] - J[\rho])$$

where $E_{XC}[\rho] = (T[\rho] - T_{S}[\rho]) + (V_{\omega}[\rho] - J[\rho])$ is the KS exchange-correlation energy. $T_{S}[\rho]$, the kinetic energy of the fictitious system has the explicit form

$$T_{S}[\rho] = \sum_{i=1}^{N} \langle \phi_{i} | (-\frac{1}{2} \nabla^{2}) | \phi_{i} \rangle,$$  \hspace{1cm} (1.96)

in terms of the orbitals. The orbitals $\{\phi_{i}\}$ satisfy equations analogous to the Hartree or Hartree-Fock equations given by

$$[-\frac{1}{2} \nabla^{2} + V_{\text{eff}}(\vec{r})] \phi_{i} = \varepsilon_{i} \phi_{i} \hspace{1cm} (i = 1, 2,\ldots, N)$$

The difference being $V_{\text{eff}}(\vec{r})$ is a local one-body potential and the sum of absolute squares of the lowest occupied $N$ orbitals yields the exact ground-state density

$$\rho(\vec{r}) = \sum_{i=1}^{N} \sum_{\omega} | \phi_{i}(\vec{r}, \omega) |^{2}.$$  \hspace{1cm} (1.98)

The effective potential $V_{\text{eff}}(\vec{r})$ in Eq. 1.97 is given by
\[ V_{\text{eff}}(\vec{r}) = V_{\text{ext}}(\vec{r}) + \frac{\delta J[\rho]}{\delta \rho(\vec{r})} + \frac{\delta E_{\text{XC}}[\rho]}{\delta \rho(\vec{r})} \]
\[ = V_{\text{ext}}(\vec{r}) + \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \ d^3 r' + V_{\text{XC}}(\vec{r}), \quad (1.99) \]

with the exchange correlation potential identified with the functional derivative:
\[ V_{\text{XC}}(\vec{r}) = \frac{\delta E_{\text{XC}}[\rho]}{\delta \rho(\vec{r})}. \quad (1.100) \]

Eqs. 1.97–1.99 constitute the KS equations. These equations are to be solved self-consistently for the orbitals that yield the desired density through Eq. 1.98. This is known as the KS scheme of DFT. The central idea is to construct the single-particle effective potential \( V_{\text{eff}}(\vec{r}) \) in such a way that the density of the fictitious non-interacting system equals the density of the physical system of interest. As in the HF theory, Eq. 1.97 can be easily derived within a variational context by minimizing \( E[\rho] \) as a functional of \( \phi_i \), subject to the orthonormality conditions on \( \{\phi_i\} \). The total energy can be determined from the density given by Eq. 1.98 through
\[ E = \sum_{i=1}^{N} \varepsilon_i - \frac{1}{2} \int \rho(\vec{r}) \rho(\vec{r}') \ d^3 r \ d^3 r' + E_{\text{XC}}[\rho] - \int V_{\text{XC}}(\vec{r}) \rho(\vec{r}) d^3 r, \quad (1.101) \]

in which the first term is the sum of the \( N \) lowest orbital energy eigenvalues for the 'canonical' Kohn-Sham theory, i.e.,
\[ \sum_{i=1}^{N} \varepsilon_i = \sum_{i=1}^{N} \langle \phi_i | \left( -\frac{1}{2} \nabla^2 + V_{\text{eff}}(\vec{r}) \right) | \phi_i \rangle \]
\[ = T_S[\rho] + \int V_{\text{eff}}(\vec{r}) \rho(\vec{r}) d^3 r. \quad (1.102) \]

To summarize; (a) The KS equations are one-electron equations, just as the Hartee or Hartree-Fock equations, to be solved iteratively until self consistency in \( V_{\text{eff}}(\vec{r}) \) is reached. The price to be paid for the incorporation of electron correlation is the appearance of the exchange correlation potential \( V_{\text{XC}}(\vec{r}) \) the exact form of which is unknown and for which no unique systematic strategy for improvement is available as an explicit functional of ground state density. The gradient expansion of exchange-correlation energy \( E_{\text{XC}} \) can be carried out but convergence of such an expansion is not guaranteed. (b) Neither the exact KS orbitals \( \{\phi_i\} \) nor the energies \( \{\varepsilon_i\} \) have any known, directly tangible meaning, except for (i) the connection (Eq. (1.98)) between the lowest occupied orbitals \( \{\phi_i\} \) yielding the
true, physical ground state density $\rho(\vec{r})$ as well as the total energy, and (ii) the fact that the magnitude of the highest occupied $e_i$, relative to the vacuum equals the exact ionization energy.\(^{25}\) (c) The essence of the KS scheme is to extract a major portion of actual kinetic energy $T[\rho]$ incarnating in the KS framework as $T_s[\rho]$ which is explicitly known in terms of the KS orbitals. The constrained-search definition of $T_s[\rho]$ dictates

$$
T_s[\rho] = \text{Min}_{\{\phi_i\}} \left[ \sum_{i=1}^{N} \sum_{\omega} \int \phi_i^*(\vec{r}, \omega) \left(-\frac{1}{2} \nabla^2\right) \phi_i(\vec{r}, \omega) \, d^3r \right] : \sum_{i=1}^{N} \sum_{\omega} |\phi_i(\vec{r}, \omega)|^2 = \rho(\vec{r}),
$$

where search is over all the KS 'wave functions' $\Psi^{\text{KS}}$, tacitly assumed to be single Slater determinants yielding at the given density $\rho(\vec{r})$. The second line in the above equation shows that the sum of orbital kinetic energies yields the noninteracting kinetic energy. The necessary condition must satisfy the $N$ orbitals so as to be the optimal orbitals for Eq. 1.103 is that they must be the eigenfunctions of the KS equations, Eq. 1.97.

The aforementioned constrained search definition of the kinetic energy functional provides an unprecedented elegant approach to determine the KS orbitals, and hence the KS estimate for the kinetic energy from a given density as established by Zhao and Parr.\(^{26,27}\) Their scheme of obtaining electronic wave function from a given density $\rho_0(\vec{r})$ involves minimization of $T_s[\rho]$ given by Eq. 1.103 with respect to noninteracting orbitals $\{\phi_i\}$ under the constraint that the density $\rho(\vec{r})$ associated with $\Psi^{\text{KS}}$ satisfies $\rho(\vec{r}) = \rho_0(\vec{r})$ for all $\vec{r}$, i.e., $\rho(\vec{r}) - \rho_0(\vec{r}) = 0$, or $\int \left[\rho(\vec{r}) - \rho_0(\vec{r})\right] d^3r = 0$; whence, the positive semi-definite functional

$$
C[\rho] = \iint \frac{[\rho(\vec{r}) - \rho_0(\vec{r})][\rho(\vec{r}') - \rho_0(\vec{r}')]}{2|\vec{r} - \vec{r}'|} \, d^3r \, d^3r',
$$

vanishes exclusively for the correct choice $\rho(\vec{r}) = \rho_0(\vec{r}) \forall \vec{r}$, with the constraint that the orbitals $\{\phi_i\}$ are orthonormal,

$$
\int \phi_i^*(\vec{r}) \phi_j(\vec{r}) \, d^3r = \delta_{ij}.
$$

We define a functional of $N$ orbitals as
where $\lambda$ and $\epsilon_{ij}$ are the Lagrange multipliers for the constraints of vanishing of $C[\rho]$ and the orthonormality of orbitals as expressed by Eq. 1.105. For $T_S[\rho]$ to be minimum it is necessary that $\delta \Omega[\{\varphi_i(\vec{r})\}] = 0$; or

$$\frac{\delta}{\delta \varphi_i(\vec{r})} \Omega[\{\varphi_i(\vec{r})\}] = 0. \tag{1.107}$$

Defining a 'local' quantity $\nu(\vec{r})$ vide

$$\nu(\vec{r}) = \int \frac{[\sum_i |\varphi_i(\vec{r}')]^2 - \rho_0(\vec{r}')] |\vec{r} - \vec{r}'| d^3r', \tag{1.108}$$

the minimization results into a set of $N$ equations

$$\left[-\frac{1}{2}\nabla^2 + \lambda \nu(\vec{r})\right] \varphi_i(\vec{r}) = \epsilon_i \varphi_i(\vec{r}) \quad (i = 1, 2, \ldots, N). \tag{1.109}$$

The orbitals minimizing $T_S[\rho]$ must satisfy these KS-type equations given by Zhao and Parr.\textsuperscript{26,27} It is clear from Eqs. 1.108 and 1.109 that the Lagrange multiplier $\lambda$ must go to infinity at the solution point ($\rho(\vec{r}) = \rho_0(\vec{r})$) because $\nu(\vec{r})$ goes to zero such that the product $\lambda \nu(\vec{r})$ remains finite and meaningful, converging on to a well-defined value for each $\vec{r}$. The effective potential is then expressed explicitly in terms of the density difference and the $\lambda$. In fact, in the limit $\lambda \to \infty$ the effective potential ($\lambda \nu(\vec{r})$) approaches the KS potential,\textsuperscript{28} i.e.,

$$\lim_{\lambda \to \infty} \lambda \nu(\vec{r}) = \nu_{\text{eff}}^{\text{KS}}(\vec{r}). \tag{1.110}$$

From this, the exchange-correlation potential $V_{\text{xc}}(\vec{r})$ for several atoms and molecules can be extracted by subtracting the external and the Hartree potentials.\textsuperscript{29,30} The KS orbitals, and hence the KS kinetic energy is determined from a given density by solving the KS-type equations for some large $\lambda$. The DFT described above is restricted to systems with spin independent potential $V_{\text{coul}}(\vec{r})$; incorporation of spin dependent potential $V(\vec{r}, \omega)$ leads to the spin density...
functional theory (SDFT)$^{12}$ that uses the partial densities of electrons with different spins $\omega = (\alpha, \beta)$ as independent variables, i.e.,

$$\rho^\omega(\vec{r}) = \sum_{i=1}^{N} |\varphi_i(\vec{r}, \omega)|^2,$$

rather than using the total density of Eq. 1.98. The single-particle orbitals $\{\varphi_i\}$ are the solutions of the KS equations (Eq. 1.97) and the effective potential in Eq. 1.99 now becomes spin-dependent

$$V_{\text{eff}}^\omega(\vec{r}, \rho^\omega) = V_{\text{ext}}(\vec{r}) + V(\vec{r}, [\rho^\omega]) + \int \frac{\rho^\alpha(\vec{r})}{|\vec{r} - \vec{r}'|} d^3r' + V_{\text{XC}}^\omega(\vec{r}, [\rho^\alpha, \rho^\beta]); \quad (1.112)$$

the exchange-correlation potential here is given by

$$V_{\text{XC}}^\omega(\vec{r}, [\rho^\alpha, \rho^\beta]) = \frac{\delta E_{\text{XC}}[\rho^\alpha, \rho^\beta]}{\delta \rho^\omega(\vec{r})}. \quad (1.113)$$

SDFT allows improving the exchange-correlation functional through its spin dependence.

### 1.3.4 Approximations to the Exchange-Correlation Energy $E_{\text{XC}}[\rho(\vec{r})]$  

For solving the KS equations, Eqs. 1.97–1.99, the exchange-correlation energy functional $E_{\text{XC}}[\rho]$ has to be explicitly supplied. DFT itself does not offer any direct and simple prescription for construction of the exchange-correlation functional. Finding accurate $E_{\text{XC}}[\rho]$ encountered great difficulties and thus it is greatest challenge in DFT. A simplest approximation to $E_{\text{XC}}[\rho]$, called the local density approximation is discussed first followed by approximations involving the gradient corrections and modified functionals with the exact HF exchange functional added.

