Chapter 2

Convergence Of Asymptotically Quasi Nonexpansive Mappings

2.1 Abstract

The object of this chapter is to establish certain weak and strong convergence results using several types of asymptotically quasinonexpansive mappings for Das and Debata type iterative sequence and Ishikawa iterative sequences of rank r respectively in Banach space under varied conditions.

2.2 Introduction

We have told in chapter I that the class of nonexpansive mapping made an independent path of investigation within the fold of nonlinear analysis for the reason that it exhibited different character from its origin i.e. the contraction mapping used to obtained fixed point. Consequently, several types of nonexpansive mappings came in to existence. Apart from proving the existence of fixed points for such mappings, various types of iteration schemes were applied for proving the convergence of their
sequences.

As we have also told that quasinonexpansive mapping is entirely different from remaining class of nonexpansive mappings because it includes fixed point within its structure. In recent years, Mann and Ishikawa iterative schemes [72, 81] have been studied extensively by many authors for the purpose of establishing their convergence for various types of nonexpansive mappings. Among them, two types of mappings - uniform \((L_i - \alpha_i)\)-Lipschitz mapping [44] and asymptotically quasi-nonexpansive map [43] are the object for study for their iterative convergence in this chapter. Before doing so, let us recall the definitions of those mappings.

**Definition 2.2.1.** Let \(D\) be a nonempty subset of normed space \(X\). \(T\) is said to be an asymptotically quasi-nonexpansive map [43], if \(F(T) \neq \phi\) and there exists a sequence \(\{u_n\}\) in \([0, +\infty)\) with \(\lim_{n \to \infty} u_n = 0\) such that

\[
||T^n x - p|| \leq (1 + u_n)||x - p|| \text{ for all } x, y \in D \text{ and } n \in N.
\]

**Definition 2.2.2.** \(T\) is said to be an asymptotically nonexpansive [25] if there exists a sequence \(\{u_n\}\) in \([0, +\infty)\) with \(\lim_{n \to \infty} u_n = 0\) such that

\[
||T^n x - T^n y|| \leq (1 + u_n)||x - y|| \text{ for all } x, y \in D \text{ and } n \in N.
\]

Recently, Qihou [44] defined \(L - \alpha\) uniform Lipschitz mapping.

**Definition 2.2.3.** \(T\) is said to be \(L - \alpha\) uniform Lipschitz mapping [44] if there exist constants \(L > 0\) and \(\alpha > 0\) such that

\[
||T^n x - T^n y|| \leq L||x - y||^\alpha \text{ for all } x \in D, p \in F(T) \text{ and } n \in N.
\]
Next, we recall the structure of the Mann and Ishikawa iterative schemes and these schemes with error terms as below:

For a nonempty convex subset $C$ of a normed space $E$ and $T : C \to C$,

(a) The Mann iteration process is defined by the following sequence $\{x_n\}$:

$$
\begin{align*}
\begin{cases}
  x_1 \in C \\
  x_{n+1} = (1 - b_n)x_n + b_nTx_n, & n \geq 1
\end{cases}
\end{align*}
$$

Where $\{b_n\}$ is a sequence in $[0,1]$.

(b) The sequence $\{x_n\}$, defined by

$$
\begin{align*}
\begin{cases}
  x_1 \in C \\
  x_{n+1} = (1 - b_n)x_n + b_nTy_n + u_n, \\
  y_n = (1 - b'_n)x_n + b'_nTx_n, & n \geq 1
\end{cases}
\end{align*}
$$

Where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$, is known as the Ishikawa [72] iteration process.

In 1995 Liu [39] introduced iterative schemes with errors as follows:

(c) The sequence $\{x_n\}$ in $C$ iteratively defined by:

$$
\begin{align*}
\begin{cases}
  x_1 \in C \\
  x_{n+1} = (1 - b_n)x_n + b_nTy_n + u_n, \\
  y_n = (1 - b'_n)x_n + b'_nTx_n + v_n, & n \geq 1
\end{cases}
\end{align*}
$$

Where $\{b_n\}, \{b'_n\}$ are sequences in $[0,1]$, and $\{u_n\}, \{v_n\}$ are sequences in $C$; satisfying $\sum_{n=1}^{\infty} ||u_n|| < \infty$; $\sum_{n=1}^{\infty} ||v_n|| < \infty$ is known as Ishikawa iterative scheme with errors.