#### A. Local Density Approximation (LDA)  

The key principle underlying this approximation is to calculate the exchange and correlation energies per particle, i.e., $\varepsilon_x(\rho(\vec{r}))$ and $\varepsilon_c(\rho(\vec{r}))$ of the homogeneous electron gas as a function of the density. These functions are then simply translated by introducing “$\vec{r}$” dependence through $\rho(\vec{r})$ (hence “local”) to estimate the exchange and correlation energies per particle of the inhomogeneous system. The exchange-correlation energy is then expressed as
Thus dependence of $\varepsilon_X(\rho(\vec{r}))$ and $\varepsilon_C(\rho(\vec{r}))$ on density is used locally at any point for the estimation of the exchange-correlation energy $E_{xc}$ of the inhomogeneous system. The LDA has proved to be remarkably accurate for a wide variety of systems although it is suitable only for systems with spatially slowly varying densities (real systems, however, have relatively large density gradients). The exchange part (for closed-shell system) had already been proposed by Dirac (1930)$^{31}$ as

$$\varepsilon_X(\rho(\vec{r})) = - C_X \rho(\vec{r})^{1/3}, \quad C_X = \frac{3}{4} \left( \frac{3}{\pi} \right)^{1/3}. \quad (1.115)$$

Thus

$$E_{xc}^{LDA} = - C_X \int \rho(\vec{r})^{4/3} d^3 r. \quad (1.116)$$

Long before the KS idea of LDA for $E_{xc}$, there existed a method due to Slater (1951)$^{32,33}$ for simplification to the complicated HF method, wherein the nonlocal Fock operator was replaced by an approximate local operator with the exchange potential given by

$$V_{xa}(\vec{r}) = - \frac{3}{2} \alpha \left( \frac{3}{\pi} \right)^{1/3}, \quad (1.117)$$

where the parameter $\alpha$ was set equal to unity in the original formulation. The Hartree-Fock-Slater (HFS) method of solving many electron problems with such an adjustable parameter is called the $X_a$ method. The HF equations with $X_a$ exchange potential are equivalent to the KS equations with the LDA exchange functional given by Eq. 1.116 and the correlation component ignored. However, the $X_a$ exchange potential obtained by Slater is not the same as the exchange potential obtained from the functional derivative of Eq. 1.116. The $X_a$ method may be regarded as density-functional scheme with the exchange-energy functional approximated by

$$E_{xa}^{LDA} = - \frac{9}{8} \alpha \left( \frac{3}{\pi} \right)^{1/3} \int \rho(\vec{r})^{4/3} d^3 r. \quad (1.118)$$
with the correlation component ignored. It has been found that \( \alpha = 2/3 \) gives slightly better total energies than the original Slater value, \( \alpha = 1 \). It is to be noted that \( E_{\text{LDA}}^X \) so described is suitable only for spin-compensated systems, while for spin-uncompensated systems such as open-shell atoms and molecules, spin dependency of \( E_X \) must be included. This leads to the local spin-density approximation (LSDA) for the exchange energy functional:

\[
E_{\text{LSDA}}^X [\rho^a, \rho^b] = 2^{1/3} C_X \left[ (\rho^a)^{4/3} + (\rho^b)^{4/3} \right] d^3 r.
\]  
(1.119)

Similar to \( E_{\text{LDA}}^X \), several local approximations were proposed to estimate the correlation component of \( E_{XC} \). These include the LSDA due to Perdew and Wang (PW, 1991)\textsuperscript{34,35} which is very closely related to the functional proposed by Vosko, Wilk, and Nusair (VWN)\textsuperscript{36} and that by Perdew and Zunger (PZ).\textsuperscript{37} The local part of correlation functional is given by

\[
E_{\text{C}}^{\text{LSDA}} [\rho^a, \rho^b] = \int \rho(\mathbf{r}) \varepsilon_{C}(r_S, \xi) \, d^3 r,
\]  
(1.120)

where \( \rho(\mathbf{r}) = \rho^a + \rho^b \), \( r_S = (3/(4\pi \rho))^{1/3} \) is the local Seitz radius, and \( \xi = (\rho^a - \rho^b)/\rho \) is relative spin polarization. The correlation energy per particle \( \varepsilon_{C}(r_S, \xi) \) proposed by VWN by fitting the random phase approximation\textsuperscript{38} solution for \( \varepsilon_{C}(r_S, \xi) \) to the uniform electron gas forms the VWN3 correlation functional\textsuperscript{36}

\[
\varepsilon_{C}(r_S, \xi) = \varepsilon_{C}(r_S, 0) + \alpha_{C}(r_S) \frac{f(\xi)}{f(0)} (1 - \xi^4)
\]

\[+ \left[ \varepsilon_{C}(r_S, 1) - \varepsilon_{C}(r_S, 0) \right] f(\xi) \xi^4,\]  
(1.121)

wherein \( f(\xi) = [(1 + \xi) \, ^{4/3} + (1 - \xi) \, ^{4/3}] - 2/(2^{4/3} - 2) \) and \( \alpha_{C}(r_S) = f^*(0)[\varepsilon_{C}(r_S, 1) - \varepsilon_{C}(r_S, 0)] \) with \( f(0) = 0, \ f(1) = 1, \) and \( f^*(0) = 1.709921 \). The other one which fits the Ceperly-Alder solution\textsuperscript{39} to the uniform electron gas makes up VWN5 functional.\textsuperscript{36}

Although LSDA provides reasonably good results for gross properties for the solid state, it does not yield satisfactory results for atoms and molecules where the density falls off rapidly (exponential decay) in the asymptotic regions. For more accurate results, one must go beyond LDA by incorporating density fluctuations. One such prescription is proffered by the generalized gradient approximation, as discussed below.
B. Generalized Gradient Approximation (GGA)

In improving the exchange-correlation functional, an important breakthrough was achieved by incorporating the density gradient $\nabla \rho$ in the functional,\(^{40}\) which led to the development method with a series of approximations called the gradient corrected or generalized gradient approximations (GGAs) method. The most widely used gradient correction (to the LDA) for exchange is that proposed empirically by Becke in 1988 with the prescription\(^{41}\)

$$\Delta E_{x}^{\text{BSS}}(\rho) = E_{x}^{\text{LDA}} - \beta \int \frac{\rho^{4/3} x^2}{(1 + 6 \beta x \sinh^{-1} x)} \, d^3r, \quad (1.122)$$

where $x = |\hat{\nabla} \rho|^1/3$. The parameter $\beta$ was chosen to fit the gradient corrected energies to the known exact exchange energies of inert gas atoms, and Becke determined it to be 0.0042 Hartrees corresponding to the best fit. The Becke’s gradient-corrected exchange-energy functional reproduces\(^{41}\) the exact asymptotic behavior of the exchange-hole potential of any closed-shell many-electron system;\(^{42}\)

$$\lim_{r \to \infty} U_x = -\frac{1}{r}, \quad (1.123)$$

where $U_x$ is the Coulomb potential due to exchange hole or exchange charge density $\rho_x(\vec{r}, \vec{r}')$ at reference point $\vec{r}'$, i.e.,

$$U_x = \int \frac{\rho_x(\vec{r}, \vec{r}')}{|\vec{r} - \vec{r}'|} \, d^3r', \quad (1.124)$$

and is related to the total exchange energy through

$$E_x(\rho) = \frac{1}{2} \int \rho(\vec{r}) U_x \, d^3r. \quad (1.125)$$

The exchange-hole density is defined (for spin saturated systems) as

$$\rho_x(\vec{r}, \vec{r}') = -\frac{|\rho_x(\vec{r}, \vec{r}')|^2}{\rho(\vec{r})}, \quad (1.126)$$

$\rho_x(\vec{r}, \vec{r}')$ in Eq. 1.126 is the first-order reduced density matrix. A detailed account of ‘exchange’-correlation hole density may be found in Refs. 12 and 16.

The GGA exchange functional due to Perdew and Wang (PY)\(^{43}\) has the form

$$E_x^{\text{PY}}(\rho) = -\frac{3}{4} \left( \frac{3}{\pi} \right)^{1/3} \int \rho^{4/3} F(s) \, d^3r, \quad (1.127)$$
where \( s = \left| \nabla \rho(\vec{r}) \right| / (2k_F \rho) \), a dimensionless parameter, is a scaled density gradient, and \( k_F = (3\pi^2 \rho)^{1/3} \), the magnitude of the local Fermi wave vector. The function \( F(s) \) emerges as \( F(s) = (1 + 1.296 s^2 + 14 s^4 + 0.2 s^6)^{1/15} \). The LDA corresponds to \( F(0) = 1 \).

Numbers of other GGA functionals for correlation and exchange have been developed of which the most popular are the Perdew86 correlation functional, the four-parameter Lee, Yang and Parr (LYP) correlation functional, and the Perdew-Wang (PW91) exchange- and correlation functional. The Perdew86 correlation functional was introduced to improve upon the GGA functional due to Langreth and Mehl:

\[
E_C^{\text{P86}} [\rho^a, \rho^\delta] = \int \rho(\vec{r}) \epsilon_c(\rho^a, \rho^\delta) \, d^3r + \int d^3r e^{-\Phi(\rho)} |\nabla \rho(\vec{r})|^2 / \rho^{4/3} \, d^3r, \tag{1.128}
\]

where \( \rho(\vec{r}) = \rho^a + \rho^\delta \) and \( \epsilon_c(\rho^a, \rho^\delta) \) is the correlation energy per particle. First term in Eq. 1.128 is the local part of correlation functional (Eq. 1.120). The function \( \Phi \) in the second term is

\[
\Phi = 1.745 \sqrt{7} [C(\infty) / C(\rho)] |\nabla \rho| / \rho^{3/6} \tag{1.129}
\]

in which the function

\[
C(\rho) = 0.001667 + \left( \frac{0.002568 + \alpha r_s + \beta r_s^2}{1 + \gamma r_s + \delta r_s^2 + 10^4 \beta r_s^4} \right) \tag{1.130}
\]

and \( r_s = (3/(4\pi \rho))^1/3 \) as before is the local Seitz radius. The constants in Eq. 1.130 are:

\( \alpha = 0.023266 \), \( \beta = 7.389 \times 10^{-6} \), \( \gamma = 8.723 \), and \( \delta = 0.472 \). The correlation energy per particle \( \epsilon_c(\rho^a, \rho^\delta) \) is taken from a parameterization of the Ceperley-Alder results. The parameter \( \sqrt{7} \) in Eq. 1.129 was chosen to fit the gradient corrected correlation energy to the exact correlation energy of the neon atom, which turned out to be 0.11 for the best fit.

The PW91 correlation functional is given by

\[
E_C^{\text{PW91}} [\rho^a, \rho^\delta] = \int \rho(\vec{r}) [\epsilon_c(r_s, \xi) + H(t, r_s, \xi)] \, d^3r, \tag{1.131}
\]

where \( t = |\nabla \rho(\vec{r})| / (2g k_s \rho) \) with \( g = [(1 + \xi)^{2/3} + (1 - \xi)^{2/3}] / 2 \) is a scaled density gradient and \( k_s = (4k_F / \pi)^{1/2} \) is the local screening wave vector. The function \( H \) in the second term equals \( H_0 + H_1 \), where
\[ H_0 = g^3 \frac{\beta^2}{2} \ln \left[ 1 + \frac{2a}{\beta} \frac{t^2}{1 + At^2 + At^4} \right], \quad (1.132) \]
\[ A = \frac{2a}{\beta} \exp \left[ -2a \frac{e_c(r_s, \xi)}{(g^3 \beta^2)} \right] - 1, \quad (1.133) \]
\[ H_1 = v[C(r_s) - C_s(0)] - 3C_s / 7] g^3 t^2 \exp \left[ -100 g^4 \frac{g_s^2}{g_s^2} t^2 \right] \quad (1.134) \]

with the constants, \( a = 0.09 \), \( \beta = v C_s(0) \), \( v = (16/\pi)(3\pi^2)^{1/3} \), \( C_s(0) = 0.004235 \), and \( C_s = -0.001667 \). The correlation energy per particle, i.e., \( e_c(r_s, \xi) \) in Eq. 1.128 is that due to VWN, \(^{36}\) (Eq. 1.121).

The LYP correlation functional\(^{45}\) constructed from the Colle-Salvetti correlation energy formula\(^{48}\) is widely used today. The LYP functional for closed-shell system is
\[ E_{C_{LYP}}^{C} = -a \int \frac{1}{1 + \rho^{1/3}} \left[ \rho + b \rho^{-2/3} [C_{C} \rho^{5/3} - 2t_w \right. \]
\[ + \left( \frac{1}{2} t_w + \frac{1}{18} \nabla^2 \rho \right) e^{-\rho^{3/3}} \right] d^3 r, \quad (1.134a) \]

which for an open-shell system is:
\[ E_{C_{LYP}}^{O} = -a \int \frac{\gamma(\tilde{r})}{1 + \rho^{1/3}} \left[ \rho + 2b \rho^{-5/3} [2^{2/3} C_{C} \rho^{8/3} + \rho^{8/3}] - \rho t_w \right. \]
\[ + \left( \frac{1}{2} \rho \nabla^2 \rho + \rho \nabla \rho + \frac{1}{18} (\nabla^2 \rho_a \rho_a \rho^2 \rho_a \nabla^2 \rho_a) \right) e^{-\rho^{3/3}} \right] d^3 r, \quad (1.134b) \]

where \( C_{C} = (3/10) (3\pi^2)^{2/3} = 2.871 \) is the Thomas-Fermi coefficient in the kinetic energy functional (Eq. 1.66)), \( t_w(\tilde{r}) = \frac{1}{8} \left[ \nabla \rho(\tilde{r}) \right]^2 / \rho(\tilde{r}) - \frac{1}{8} \nabla^2 \rho(\tilde{r}) \) is a local Weizsäcker kinetic-energy density, and \( \gamma(\tilde{r}) = 2 \left[ 1 - (\rho_a^2(\tilde{r}) + \rho_b^2(\tilde{r}) / \rho^2(\tilde{r}) \right] \). The four constants \( a = 0.04918, b = 0.132, c = 0.2533, \) and \( d = 0.349 \) introduced by Colle and Salvetti\(^{48}\) were obtained by a fitting of correlation energy calculated using only the Hartree-Fock orbitals for the helium atom.

Though the gradient corrected schemes brought significant improvement over LDA, the results were still far from the desired accuracy. The goal of obtaining highly accurate exchange-correlation functional is difficult, but further improvement over GGA can be achieved through the hybrid exchange-correlation functionals which are the combination of the exact HF exchange and the DFT exchange-correlation functionals. However, it must be mentioned here that attempts to construct a better GGA functional have faced a dilemma.\(^{49}\)

An enhanced gradient dependence\(^{50}\) of GGA functional improves atomization and total...
energies while the reduced gradient dependence deteriorates in solids\textsuperscript{51} these energies.\textsuperscript{52} Despite accuracy atomization and total energies in former case bond lengths are worsen, while lattice parameters and surface energies improved in reduced gradient dependence GGA functional. Perdew \textit{et al.}\textsuperscript{53} showed in very recent work that no GGA can accomplish both as the GGA and LDA have their artifacts rendering them wayward towards localization and delocalization, respectively. Thus, in favour of accurate exchange energies of atoms any GGA must strongly violate the gradient expansion for slowly varying densities over space.

With this, let us turn to the hybrid exchange-correlation functionals.

\section*{C. Hybrid Exchange-Correlation Functionals}

The DFT exchange-correlation functionals that employ the exact exchange functional of Hartree-Fock-Kohn-Sham (HFKS) method are referred to as the exchange-correlation hybrid functionals. The HF exchange, although 'exact' in a mathematical sense, is a very poor representation of the exchange effect in molecular systems. Becke\textsuperscript{54} found that better results are obtained for atoms and molecules, if a small fraction of exact exchange component is used in the DFT exchange-correlation functionals, and thus proposed the three parameters hybrid functional of the form

\[ E_{XC}^{\text{B3PW91}} = E_{X}^{\text{LSDA}} + a_0 (E_{X}^{\text{exact}} - E_{X}^{\text{LSDA}}) + a_x \Delta E_{X}^{\text{BS88}} + a_C \Delta E_{C}^{\text{PW91}}, \tag{1.135} \]

where \( E_{X}^{\text{exact}} \) is the exact exchange energy, \( \Delta E_{X}^{\text{BS88}} \) is his own gradient correction (to the LSDA) for exchange (Eq. 1.122), and \( \Delta E_{C}^{\text{PW91}} \) is the gradient correction for the correlation component due to Perdew and Wang (1991),\textsuperscript{35} as given in Eq. 1.135. The three semiempirical parameters \( a_0 = 0.20, \ a_x = 0.72 \) and \( a_C = 0.81 \) were determined by an appropriate fit of atomization energies, ionization potentials, proton affinities and the total atomic energies of the first row atoms of the G1 database.\textsuperscript{55} The other hybrid functional is \( E_{XC}^{\text{B3P86}} \) that uses correlation correction functional due to Perdew (1986),\textsuperscript{44} given by Eq. 1.128. These hybrid functionals did not find much use, but the idea was used in the construction of other hybrid functionals such as the B3LYP functional that uses same parameters as specified by Becke. The B3LYP functional is

\[ E_{XC}^{\text{B3LYP}} = \left( E_{X}^{\text{LSDA}} + E_{C}^{\text{VWN3}} \right) + a_0 (E_{X}^{\text{exact}} - E_{X}^{\text{LSDA}}) + a_x \Delta E_{X}^{\text{BS88}} + a_C (E_{C}^{\text{LYP}} - E_{C}^{\text{VWN3}}). \tag{1.136} \]

It uses VWN3 local spin density correlation functional\textsuperscript{36} and the four parameters LYP correlation functional\textsuperscript{45} presented in the preceding section. Today, B3LYP is probably the
most widely used exchange-correlation functional in chemistry, and the reason for the growing popularity of the DFT calculations.\textsuperscript{56} Throughout the DFT calculations presented in the thesis we have employed B3LYP prescription for the exchange-correlation functional. There exist, for example, Harbola-Sahni method of “local” exchange-correlation\textsuperscript{57} Perdew-Burke-Ernzerhof (PBE) generalized gradient approximation,\textsuperscript{58} the optimized effective potential (OEP) method,\textsuperscript{59,60} Janak-type\textsuperscript{61} theorem for OEP model,\textsuperscript{62} which we have not discussed despite their elegance. One is referred to recent literature for other significant developments.\textsuperscript{63-65}

In summary, DFT provides an economical alternative for treating molecules at correlated level of theory. However, disadvantage of DFT is the exchange-correlation functional, which cannot be derived rigorously from first principles. As of today, DFT remains an attractive alternative to more rigorous methods, such as CI and MP2.

1.4 Geometry Optimization

This section reviews some generic criteria for implementation of the SCF scheme in a molecular calculation. The very first is geometry optimizations which is uncomplicated for isolated single molecules but relatively difficult for a molecular complex or cluster, particularly weakly bound due to more or less flat potential energy surface sometimes with unclear features (maxima, minima, saddle etc.) with a number of different local minima strewn over it. The most important quantity characterizing the stable equilibrium structure of a molecular system is the total energy of the geometrical structure (configuration) that possesses the minimum possible value compared to its immediate neighbouring structures. Any change in the geometrical parameters (bond lengths, bond angles, dihedral angles) of system would shift energy of system from the minimum. The energy variations with small changes in its structure is specified by its multi-dimensional potential energy function, \( U (\vec{q}) \), where \( \vec{q} \) being \( n \)-dimensional generalized coordinate vector. The plot of this energy function engenders the potential energy surface (PES), each point on which represents a particular configuration with energy corresponding to its position on the PES.

The PES is described in terms of three types of critical points (CPs), viz., maxima, minima and saddle points. A maximum on PES is a point where small change in geometry of molecule with respect to any \( \vec{q} \) lead to decreased energy of system and the corresponding molecular geometry is unstable. On the other hand, a minimum on PES forms a basin a viewed from all directions and the corresponding molecular configuration is an equilibrium
structure. However, a minimum can be a 'local' when it is in some confined region of PES, and it can be the 'global' if it represents the lowest energy point on the entire PES. A saddle point is maximum in one direction and minimum in all other directions; chemically it represents a transition structure connecting the two equilibrium structures. These three CPs (maximum, minimum and saddle points) are schematically shown in Figure 1.1 for a potential energy surface depending on two variables \((q_1, q_2)\). At a CP, all the first derivatives of energy function with respect to the geometrical parameters vanish and thus suggest the generalized forces (negative of derivatives of energy) are zero at the CPs. The nature of CPs is characterized by a matrix, obtained of the second order derivatives of energy function with respect to the molecular coordinates, called the Hessian matrix, i.e.,

\[
H^{(K)}_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \bigg|_{\vec{q} = \vec{q}_K}
\]

(1.137)

at the reference point \(\vec{q}_K\). The CPs are designated by a notation \((\text{Rank}, \text{Signature}) \equiv (\omega, \sigma)\). The 'Rank' is the number of non-zero eigenvalues of the Hessian matrix, and the 'Signature' is the algebraic sum of the signs of the eigenvalues. A \((3, -3)\) CP signifies a local maximum because all its eigenvalues (slopes) are negative implying that the energy function is decreasing in all three directions from that point. Similarly, a \((3, +3)\) CP is a local minimum and \((3, +1)\) as well as \((3, -1)\) CPs are saddle points.

Figure 1.1. A schematic potential energy surface depending on \((q_1, q_2)\) with maxima, minima and saddle points.
Geometry optimization is a systematic procedure of locating a minimum on the PES of a given molecular framework with equilibrium configuration. It is desirable to optimize all parameters affecting the energy function of the system simultaneously, yielding a minimum energy structure. The efficiency of convergence of geometry optimization of a given structure toward a minimum depends on the optimization algorithms employed, which are either based on semi-empirical methods (AM1, PM3, MNDO etc.) or *ab initio* (HF, MPPT, CI, DFT etc.) methods. A very first, the energy and the energy gradient of the given molecular structure, at some point on its PES, are computed. The gradient indicates the direction in which the energy decreases most rapidly from the current point and also the slope of PES at that point. Thus the calculated energy gradient determines in which direction to make a next move. Most of the algorithms compute the second order derivatives of the energy function with respect to the molecular coordinates which constitute the Hessian matrix, or the force constant matrix. The computed second derivatives specify the curvature of PES at that point. It then becomes straightforward to judge whether the current point is a minimum or a transition state. The stationary minimum is distinguished by the Hessian with all positive eigenvalues (curvatures). However, it must be mentioned here that there is no guarantee that the minimum obtained after optimization will necessarily be the global minimum; it could just as well be equally a local minimum. This is quite common in systems involving weak molecular interactions, for example, H-bonded complexes, which have primarily a flat PES with indiscernible features (max, min, saddle etc.) on their PES. Locating global minimum is, in fact, an ultimate goal of all optimization methods, which for clusters, in particular, required a set of good input 'guess' geometries that are usually obtained from some semi-empirical calculations.

The convergence criteria of the optimization algorithms are usually defined in terms of forces, displacements and their rms values with threshold and percentage tolerance, as least as possible. It is desirable that the forces be essentially zero. Once the stationary point is found, several quantities of interest can be calculated. For example the characteristic vibrational spectrum of a molecular system can be calculated from its Hessian matrix at the stationary point. If all frequencies are real, then the stationary point is minimum (local or global) but if one or more vibrational frequencies are imaginary, then the stationary point are first or higher order saddle points. The derivative properties like polarizabilities and hyperpolarizabilities, which depend on the second and higher derivative of energy function with respect to electric fields, are also routinely calculated. For a given level of theory, the
accuracy of the calculated geometrical parameters, energies, frequencies and all other properties depend on the basis set used.

1.5 Basis set

In order to solve the spatial HF equation molecular orbitals (MO) \{\Psi_i(\vec{r})\} are constructed from a set of atomic orbitals (AO) \{\phi_m\}. These MOs are expanded as linear combinations of AOs. In the early days, Slater type orbitals\(^{71}\) (STOs) were used to mimic AOs since it was not possible to obtain the exact AOs for many-electron atoms.\(^{72,73}\) The computation of two- and other multi-center integrals using STOs is quite difficulty, to overcome these issue Boys\(^{74}\) suggested the use of standard Gaussian functions,\(^{75}\) called Gaussian type orbitals (GTO), centered on atoms. A Cartesian Gaussian function used in electronic structure calculations has the form:

\[
\phi_{p}^{GF} = x^n y^m z^l \exp(-\alpha |\vec{r}|^2)
\]  

(1.138)

where, \(\alpha\) is the orbital exponent and \(l, m, n\) are the powers of Cartesian components \(x, y\) and \(z\) respectively. The center of a Gaussian function is denoted by \(\vec{r}_A = (x_A, y_A, z_A)\). These Gaussian functionas are called as primitive Gaussian functions \(\phi_{p}^{GF}\). The product of two GTOs is also a Gaussian function centered at the weighted midpoint of the two functions known as Gaussian product theorem.\(^{76}\) Thus it primarily offer the computational advantage of GTOs over STOs, along with the ability of analytically evaluation of the resulting integrals. It is a common practice\(^{77}\) to bunch together a set of GTOs with fixed coefficients for the sake of computational convenience. Thus computationally efficient basis functions will have the form

\[
\phi_{\mu}^{CGF}(\alpha, \vec{r} - \vec{R}_A) = \sum_{p=1}^{L} d_{p\mu} \phi_{p}^{GF}(\alpha_{p\mu}, \vec{r} - \vec{R}_A) 
\]  

(1.139)

where \(L\) is the number of primitive Gaussian functions \(\phi_{p}^{GF}\) used in constructing the basis function \(\phi_{\mu}\), engender a contracted Gaussian function (CGF) \(\phi_{\mu}^{CGF}\). The proper choice of contraction “length” (\(L\)), contraction coefficients \((d_{p\mu})\), and orbital exponents \((\alpha_{p\mu})\), the basis functions \(\phi_{\mu}^{CGF}\) leads to fit the desired atomic orbitals (Slater-type-orbitals).
The minimal basis set uses minimum number of basis functions is the STO basis set labelled as STO–nG. The STO–3G basis set uses three primitive Gaussian functions \( \varphi_p^{GF} \) (3G) per basis functions \( \varphi^{CGF} \) to represent each orbital for each shell whether full or partially filled. Systematic improvement of this basis set involves increased number of basis functions per orbital in each shell and also with different orbital exponents \( (\zeta) \), followed by inclusion of polarized and diffuse functions. The most frequently used Pople’s split valence (SV) basis sets, for the elements from Li to Ne those do not containing d orbital electrons, such as 3–21G, 6–31G and 6–311G, respectively use 3, 6 and 6 (the very first digits) \( \varphi_p^{GF} \) to describe the inner orbital, the second digits 2, 3 and 3 \( \varphi_p^{CF} \) to model the inner and third digit 1 in each basis sets represents outer valence shell orbitals. The fourth digit (i.e. 1) in the 6–311G basis set adds one more \( \varphi_p^{GF} \) to the ‘middle’ valence shell orbitals. The double zeta (DZ) and triple zeta (TZ) basis sets are similar to the minimal basis sets but assign two and three primitive Gaussian functions of different orbital exponents \( (\zeta) \) to each orbital. SV basis sets allow orbitals to change size, but not the shape and thus for flexibility, polarized and diffusion functions must be added to the SV basis sets. It must be noted that an STO expressed to be represented by linear combination of a finite number of Gaussian functions will never mimic the cusp condition or the exponential asymptotic decay. Thus one has to use an optimal number of Gaussians, i.e., an optimal basis.

1.6 Quantum Theory of Atoms in Molecules

Bader’s quantum theory of atoms in molecules (QTAIM or simply AIM) exploits the topographical features of electronic charge distribution and defines a chemical bond, based on the existence of a bond critical point, an extremum of electron density function, linking the nuclei of neighboring atoms through a special density gradient path (the bond path). AIM theory is now a well-established tool for the description of various bonding interactions, including the conventional and unconventional hydrogen bonding, and provides useful information about the structure and stability of molecules, reaction mechanism, and many other issues of interest in chemical physics.
1.6.1 Molecular Electron Density

We have known about atoms, their constituents, and also molecules and molecular clusters, but how meaningful would it be to talk of ‘atoms in molecules’? The answer is in the affirmative: “An atom here, in general, is not the same as that in a molecule or in molecular clusters”. A mathematical structure of this statement constitutes the theory of AIM, pioneered by Bader et al., which offers partitioning of a molecule into atomic volumes on the basis of topographical analysis of electron density. The density gradients, further, endow with the description of bonding between the atoms, structure and stability of molecules, reaction mechanism, and so on. The key elements in AIM are electron density $\rho(\hat{r})$, and its gradient $\nabla\rho$ and Laplacian $\nabla^2\rho$; all these together characterize the topography of $\rho(\hat{r})$. The topographical features of charge density, a scalar quantity defined at each point in the space occupied by atoms and molecules, are suitably described by $\nabla\rho$ and $\nabla^2\rho$. For a given molecular structure, the electronic charge distribution is defined by the nuclear charges localized in space. In the vicinity of nuclei, the electron density $\rho(0)$ exhibits a local maximum. The cusp in the electron density in the neighbourhood of nuclei is interrelated to nuclear charge $Z_A$ by

$$\left[\nabla\rho\right]_{\hat{r}_a=\hat{0}} = -2Z_A[\rho]_{\hat{r}_a=\hat{0}} \hat{n}_a \quad (1.140)$$

where $\hat{n}$ is a unit vector. The ground state electron density decreases steadily as we go away from the nuclei and it decreases exponentially at sufficiently large distance from the nuclei, i.e.,

$$\rho(\hat{r}) \sim \exp[-(2\varepsilon)^{1/2}\hat{r}], \quad (1.141)$$

where $\varepsilon$ is the first ionization energy of the system. The density gradient $\nabla\rho$ points in the direction of steepest increase in density and always perpendicular to a constant electron density surface is very crucial in the description of topology of electron distribution. Successive alignment of infinitesimal gradient vectors in space traces loci (curves) called the gradient paths (‘trajectories’ of $\nabla\rho$). These gradient paths end at a point called an attractor and the collection of gradients paths in space that ends at each nucleus is known as an atomic basin. An atom in AIM theory is thus defined as the union of an attractor (nucleus) and its associated basin (the portion of space allocated to an atom, i.e., atomic volume). A molecule can thus be uniquely divided into a set of atomic volumes bounded by
Figure 1.2. Map of the gradient vector field of the charge density $\rho(\vec{r})$ on HF molecular plane. Lines represent a trajectory of $\vec{\nabla}\rho(\vec{r})$. The ‘blue color’ gradient paths which originate at (3, −1) CP define bond path. The ‘red color’ gradient paths terminate at (3,−1) CP and represent the intersection of zero flux surface with molecular plane.

surfaces through which the gradient vector field $\vec{\nabla}\rho$ have no flux. Thus an atom in AIM can be defined as a region of space bounded by surface of zero flux in the gradient vector field $\vec{\nabla}\rho$, i.e.,

$$\vec{\nabla}\rho(\vec{r}) \cdot \hat{n}(\vec{r}) = 0, \text{ for every point on the bounding surface,}$$

(1.142)

where $\hat{n}(\vec{r})$ is a unit vector normal to the surface at point $\vec{r}$. The necessary condition required to satisfy Eq. 1.142 is that the atomic surface should not be crossed by any gradients paths. The atomic properties such as atomic charge, dipole moment, energies, etc. are calculated by integrating operators in space enclosed by the atomic surface (atomic volume). One more important aspect in AIM theory is the definition of a bond; two atoms are bonded if their atomic volumes share a common inter-atomic surface with a (3,−1) criticality at a bond critical point (BCP). BCP is a critical point, common to both atomic surfaces, from which two gradient paths, shown in ‘blue color’ lines (bond path) in Figure 1.2 for HF molecule, start and terminate one at each nucleus of the bonding atoms. Along a line, linking two bonded nuclei, the electron density is maximum with respect to the neighboring lines. Thus AIM theory provides a simple definition of an atom and a bond in a molecule.
The \((3, -1)\) critical point has originated from the Hessian matrix \((H)\) of charge density \(\rho(\vec{r})\). This is a real \(3 \times 3\) symmetric matrix formed by the nine components of second derivatives of \(\rho(\vec{r})\), which describes the curvature of \(\rho(\vec{r})\) at a point in space, i.e.,

\[
H_{ij}^{(C)} = \left. \frac{\partial^2 \rho}{\partial x_i \partial x_j} \right|_{\vec{r} = \vec{r}_c} \quad (i = 1, 2, 3)
\]

(1.143)

at the critical point, \(\vec{r}_c\). After carrying out diagonalization of the Hessian, the diagonal elements form the eigenvalues of the Hessian matrix. The trace of Hessian matrix (sum of eigenvalues) is the Laplacian of \(\rho(\vec{r})\), and the rank of a CP, denoted by \(\omega\), is the number of non-zero eigenvalues (i.e. curvatures) of \(\rho(\vec{r})\). The signature \((\sigma)\) is the sum of the signs of the eigenvalues at the CP. The CP is then described by \((\omega, \sigma)\). Thus \((3, -1)\) CP is defined as a CP at which two of the curvatures are negative, while the other one is positive, i.e., \((3, -1)\) CP represents a saddle point at the critical point, \(\vec{r}_c\). The other three possible CPs of rank three are, \((3, -3)\), \((3, +1)\) and \((3, +3)\). At \((3, -3)\) all three curvatures are negative and thus \(\rho(\vec{r})\) is maximum at the CP, \(\vec{r}_c\). Analogously \((3, +3)\) and \((3, +1)\) CPs represents, respectively, minima in \(\rho(\vec{r})\) and a saddle point with two positive curvatures and one negative.

The nature of BCP given by \((\omega, \sigma)\), the density value \(\rho(\vec{r}_c)\) and the Laplacian density \(\nabla^2 \rho(\vec{r}_c)\) at the critical point \(\vec{r}_c\) all together characterize the nature of bonds. The electron density at the BCP i.e. \(\rho(\vec{r}_c)\) is a measure of bond strength, while the bond ‘strain’ can be gauged by examining the deviation of BCP from the inter-atomic axis joining the nuclei of the two atoms; a large deviation implies larger bond strain from the reference geometry.

The magnitude of \(\rho(\vec{r})\) and the sign of \(\nabla^2 \rho\) at BCP both characterize the atomic interactions, namely, (a) covalent bonding (shared interactions), and (b) ionic, hydrogen and van der Waals bonding (closed-shell interactions). In covalent bonding (when electron density is shared by both the nuclei); the density Laplacian at bcp is negative, i.e., \(\nabla^2 \rho(\vec{r}_c) < 0\), and both its magnitude and density \(\rho(\vec{r}_c)\) have appreciable values. While in closed-shell interactions \(\rho(\vec{r}_c)\) is relatively low and \(\nabla^2 \rho(\vec{r}_c) > 0\). The \((3, -1)\) bond criticality is common to both the interactions. The AIM based criteria for the existence of hydrogen bond have been proposed.\textsuperscript{80,85}
1.6.2 Molecular Electrostatic Potential

In this section the molecular electrostatic potential (MESP) and its importance have been discussed briefly. The molecular electrostatic potential \( V(\vec{r}) \) is also a scalar quantity (as \( \rho(\vec{r}) \) do), and has finite values in the region of space occupied by a molecule barring the nuclei. If a molecule consisting of \( M \) atoms with atomic number \( Z_A \), the MESP at point \( \vec{r} \) is given by

\[
V(\vec{r}) = \sum_{A=1}^{M} \frac{Z_A}{|\vec{R}_A - \vec{r}|} - \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d^1r',
\]

(1.144)

where \( \vec{R}_A \) is the position vector of nucleus \( A \). An important feature of MESP is that it can be positive or negative, depending on the dominating term in Eq. 1.144 at the point of interest. MESP is positive near the nuclei due to preponderance of the nuclear potential (first term) over the electronic potential while in the regions of lone pairs or \( \pi \) bonds MESP is negative indicating the electron-rich region of the molecules. So the magnitude and sign of MESP together provide insight into electrophilic and nucleophilic sites in molecular system. Study of MESP is important in understanding number of phenomena such as molecular reactivities, intermolecular interactions, and solvent effects.\(^{86,89}\) The topographical information of MESP is important in recognition of molecules.\(^{86}\)

The topography of molecular electrostatic potential can be characterized in terms of critical point \( \text{viz.} \) minima, saddle and maxima.\(^{90-93}\) MESP brings out electron rich regions in the form of \( (3, +3) \) minima \( \text{vis-à-vis} \) lone-pairs of electrons and \( \pi \)-bonds exhibited as minima at the corresponding position near the van der Waals surface. Similar to the MED topography, these BCPs bring out the strain in bonds by deviating from the line joining the corresponding atoms. However, unlike the BCPs of MED, the MESP bond critical points of strained rings and cages lie within the ring/cage. The significance of the negative-valued MESP and their critical points in molecules and their anionic species have been widely dealt in the literature.\(^{94-97}\)

1.7 Population Analysis

A molecular polarity with charge build up or depletion on the individual atoms is very old concept in chemistry. The driving force for this concept is that it allows one to conveniently ignore the wave character of the electrons and deal with the more particulate atoms. These atoms reflecting electronic distribution by the degree to which they carry positive or
negative charge. The well-defined quantum mechanical operators are able to compute partial atomic charges, however none of procedure for computing partial atomic charges is accepted universally best. This is because the partial atomic charges are used in various ways for the individual models in chemistry. Thus, out of many methodologies promulgate for computing partial charges a few are discussed here.

Mulliken98,99 proposed a direct partitioning of the molecular wave function into atomic contributions following some arbitrary orbital-based scheme; this method of population analysis known as Mulliken population analysis. Here, the electrons are divided amongst the atoms depending on the contribution of the different atomic AO basis functions to the overall wavefunction. The total number of electrons is given by expression

$$N = \sum_{j}^{e l e c t r o n s} |\psi_{j}(r_{j})\psi_{j}(r_{j})|dr_{j}$$  \hspace{1cm} (1.145)

when expanded into its AO basis set

$$N = \sum_{j}^{e l e c t r o n s} \left( \sum_{r,s}^{c_{j,r}} \phi_{r}(r_{j})c_{j,s}\phi_{s}(r_{j})|dr_{j} \right)$$

$$= \sum_{j}^{e l e c t r o n s} \left( \sum_{r}^{c_{j,r}} c_{j,r}^{2} + \sum_{r,s}^{c_{j,r}} c_{j,r}c_{j,s}S_{rs} \right)$$  \hspace{1cm} (1.146)

where 'r' and 's' index AO basis functions $\phi$, $C_{j,r}$ is the coefficient of basis function 'r' in MO 'j', and $S_{r,s}$ is the overlap matrix element. Thus the total number of electrons are divided into two sums, one including only squares of single AO basis functions, and the other including products of two different AO basis functions. Electrons associated with only a single basis function belongs entirely to the atom on which that basis function is centred. Electrons shared between basis functions may be divided evenly between the two atoms on which basis functions 'r' and 's' reside. Thus the atomic population $N_{k}$ can be calculated as

$$N_{k} = \sum_{j}^{e l e c t r o n s} \left( \sum_{r,k}^{c_{j,r}} c_{j,r}^{2} + \sum_{r,k,r,e}^{c_{j,r}c_{j,s}S_{rs}} \right)$$  \hspace{1cm} (1.147)

When 'r' and 's' are different AO basis function both residing on 'k' drop out, since the orthonormality, of the atomic basis functions on a single centre, makes all the corresponding overlap integrals zero. The Mulliken atomic charge is then defined as

$$q_{k} = Z_{k} - N_{k}$$  \hspace{1cm} (1.148)
where, $Z_k$ is the nuclear charge of $k^{th}$ atom.

Mulliken charges estimated using minimal or small split valence basis sets tend to be reasonably intuitive. Mulliken partial charges are sensitive to the basis set size and thus limiting comparison of the charges within the same levels of theory. Furthermore use of very complete basis sets may engender unphysically large Mulliken charges. The best use of Mulliken charges is to analyze the variations in atomic charge as function of substitution or geometric change tends.

One of the observables from which charges are derived is the electrostatic potential (ESP). The molecule-molecule interactions can be modeled through partial atomic charges calculated using ESP. All ESP charge-fitting schemes involve determining atomic partial charges $q_k$, when used as a multipole expansion according to

$$V_{ESP}(r) = \sum_{\text{nuclei}} \frac{q_k}{|r-r_k|}, \quad \text{(1.149)}$$

minimize the difference between $V_{ESP}(r)$ and the correct $V_{MEP}(r)$ calculated as

$$V_{MEP}(r) = \sum_{\text{nuclei}} \frac{Z_k}{|r-r_k|} \int \frac{\psi(r')^2}{|r-r'|} dr'.$$

Typical algorithms select a large number of evenly spaced points on a cubic grid nearby the van der Waals surface of the molecule.

The idea of unimpaired validity is central to any population analysis, such that divide a molecule into atoms for its better description and examining how it differ from the free atom. For a straightforward dissection of molecular charge, the sophisticated partitioning scheme by Politzer and Harris has two advantages: (i) It makes the charge on each atom an unambiguous property of the molecular charge distribution, independent of the mathematical formalism used to derive this distribution. (ii) When applied to the promolecule, made up of overlapping ground state atoms prior to any charge migration between or within these atoms, it yields neutral atomic charges. The atomic fragmentation by Politzer and Harris has some shortcoming as they are bounded fairly artificially to an array of partition planes. While yielding acceptable atomic charges, they become ill-suited to the definition of other properties needed for a more complete specification of the molecular charge distribution. These shortcomings have been rectified using Hirshfeld partitioning scheme. In the promolecule here, the total density is a sum of well-defined contributions from all the constituent atoms. The atomic composition of the promolecule...
density \( \rho^{\text{pro}}(r) \) is mimicked while allocating the actual molecular density among the atoms. Therefore, the molecular density at each point is divided among the atoms of the molecule in proportion to their respective contributions to the promolecule density at that point ‘r’ is given by

\[
\rho^{\text{pro}}(r) = \sum_i \rho^{\text{at}}_i(r)
\]  

(1.151)

where the functions \( \rho^{\text{at}}_i(r) \) are properly located spherically averaged ground state atomic densities. For each atom a sharing function defined as

\[
w_i(r) = \frac{\rho^{\text{at}}_i(r)}{\rho^{\text{pro}}(r)}
\]  

(1.152)

specifies its relative share in the promolecule density at ‘r’. Functions \( w_i(r) \) are all positive and their sum equals one everywhere. The charge density of the bonded atom \( i \) is given by

\[
\rho^{\text{b.a.}}_i(r) = w_i(r) \rho^{\text{mol}}(r)
\]  

(1.153)

has \( \rho^{\text{mol}}(r) \) as the actual molecular density. Thus overlapping continuous bonded-atom distributions thus obtained have similar advantages as that of Politzer and Harris partitioning scheme. Furthermore, it produces well-defined atomic fragments that differ from the free atoms only to the degree that the molecule itself differs from a superposition of free atom densities.

The atomic deformation density then can be obtained by subtracting the density of the free atom from that of the bonded atom as

\[
\delta \rho_i(r) = \rho^{\text{b.a.}}_i(r) - \rho^{\text{at}}_i(r)
\]  

(1.154)

Now, the total electronic charge in our bonded atom is given by

\[
Q_i = -\int \rho^{\text{b.a.}}_i(r) dv
\]  

(1.155)

Adding the nuclear charge \( Z_i \) gives the net atomic charge

\[
q_i = Q_i + Z_i
\]  

(1.156)

In practice the integrand \( \rho^{\text{b.a.}}_i(r) \) varies too steeply for easy numerical integration; thus it is generally more convenient to integrate the atomic deformation density, which yields directly

\[
q_i = -\int \delta \rho_i(r) dv ,
\]  

(1.157)

a net atomic charge on \( i^{th} \) atom.
1.8 Natural Bond Orbital Analysis

Weinhold and co-workers\textsuperscript{101} proposed the scheme for natural atomic orbital (NAO) and natural bond orbital (NBO) analysis, where the one-electron density matrix was utilized to defining the shape of the atomic orbitals in the molecular environment, and thus derive molecular bonds from electron density between atoms.

If $\chi_A, \chi_B, \chi_C, \ldots$ represents the basis functions arranged for all the orbitals on the center $A$, $B$, $C$, etc., then the density matrix can be written in terms of blocks of basis functions belonging to specific center as

\[
D = \begin{pmatrix}
D_{AA} & D_{AB} & D_{AC} & \cdots \\
D_{AB} & D_{BB} & D_{BC} & \cdots \\
D_{AC} & D_{BC} & D_{CC} & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

(1.158)

The NAO for atom $A$ in the molecular environment may be defined as those which diagonalize the $D_{AA}$ block, NAOs for the atom $B$ as those which diagonalize the $D_{BB}$ block and so on. These NAOs will in general not be orthogonal, and the orbital occupation numbers will therefore not sum to the total number of electrons, and thus to achieve a well-defined division of the electrons, the orbitals are necessarily orthogonalized.

The NAOs, normally resemble the pure AOs, may be divided into a “natural minimal basis” (corresponding to the occupied AOs for an isolated atom), and a remaining set of natural “Rydberg” orbitals based on the magnitude of occupation numbers. In general, the natural minimal basis NAOs are strongly occupied whereas the Rydberg NAOs are weakly occupied. The number of NAOs is equal to size of the atomic basis set, and the number of Rydberg NAOs increases as the basis set is enlarged. Thus the orthogonalization procedure preserves the form of the strongly occupied orbitals as much as possible by using the occupancy-weighted orthogonalizing matrix. The diagonal elements of the density matrix in the basis give the orbital populations. Sum of all contributions from orbitals on specific atomic center produces the atomic charge. Natural minimal NAOs contribute most of the electron density and thus give a compact representation of the wave function in terms of atomic orbitals. Further since the NAOs are defined from the density matrix, it promises a convergence to well defined values as the size of the basis set is increased.
1.9 Vibrational Frequencies

Within the framework of Born-Oppenheimer approximation, the total electronic energy $E_e$ for fixed nuclear coordinates (nuclear spins are ignored)

$$X = (R_1, R_2, \ldots, R_M) = (R_{1x}, R_{1y}, R_{1z}, R_{2x}, R_{2y}, R_{2z}, \ldots, R_{Mx}, R_{My}, R_{Mz})$$

is given by

$$E = E_e + \frac{1}{2} \sum_{k,l=1}^{M} \frac{Z_i Z_j}{|R_k - R_l|}$$

$$E_{tot}(\{\tilde{R}_A\}) = E_e + \sum_{a=1}^{M} \sum_{b \neq a}^{M} \frac{Z_A Z_B}{R_{AB}}.$$  (1.160)

The total energy is a function of $\tilde{X}$,

$$E = E(X) = E(R_{1x}, R_{1y}, R_{1z}, R_{2x}, R_{2y}, R_{2z}, \ldots, R_{My}, R_{Mx}, R_{Mz})$$

For the optimized structure $X^e$, $E$ has a minimum, and by expanding $E$ to second order about the minimum one finds

$$E(X) \approx E(X^e) + \frac{1}{2} \sum_{k,l=1}^{M} \sum_{a_i, a_j} \frac{\partial^2 E(X^e)}{\partial R_{k_{a_i}} \partial R_{k_{a_j}}} (R^e_{k_{a_i}} - R^e_{k_{a_j}})(R^e_{k_{a_j}} - R^e_{k_{a_j}})$$

the Taylor series. The Eq. 1.162 thus obtained, in principal, may include the terms of higher order. The matrix

$$H = \left( \frac{\partial^2 E(X^e)}{\partial R_{k_{a_1}} \partial R_{k_{a_2}}} \right)$$

is called Hessain. For the optimized structure $X^e$ the forces $\frac{\partial E}{\partial R_{k_{a}}}$ vanishes, these stationary point geometries are thus confirmed to be local minima when all eigen values of the Hessian matrix are non negative. Thus one first determines the structure of the total-energy minimum and the changes in the total energy due to slightly dislocating the various atoms considering different displacements are calculated. The results of these calculations finally fitted with in an expression like Eq. 1.162. We then treat the nuclei as quantum particles and consider accordingly following nuclear Hamiltonian operator.
\[
\hat{H}_n = -\sum_{k=1}^{M} \frac{1}{2M_k} \nabla^2_{R_k} + \frac{1}{2} \sum_{k, k^\prime = 1}^{M} \sum_{a_k, a_{k^\prime}} \frac{\partial^2 E(X^e)}{\partial R_{k, a_k} \partial R_{k^\prime, a_{k^\prime}}} \times (R_{k, a_k} - R_{k^\prime, a_{k^\prime}}) (R_{k, a_{k^\prime}} - R_{k^\prime, a_k})
\]

(1.164)

The eigenvalues and eigenfunctions to the Schrödinger equation
\[
\hat{H}_n \Psi_n = E_n \Psi_n
\]
then define the energies and modes of the vibrations of the system of interest.

In order to solve this equation it is advantageous to introduce new coordinates
\[
u_k = \sqrt{M_k} (R_k - R^e_k)
\]
that are the derivation from the equilibrium position but scaled with the square root of the masses. The nuclear Hamilton operator of Eq. 1.164 then takes the form
\[
\hat{H}_n = \sum_{k=1}^{M} \frac{1}{2M_k} \nabla^2_{\tilde{u}_k} + \frac{1}{2} \sum_{k, k^\prime = 1}^{M} \sum_{a_k, a_{k^\prime}} \frac{1}{\sqrt{M_k M_{k^\prime}}} \frac{\partial^2 E(X^e)}{\partial R_{k, a_k} \partial R_{k^\prime, a_{k^\prime}}} \tilde{u}_{k, \alpha_k} \tilde{u}_{k^\prime, \beta_{k^\prime}}
\]

(1.167)

As for the Hessian, this leads naturally to definition of matrix
\[
D = \begin{pmatrix}
\frac{1}{\sqrt{M_k M_{k^\prime}}} & \frac{\partial^2 E(X^e)}{\partial R_{k, a_k} \partial R_{k^\prime, a_{k^\prime}}}
\end{pmatrix}
\]

(1.168)

referred as dynamical matrix

Also in this case we should in principle solve the nuclear Schrödinger equation (Eq. 1.165).

The problem with this equation, however, is that all different \(u\)'s mix, i.e., is a true 3M-dimensional problem. we diagonalize the dynamic matrix as
\[
D = U^\dagger \Lambda U
\]

(1.169)

where \(U\) is a unitary matrix and \(\Lambda\) is a diagonal matrix containing the eigenvalues of \(D\), we then define new coordinates
\[
\tilde{U}_i = \sum_{k=1}^{M} U_{i, k} \tilde{u}_k
\]

(1.170)

Here, we simplify the notation by letting
Using this nuclear Hamilton operator can be rewritten. First, we will write the kinetic energy part,

\[
\frac{\partial^2}{\partial u_i^2} = \frac{\partial}{\partial u_i} \left[ \sum_{k} \frac{\partial \tilde{u}_k}{\partial u_i} \frac{\partial}{\partial \tilde{u}_k} \right]
\]

\[
\frac{\partial^2}{\partial u_i^2} = \frac{\partial}{\partial u_i} \left[ \sum_{k} U_{ik} \frac{\partial}{\partial \tilde{u}_k} \right] \quad (1.172)
\]

\[
= \sum_{k} U_{ik} \sum_{l} \frac{\partial \tilde{u}_l}{\partial u_i} \frac{\partial^2}{\partial \tilde{u}_l \partial \tilde{u}_k} \frac{\partial}{\partial \tilde{u}_k}
\]

\[
= \sum_{k} U_{ik} U_{i\ell} \frac{\partial^2}{\partial \tilde{u}_k \partial \tilde{u}_l} \quad (1.173)
\]

then

\[
-\frac{1}{2} \sum_{i} \frac{\partial^2}{\partial u_i^2} = -\frac{1}{2} \sum_{i} \sum_{k} U_{ik} U_{i\ell} \frac{\partial^2}{\partial \tilde{u}_k \partial \tilde{u}_l}
\]

\[
= -\frac{1}{2} \sum_{k} \frac{\partial^2}{\partial \tilde{u}_k \partial \tilde{u}_l} \sum_{i} U_{ik} U_{i\ell}
\]

\[
= -\frac{1}{2} \sum_{k} \frac{\partial^2}{\partial \tilde{u}_k \partial \tilde{u}_l} \delta_{k,l}
\]

\[
-\frac{1}{2} \sum_{i} \frac{\partial^2}{\partial u_i^2} = -\frac{1}{2} \sum_{k} \frac{\partial^2}{\partial \tilde{u}_k^2} 
\]

\[
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i.e., there is no mixing between the different components (the different k). In this form, it is straightforward to calculate the eigenvalues and eigenfunctions. To this end we observe that the form of the nuclear Hamilton operator in Eq. 1.176 makes it possible to write the nuclear wavefunction as a product,

\[ \Psi_n(R_1, R_2, \ldots, R_M) = \Psi_1(\vec{u}_1) \cdot \Psi_2(\vec{u}_2) \ldots \cdot \Psi_M(\vec{u}_M) \]

(1.177)

Inserting this into the nuclear Schrödinger equation gives 3M independent equations of the form

\[ \left[ -\frac{1}{2} \frac{\partial^2}{\partial \vec{u}_k^2} + \frac{1}{2} A_{kk} \right] \Psi_k(\vec{u}_k) = \hat{E}_k \Psi_k(\vec{u}_k) \]

(1.178)

the eigenvalues are then

\[ E_k = m_k + \frac{1}{2} \sqrt{A_{kk}} \]

(1.179)

where the integers \( m_k \) distinguish different eigenvalues and eigenfunctions.

Thus \( m_k \) describes how different atoms move when the \( k \)-th variational mode is excited and the eigenvalues \( \Lambda_{kk} \) of the dynamic matrix are the square roots of the vibrations excitation energies. We may translate or rotate the molecule without any energy cost, and thus have 5 (for linear) or 6 (for non-linear) modes of zero energy.

### 1.10 NMR Chemical Shifts

NMR chemical shifts (\( \delta \)) of atoms can be computed using ab initio QM methods. To be precise, these methods give the shielding tensors for the atoms in the given molecular system. The chemical shifts are the values of the isotropic shielding tensor of atoms minus the isotropic shielding tensor in the reference material. The \( ^1H \) NMR chemical shifts are calculated with tetramethyl silane (TMS) as a reference. The computation of shielding tensors must be performed for equilibrium geometries and with the same method as well as basis used for the geometry optimization. However, due to gauge problem, dependence on the coordinate system’s origin, computing shielding tensors become difficult task. A number of techniques for correcting this are in use. One of the most popular techniques employed for correcting is gauge invariant atomic orbitals (GIAO). Recent versions have included ways to relax this condition without loss of accuracy and subsequently the same acronym was forwarded for gauge including atomic orbitals. The GIAO method is based on
perturbation theory\textsuperscript{102} and it is used to compute the shielding tensors from HF or DFT wave functions.

In $\chi_e$, the gauge-invariant atomic orbitals (GIAO), the real atomic basis functions $\varphi_e$ are modified by a multiplicative complex factor which depends on the gauge of the vector potential $A(r)$

$$\chi_e = \exp\left(-\frac{i}{c} A_x r\right) \varphi_e$$ (1.180)

The electronic Hamiltonian describing a closed-shell molecule in the total magnetic field due to a uniform external magnetic field $H$ and the dipole field arising from nuclear magnetic moments $\mu_B, \mu_D, \ldots$ situated at fixed nuclear positions $R_B, R_D, \ldots$ has the form (in atomic units)

$$\hat{H}(H, \mu) = \left\{ \frac{1}{2} \sum_j \left[ (p_j + (1/c) A(r_j))^2 - 2 \sum B Z_B r_j^{-3} \right] + \sum j \sum m r_j^{-1} + \sum \sum Z_B Z_B R_D^{-1} \right\}$$ (1.181)

Here, $c$ is the velocity of light, $\sum_j$ and $\sum B$ represents a sum over all electrons and nuclei, respectively. Last three terms represent the electron-nucleus, electron-electron and nucleus-nucleus contributions to the potential energy.

The vector potential $(A'(r_j))$ describing the total magnetic field at the position of $j^{th}$ electron, is given by

$$A'(r_j) = \left\{ \frac{1}{2} H \times r_j + \sum B (\mu_B \times r_{j,B}^{-1}) r_j^{-3} \right\} + \sum j (\mu_B \times r_{j,B}) r_j^{-3}$$ (1.182)

In the above equation $r_j$ is the distance vector from $j^{th}$ electron to some arbitrary origin, $r_{j,B}, r_{j,D}$ are the vectors $(r_j - R_B), (r_j - r_j)$ and $(R_B - R_D)$ and $Z_B$ is the charge on nucleus B.

The two equation in the coulomb gauge (div=0) leads to equation

$$\hat{H}(H, \mu_B) = H^{(0)} + \sum \sum H_{a} H_{a}^{(0)} + \sum \sum \mu_{B,a} H_{a,B}^{(0)} + \sum \sum \mu_{B,a} H_{a,B}^{(0)} + \sum \sum \mu_{B,a} H_{a,B}^{(0)}$$ (1.183)

The energy $E(H, \mu_B)$ associated with this Hamiltonian can be found by solving the Schroedinger equation

$$\hat{H}(H, \mu_B) \Psi(H, \mu_B) = E(H, \mu_B) \Psi(H, \mu_B)$$ (1.184)

here $\Psi(H, \mu_B)$ is the wavefunction describing the molecule in the presence of $H$ and $\mu_B$. 

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The energy expression then obtained will be
\[
E(H, \mu_B) = E^{(0)} + \sum_a \gamma_a H_a - \sum_a \mu_{B,a} H_a - \left( \frac{1}{2} \right) \sum_a \sum_p H_a \chi_{a\beta} H_{\beta} + \sum_a \sum_p H_a \sigma_{B,a\beta} \mu_{B,\beta} + \ldots \tag{1.185}
\]

Where \( \gamma_a \) is a component of the permanent magnetic moment of the molecule. For singlet state molecules with zero orbital angular momentum \( \gamma \) is zero. The interaction between the nuclear magnetic moment and the external magnetic field is given by third term. The fourth term describes the diamagnetic polarization of the molecule; the total magnetic moment (in direction \( \alpha \)) associated with the electronic currents induced by the external magnetic field is \( \sum_{\beta} H_{\alpha} \chi_{\alpha\beta} H_{\beta} \), where \( \chi_{\alpha\beta} \) is a component of the molecular diamagnetic susceptibility tensor \( \chi \). The secondary magnetic field at nucleus B due to these electronic currents is \( \sum_{\beta} H_{\alpha} \sigma_{B,a\beta} \), being a component of the magnetic shielding tensor \( \sigma_B \). Thus the total magnetic field experienced by nucleus B which determines its NMR frequency is given by
\[
H_B = H(1 - \sigma_B) \tag{1.186}
\]

In absence of a magnetic field, usually each MO is written in LCAO form, where the \( \phi \) are taken in real form. In presence of a magnetic field such a theory does not give a correct description of an isolated atom. Hence, each MO is written as a linear combination of GIAO, \( \chi \).
\[
\Psi\left(\gamma, \mu_B\right) = \sum c_{ij}(H, \mu_B) \chi_i(H) \tag{1.187}
\]

The energy, \( E\left(\gamma, \mu_B\right) \), is associated with the MO \( \Psi\left(\gamma, \mu_B\right) \). It is worth to note that the use of GIAO leads to one – and two- electron integrals with explicit magnetic field dependence.

Now chemical shielding tensors of the nucleus B can be calculated by taking the second order partial derivative of energy given by Eq. 1.185 with respect to the nuclear magnetic moment \( \mu_B \) and uniform external magnetic field, \( H \) as;
\[
\sigma_{\gamma B} = \left( \frac{\partial^2 E\left(H, \mu_B\right)}{\partial H \partial \mu_B} \right)_{H=0} \tag{1.188}
\]

This equation can be written more explicitly as
\[
\sigma_{\gamma B} = \sum_{xy} D_{xy} \frac{\partial^2 h_{xy}}{\partial H \partial \mu_B} + \sum_{xy} \frac{\partial D_{xy}}{\partial H} \frac{\partial h_{xy}}{\partial \mu_B} \tag{1.189}
\]

Where, \( D_{xy} \) is density matrix and \( h_{xy} \) is one electron Hamiltonian in AO representation.
1.11 Self Consistent Reaction Field methods

The effect of a solvent can be evaluated by using the methods those are broadly divided into two types: (a) one describing the individual solvent molecules and the other (b) treat the solvent as a continuous medium. The latter one is accountable for “macroscopic” or “long range” effects showing screening of charges, i.e. solvent polarization, responsible for generating an altered dielectric constant. Several methods are available for modeling the long-range solvation usually employing the concept of a “reaction field” in some way or another. One of these methods the self-consistent reaction field (SCRF) model that considers an uniform polarizable solvent with a dielectric constant of ε and the solute M is placed in a appropriate shaped hole in the medium. Creation of cavity in the medium need breaking of uniform solvent field and thus costs energy, i.e. this is a destabilization, while dispersion interactions between the solvent and solute add to stabilization. The electric charge distribution of M further polarizes the medium, which in turn acts on the molecule thereby engendering an electrostatic stabilization. The solvation (free) energy may thus be written as

$$\Delta G_{\text{solvation}} = \Delta G_{\text{cavity}} + \Delta G_{\text{dispersion}} + \Delta G_{\text{electrostatic}}$$

The energy required for creating the cavity and the stabilization resulting from the van der Waals interactions between the solute and solvent, is usually assumed to be proportional to the solute surface-area. The corresponding energy terms is simply proportional to the total surface area, or parametrized by having a constant $\xi$ specific for each atom type, with the $\xi$ parameters being determined by fitting to experimental solvation data. Thus,

$$\Delta G_{\text{cavity}} + \Delta G_{\text{dispersion}} = \sum_{i} \xi_{i} S_{i}$$

The shape of the cavity therefore becomes important. Although, more realistic models employ molecular shaped cavities, the simplest cavity shape is a sphere or an ellipsoid which ease the calculation of the electrostatic interaction between M and the dielectric medium. The atomic radius (as a suitable factor) times a van der Waals radius defines a van der Waals surface, it may have small “pockets” wherein no solvent can enter. A better descriptor may be the surface traced out by a spherical particle of a given radius rolling on the van der Waals surface denoted as solvent accessible surface (SAS).

The charge density on the surface of the hole, $\sigma (r_s)$, is given by standard electrostatics in terms of the dielectric constant, $\varepsilon$, and the electric field perpendicular to the surface, $F$, generated by the charge distribution within the cavity.
\[ 4 \pi \varepsilon \sigma(r_S) = (\varepsilon - 1)F(r_S) \]  

Eq. 1.192 can be solved analytically for spherical or ellipsoidal cavities, however for molecular shaped surface it needs to evaluate numerically. Once knowing \( \sigma(r_S) \), the associated potential is added as an extra term to the Hamilton operator.

\[ H = H + V_\sigma \]  

The potential from the surface charge \( V_\sigma \) is given by the molecular charge distribution,

\[ V_\sigma(r) = \int \frac{\sigma(r_S)}{|r - r_S|} dr_S, \]  

which have also been incorporated in the Hamiltonian and as a consequence influences the molecular wave function. The procedure is therefore iterative.

The simplest reaction field model is a spherical cavity, where only the net charge and dipole moment of the molecule are taken into account. For a net charge \( q \) in a cavity of radius \( a \), the difference in energy between vacuum and a medium with a dielectric constant of \( \varepsilon \) is,

\[ \Delta G_{\alpha}(q) = - \left( 1 - \frac{1}{\varepsilon} \right) \frac{q^2}{2a}, \]  

given by the generalized Born model\(^{105}\) using a set of partial atomic charges. The spherical cavity with dipole only, SCRF model is known as Onsagar model.\(^{106}\) The Kirkwood model\(^{107}\) utilizes the general multipole expansion whereas the cavity is elliptical in the Kirkwood-Westheimer model.\(^{108}\) The Polarizable continuum Model (PCM) employs a van der Waals surface type cavity, a detailed description of the electrostatic potential, and parameterizes the cavity/ dispersion contributions based on the surface area.

The tenets of theory reviewed herein will be employed to understand intermolecular interactions, and in host-guest chemistry within the realm of density functional theory, in subsequent chapters.

### 1.12 Introduction to Cyclodextrins and Cucurbit[n]uril hosts

#### 1.12.1 Cyclodextrins

Cyclodextrins (CD) or cycloamylose (CA) represent a family of macrocyclic oligosaccharides consisting of six or more D-glucose units linked by \( \alpha(1 \rightarrow 4) \) glycosidic bonds. The D-glucose and the cavity formed by the CD have been shown in Figure 1.3. The CD are labelled depending on the number of monomer units in the particular host; thus the host containing 6, 7, 8, 9, ... D-glucose units are recognized as \( \alpha \)-CD, \( \beta \)-CD, \( \gamma \)-CD, \( \delta \)-CD... . Although CD
Figure 1.3: The molecular structure of the glucose (i) and the cavity formed in CD, a cyclic oligomer of glucose.

Figure 1.4: Cyclodextrines containing (i) six (α-CD), (ii) seven (β-CD) and (iii) eight (γ-CD) D-glucose units.

Having up to 14 D-glucose unit have been synthesized, only α-, β-, and γ-CD (cf. Figure 1.4) posses the regular cavity shape are commercial important and attracted great interest since few decades. The specific coupling of glucose monomers engenders a rigid conical molecular structure with a hydrophilic exterior and hollow lipophilic interior of a specific volume. This internal cavity is capable of accommodating a wide range of guest molecules, ranging from polar compounds such as alcohols, acids, amines as well as small inorganic anions to nonpolar aliphatic and aromatic hydrocarbons. The cavity of appropriate dimension is conducive to binding to various guest molecules to form inclusion complexes in aqueous solution which find potential applications in pharmaceutical science, catalysis, separation technology and in affinity chromatography as chiral discriminators against optically active molecules. Owing to multifunctional characteristics and bioadaptability, α-, β- and γ-CD have been explored in biochemistry and drug research. Recent investigations have shown that inclusion complexes of CD can act as carriers for biologically active substances. The principal advantages of natural CD’s as drug
carriers may be traced back to their well-defined chemical structure, offering a multitude of potential sites for chemical modification, availability of different cavity sizes, low toxicity and low pharmacological activity and protection of the guest molecule from biodegradation. Inclusion complexes of CD serve as ideal model mimicking the enzyme-substrate interactions. The α-, β- and γ- cyclic oligomers are useful for phase transfer in catalysis and have further been explored as building blocks in supramolecular structures and functional units and in asymmetric catalysis. In order to understand host-guest interactions structural elucidation of CD and their inclusion complexes in the liquid and solid state are studied using the NMR spectroscopy. Crystal structure of CD hydrates and its inclusion complexes have been determined by X-ray and neutron diffraction. Effect of alkali metal chlorides on the self-association of decanoic acid and its inclusion in β-CD has been studied with the help of 1H NMR spectroscopy. Solid-state inclusion complex of β-CD and 2-phenoxy-ethanol and its properties have been characterized for different humidities using the powder X-ray diffraction (PXRD), thermogravimetry, FT-Raman and 13C CP MAS NMR. On the contrary synthesis of CD hydrates and the complexes with no guest molecule inside the cavity, for example, α-CD complexes of benzene and iodoanilide trihydrate, are also reported in the literature. Binding constants of a variety of guests with cycloextrinsics in water are reported in the literature. Association binding constants of phenol and CD have been obtained by near infrared spectroscopic measurements. It has also been observed that phenol and CD interaction in the ionic liquids turns out to be external adsorption. Thermodynamic studies on cyclodextrin complexation of aromatic guests in water and urea-water mixtures have also been reported. Binding affinity of a guest in these complexes is governed by cavity dimension, its relative orientation and non-covalent interactions which refer to both dipole-dipole and hydrogen-bonded interactions. Fluorescence, UV visible spectroscopy, calorimetry experiments enable one to measure complexation energies. These techniques, however, provide only indirect and qualitative information about inclusion modes and geometries. Thus structural characterization pertained to relative orientation of a guest in the CD cavity is of particular significance for the understanding of supramolecular host-guest complexation. Encapsulation of a guest in CD cavity has widely been studied from the NMR experiments. Thus the observed chemical shifts have emerged as an alternative means to a single crystal diffraction data derived from either the X-ray or neutron diffraction experiments, to obtain the insights for orientation of a guest in the complex.
These NMR experiments have shown that hydrogen bonded interactions of secondary hydroxyl groups from the bottom rim render less flexibility to β-CD than α or γ-CD. Along with these experiments the theoretical methods including molecular mechanics (MM) have been useful to a certain degree to understand the host-guest interactions in CD complexes. Accordingly the inclusion complexes of β-cyclodextrin with aliphatic alcohols are investigated using empirical force fields. Calculated vibrational frequencies are subsequently used to estimate the Gibbs free energy and related thermodynamic parameters. Thermodynamic and NMR studies have shown that the reactions of α- and β-CD with acid, aliphatic amines and cyclic alcohols can be qualitatively understood in terms of van der Waals forces and hydrophobic effect. It has been shown further that the standard Gibbs free energy correlates well with the structural features of ligands. Recently, molecular dynamics (MD) simulations have widely been used to derive the thermodynamic parameters of the CD complexation. Estrada and coworkers have utilized the simulations on the complexes of α- and β-CD with substituted benzene derivatives and obtained the quantitative structure-property relations. MD simulations on β-CD hydration have shown that hydrophobicity dominates inside the cavity whereas at the top and bottom interactions with water are mostly hydrophilic in nature. Crystallographic data for the spontaneous hydration process of a CD crystal in wet atmosphere concur with the calculated results. Complexation with organic and the inorganic guest molecules have also been studied using the semi-empirical quantum chemical calculations. Calculations based on the Parametric Method-3 (PM3) have been carried out to predict the structure of the α-CD and 4-fluorophenol complex. These calculations have shown that the -OH group of the guest directs towards the primary hydroxyls of α-CD in solid state. It has also been remarked that the solvation has profound influence and may reverse the orientation of the guest. Lipkowitz, however, has questioned the use of semi-empirical methods to yield the structures of cyclodextrins. Semi-empirical quantum chemical methods have attained some success, however, failed to account for weak intermolecular interactions. The Hartree-Fock and second order Moller-Plesset (MP2) theory have been utilized to characterize the structure and normal vibrations of substrate in the inclusion complex of CH3HgCl and α-CD. Thus use of explicit water molecules favor the inclusion of CH3HgCl with the guest orienting perpendicular to the CD ring. On the contrary the structure with the guest orienting parallel to the ring has been observed experimentally. Single point hybrid density functional calculations on the structure consisting of two water molecules inside the cavity.
and four outside the cavity are used to obtain the thermodynamic parameters viz., entropy, enthalpy and Gibbs free energy of hydration of the α-CD hexahydrate, which compare well with those from experiment. Quantum chemical calculations at the electron correlation level of theory could not be utilized for optimization of such large systems. Very recently the molecular structure, stabilization energy and thermodynamic properties of the interaction of the α-CD dimers (head to head, tail to tail and head-to-tail) with water cluster are derived successfully using the density functional calculations using the PM3 optimized geometries. These calculations reveal that the inter α-CD hydrogen bonded interactions render stability to the dimeric structures and no water tetramer was found between the two α-CD subunits in the lowest minimum structure. Thus, binding of water molecules to the α-CD dimer or their inclusion complexes in aqueous media has to be accounted in the theoretical calculations. It should further be remarked here that electrostatic interactions constitute an important driving force in cyclodextrin complexation. Alternatively, electronic repulsion between frontier orbitals of the host and guest has been conjectured to be a driving force in complexes between the α-CD and benzene derivatives. Complexation of β-CD with benzene derivatives, on the other hand is controlled by topological and topographic parameters indicating relevance of the van der Waals and hydrophobic interactions. Unlike in α-CD wherein benzene ring is placed outside the cavity, the β-CD complexes exhibit deep penetrates into the hydrophobic cavity of the host. Inclusion complexes of γ-CD in contrary are not studied to a great detail. A few reports, where steady state fluorescence and the calculations based on molecular mechanics have been used to determine the stoichiometry and binding constants of the inclusion complexes of 2-methylnaphthoate and γ-CD have appeared in the literature. Complete penetration of the guest has been observed both in β- and γ-CD. Time-resolved emission spectra measurements have been carried out to understand the influence of cavity-size on the excited state dynamics of methyl 4-(dimethylamino)benzoate and cyclodextrin complexes.

1.12.2 Cucurbit[n]urils:
The molecular containers cucurbit[n]urils, cyclic oligomers of glycouril denoted by CB[n], ‘n’ being number of repeating glycouril units, have attracted significant attention owing to binding ability to form the complexes with a variety of guest molecules. Interestingly the X-ray crystal structure of CB[6] unravelled only in 1981; decades after its synthesis by
condensation of glycouril (acetyleneurea) and formaldehyde in acidic condition (cf. Figure 1.5). CB[n] are comprised of a hydrophobic cavity surrounded by hydrophilic portals lined up with the polar ureido carbonyl groups. CB[n] homologues have been characterized using the X-ray crystallography, ^1^H NMR and ESIMS experiments in the literature. To this end higher CB[n] homologues containing up to 16 glycouril units condensed together have been synthesied. Moreover, the specific reaction conditions to obtain optimum yield of isolated CB[n] homologues have recently been well documented. The salient features those make CB[n] family widely popular in host-guest chemistry include (i) the portals laced by ureido carbonyl groups, (ii) hydrophilic cavity and (iii) wide range of cavity dimensions.

![Figure 1.5: The synthesis of CB[n] using glycourea and formaldehyde in highly acidic medium.](image)

CB[n] homologues are known to efficiently bind range of molecules viz. gases, aromatic compounds, metal ions and their complexes as well. The growing interest in this class of molecules mainly stems from its remarkable affinity and high selectivity towards the guest facilitated by the inner cavity of CB[n] and its opening. The selective encapsulation of the guest facilitates use of CB[n] derivatives in separation technology, molecular recognition, catalysts and organic synthesis. In this respect, the positional isomers of the nitrotoluene, nitrophenol, nitroaniline, and methylaniline isomers are separated using CB[7] mediated capillary electrophoresis. These experiments indicated that interactions of CB[n] (n = 6, 7) with guests are strongly affected by both the position of substituent(s) on aromatic ring and ion-dipole interactions between guest molecule and CB[n]. Amongst binding of amines, diol, diacid, guanidinium, and pyridinium species toward CB[n] analogues, host shows molecular recognition toward alkaneammonium species which binds tightly in water and further exhibits length-dependent selectivity.
Catalytic activation of polymerization and simultaneous rotaxane formation yields main chain and branched polyrotaxanes, which can be achieved by self-threading CB[6] macrocycle. It has been demonstrated that the catalyst CB[6] is highly sensitive toward structure of monomers employed in polymerization. Recently, Carlqvist and Maseras have studied cycloaddition between an azide and an alkyne inside the CB[6] and elucidated the catalytic function of such a highly efficient supramolecular catalyst. Furthermore CB[n] homologues accommodating more than one guests in their cavities have been explored in organic synthesis. Accordingly CB[8] stabilizes two (E)-diaminostilbene dihydrochloride in close proximity with parallel orientation thereby accelerating the reaction with high stereoselectivity in [2+2] photodimerization. The [4+4] photochemical reaction of 2-aminopyridine hydrochloride in aqueous solution produces anti-trans and syn-trans dimers of 4,8-diamo-3,7-diazatricyclo[4.2.2.2]dodeca-3,7,9,11-tetraene in a 4:1 ratio; nevertheless they rearomatize at room temperature. It should be remarked here that along with stabilization of photodimer, CB[7] facilitates 2:1 host-guest complex yielding exclusively anti-trans isomer. Ramamurthy and co-workers investigated geometrical isomerisation and selective photodimerization of cinnamic acid and protonated azastilbenes mediated by CB[n] host cavities in water. An intramolecular photocyclomer of 2-naphthalene-labeled poly(ethyl glycol) has thus been synthesized by encapsulating both naphthalene terminals of guest in CB[8] cavity, which otherwise was not feasible. The photodimerization of alkyl 2-naphthoate in CB[n] aqueous solution further facilitates
cubane-like photodimer formation. These authors have shown that both the cavity size of CB[n] and alkyl substitutions on 2-naphthoate derivatives significantly modulate the host-guest interactions and remarkably alters photochemical reactivity. CB[n] family of hosts have also proven useful in nanotechnology and biologically important areas such as drug delivery vehicles, DNA binding and gene transfection, and selective encapsulation of peptides. Encapsulation of metal nanoparticle in CB[7] can be utilized to control its size, which concomitantly render stability to very small gold nanoparticles. Largely stabilized gold nanoparticles confined to CB[7] (compared to cyclodextrins) do not interact with cyanide or large amines; and moreover phase separation of gold from aqueous solution upon standing (or heating) or gold agglomeration, even after prolonged heating, was not observed. It may be noteworthy that CB[7] has potential as a delivery system for DNA targeting multi-nuclear platinum complexes. The NMR spectroscopy investigations have shown that some cisplatin binds in CB[n] cavity, while cis-[PtCl(NH3)2(H2O)]⁺ only binds only at host portals. Alternatively, 4,4’-dipyrazolylmethane linked multinuclear Pt complexes are quantitatively encapsulated within the cavities of CB[7] and CB[8] as well. Binding of dinuclear platinum complex inside CB[7], with slow exchange kinetics, does not significantly affect cytotoxicity of dinuclear complexes, however reduces reactivity at the platinum centre. In addition to this, Pt(II)-based complexes containing imidazole, pyrrole and β-alanine subunits recognize specific DNA base-pair sequences. Encapsulation of such Pt-based DNA intercalators by CB[6] does not significantly change cytotoxicity. In other words, aforementioned investigations reveal efficacy of CB[n] hosts as pharmacological delivering macrocycle.

Figure 1.7: Charge transfer complexes mediated by CB[n].

A macrocyclic host CB[7] in aqueous solution of rohadamine 6G dye have been utilized, with environmental and safety benefits, in unconventional practical application to develop a "supramolecular dye laser" aimed at high lasing efficiency, stability, and an impressive beam shape. In supramolecular chemistry CB[n] homologues have further been
primarily used to build charge transfer complexes, molecular machines, molecular necklaces, rotaxanes and pseudorotaxanes. CB[8] capability to selectively bind two (one electron deficient and other electron rich) guest molecules by a dynamic interlink has been used in designing of new polymeric block copolymers. Selective inclusion of a hetero-guest pair in a molecular host (CB[8]), driven and stabilized by a charge-transfer interaction between electronrich (2,6-dihydroxynaphthalene or 1,4-dihydroxybenzene) and electron-deficient (methylviologen) guests, expected to possess rich spectroscopic and electrochemical properties. A novel supramolecular assembly of a highly stable inclusion complex between 2,7-dimethyl diazapyrenium and CB[8] host form 2:1 charge-transfer complexes with catechol (or dopamine) containing π-donor moieties. The sensitivity for detecting catechol and dopamine has considerably been improved by the presence of CB[8]. Thermodynamic stability of CB[n]-stabilized charge transfer complexes allows to build supramolecular assemblies, which can readily be disassembled into their components by redox stimuli. The reversibility of charge transfer complex formation inside CB[8] have facilitated in design redox-driven molecular machines. Host-guest chemistry of systems containing a molecular triad Ru(bpy)3-MV2+-naphthol complex and CB[8] is investigated by NMR, ESI-MS, UV-vis, and electrochemistry and suggested their use in design and synthesis of photoactive molecular devices. The interactions between CB[7] and a series of linear guests, containing bis(pyridinium)-1,4-xylylene and 4,4’-bipyridinium residues those exhibit considerable binding selectivity for guests with former moiety, have been utilized in synthesis of rotaxane and pseudo-rotaxane compounds. A novel supramolecular adduct of CB[6]-based pseudo-rotaxane [N,N’-bis(2-pyridylmethyl)-1,6-diaminohexane+CB[6]] and six-hydrated Zn(II) ion [{Zn(H2O)6}Cl2], forming an infinite 2D network and 3D spatial structure facilitated by hydrogen bonding interactions have been reported. Efficient strategies to synthesize molecular necklaces, in which a number of small rings are threaded onto a large ring based on the principles of self-assembly and coordination chemistry. CB[6] "bead" threaded with a short string of pyridylmethyl substituted 1,4-diaminobutane (or 1,5-diaminopentane) to make a pseudorotaxane, which then linked with a metal complex to form a cyclic product (molecular necklace). The reaction of pseudorotaxanes with complex [Pt(ethylenediamine)(NO3)2] produces a molecular necklace, in which three molecular “beads” are threaded on a triangular framework, and/or a molecular necklace, in which four molecular “beads” are threaded on a square framework. The structures of molecular necklaces are strongly influenced by the
structures of pseudorotaxane building blocks. Pioneering work from Kim and co-workers has demonstrated a novel approach to noncovalent synthesis of molecular necklaces using host-stabilized charge transfer complexes. A variety of molecular necklaces with different sizes, shapes, and number of molecular beads can be synthesized by varying the length and angle of the linker. It has further been proposed that, such topologically intriguing self-assembly of supramolecules may offer insight into the design of well-defined structures and functions of nanoscale objects. It has been pointed out that an efficient synthesis of 1D, 2D and 3D polyrotaxanes with high structural regularity and molecular necklaces can be achieved by a combining self-assembly formation and coordination chemistry. Molecular machines controlled by either proton or electron transfer have been utilized to control position of CB[n] in the molecular complex. For example, a pH switchable and highly stable pseudorotaxane, in which the macrocycle CB[7] (wheel) exchanges rapidly between two identical sites at the extremes of the molecular axle of guests at low pH, while at neutral pH it is confined to bind the central viologen residue in axle. In the pseudorotaxanes with ferrocenyl-terminated viologen-containing guests, the location of wheel (shuttling of CB[7]) on the guest axle can be controlled by oxidation state (electron transfer) of the terminal ferrocenyl residues. Moreover, NMR spectroscopy has been used to understand encapsulation of viologen and FC at focal point functionality in newkome-type dendritic block.

On the theoretical front, ab initio and density functional calculations have been carried out to quantify stabilities of CB[n] and their methyl substituted derivatives. Electronic structure of CB[n] and their thio analogues interacting with transition metal ions have been carried out using semiemperical MO calculations. A supramolecular strategy for the synthesis of thia-cucurbituril nanotubes of a desired molecular length is proposed. A class of tubular nanostructure from thia-cucurbituril macrocycles with transition metal ions has been designed. The frontier orbitals here indicate that the growth of nanotube is likely to initiate at one end. Recent work by Carlquist on cycloaddition of azide and alkyne within the CB[6] cavity brings out its potential as a highly efficient supramolecular catalyst. Further the NMR has widely been used to gain insights for the interaction of guest with CB[n] hosts. Theoretical calculations due to Buschmann et al. have shown that the Frontier orbitals in CB[n] homologues are similar and the predicted NMR chemical shifts therein turn out to be nearly unchanged.
1.12.3 Inverted cucurbit[n]urils

In spite of a variety of cavity sizes of CB[n] homologues, a yet another attractive class known as inverted cucurbi[n]uril (iCB[n]) have shown to exist. One or more glycouril units in iCB[n] is inverted and thus the methine carbon and protons in those units are directed inside the cavity. Isolation of i-CB[6] and i-CB[7] containing only one inverted-glycouril unit has been reported by Isaacs et al. These iCB[n] were isolated as intermediates during the synthesis of CB[n] from the reaction of glycouril and formaldehyde in acidic media and were detected by the $^1$H NMR spectroscopy. Single crystal data of $i$CB[6] and $i$CB[7] have led to the conclusion that the methine protons of inverted glycouril unit placed within the cavity engender smaller cavity volume compared to their CB[n] counterpart. It is not surprising that the small cavity with open portals of iCB[6] and iCB[7] led to loose binding of guests with different selectivity compared to the CB[n] hosts. Further the inverted glycouril render large dipole moment to these iCB[n] enantiomers. The inverting of one or two glycourils as in the iCB[n] or $i$CB[n] influence significantly the hydrophobicity of cavity as well as hydrophilicity at the portals, which primarily governs interaction with the guest, that is, either accommodation within the cavity or lateral interaction with the portals.

![CB[7] and i-CB[7]](image_url)

**Figure 1.6:** Cucurbit[7]uril and its inverted diastereomer $i$CB[n].

In the following chapters we analyze electronic structure, $^1$H NMR and IR spectra of aforementioned novel hosts complexed with prototype neutral and cationic guests.
1.13 References


(2) Szabo, A.; Óstlund, N. S. Modern Quantum Chemistry (MacMillan, New York, 1982).