(d) The sequence $\{x_n\}$ iteratively defined by:

$$
\begin{align*}
\begin{cases}
  x_1 \in C \\
  x_{n+1} = (1 - b_n)x_n + b_nTx_n + u_n, & n \geq 1
\end{cases}
\end{align*}
$$
Where \( \{b_n\} \) is a sequence in \([0,1]\) and \( \{u_n\} \) a sequence in \( C \) satisfying \( \sum_{n=1}^{\infty} ||u_n|| < \infty \) is known as Mann iterative scheme with errors. Although, the inclusion of error terms in iterative scheme is an important part of the theory, it is clear that the iterative schemes with errors introduced by Liu [39] as in (c) and (d) above, are not satisfactory. The errors may occur randomly. The conditions imposed on the error terms in (c) and (d) which says that they tend to zero as \( n \) tends to infinity are, therefore, unreasonable.

In 1998, Xu [83] introduced a more satisfactory error term in the following iterative schemes:

(e) The sequence \( \{x_n\} \) iteratively defined by:

\[
\begin{aligned}
  x_1 & \in C \\
  x_{n+1} &= a_n x_n + b_n T x_n + c_n u_n, \quad n \geq 1
\end{aligned}
\]

with \( \{u_n\} \) a bounded sequence in \( C \) and \( a_n + b_n + c_n = 1 \), is known as Mann iterative scheme with errors.

(f) The sequence \( \{x_n\} \) iteratively defined by:

\[
\begin{aligned}
  x_1 & \in C \\
  x_{n+1} &= a_n x_n + b_n Ty_n + c_n u_n, \\
  y_n &= a'_n x_n + b'_n Tx_n + c'_n v_n, \quad n \geq 1
\end{aligned}
\]

with \( \{u_n\}, \{v_n\} \), bounded sequences in \( C \) and \( a_n + b_n + c_n = 1 = a'_n + b'_n + c'_n \) is known as Ishikawa iterative scheme with errors.

On the other hand, Das and Debata [17] and Takahashi and Tamura [76] developed a generalization of Ishikawa iterative schemes with errors for two mappings in Hilbert space and Banach space, respectively.
In [17], Das and Debata considered the iterative sequence \( \{x_n\} \) for two quasi-nonexpansive self-mapping \( S \) and \( T \) on \( C \) defined by

\[
\begin{align*}
x_1 & \in C \\
x_{n+1} & = (1 - a_n)x_n + a_nSy_n, \quad n \geq 1 \\
y_n & = (1 - b_n)x_n + b_nTx_n, \quad n \geq 1
\end{align*}
\]

where \( \{a_n\} \) and \( \{b_n\} \) are some sequences in \([0,1]\).

Takahashi and Tamura [76] studied the above scheme for two nanexpansive mappings.

Recently, Khan and Takahashi [69] studied the above scheme for two asymptotically nonexpansive mappings \( S \) and \( T \) through weak and strong convergence of the sequence defined by:

\[
\begin{align*}
x_1 & \in D \\
x_{n+1} & = (1 - \alpha_n)x_n + \alpha_nS^n[(1 - \beta_n)x_n + \beta_nT^nx_n]
\end{align*}
\]

for all \( n = 1, 2, \ldots \) where \( \{\alpha_n\} \) and \( \{\beta_n\} \) in \([0,1]\).

On the other hand Bose and Laskar [65] studied the following existence theorem:

**Theorem 2.2.1.** Let \( X \) be a uniformly convex Banach space and \( D \) be a nonempty closed convex bounded subset of \( X \) and let \( S, T : D \to D \) be the continuous mappings such that for each \( x, y \in D \) and \( n = 1, 2, \ldots \)

\[
||S^n x - T^n y|| \leq a_n ||x - y|| + b_n(||x - S^n x|| + ||y - T^n y||) + c_n(||x - T^n y|| + ||y - S^n x||)
\]

(2.2.1)

where \( a_n, b_n, c_n \geq 0 \) and satisfying following conditions:

(i) there is an integer \( I \) such that \( b_n + c_n < 1 \quad \forall n = 1, 2 \ldots \)

(ii) \( \lim_{n \to \infty} \frac{a_n + b_n}{1 - b_n - c_n} = 1 \),

(iii) \( a_n + 2c_n < 1 \), for at least one \( n \).

Then \( S \) and \( T \) have unique common fixed points and it is unique as fixed points of
If we put $b_n = 0$ then the condition (2.2.1) reduces to
\[
||S^n x - T^n y|| \leq a_n ||x - y|| + c_n (||x - T^n y|| + ||y - S^n x||) \tag{2.2.2}
\]
for all $x, y \in D$ and $n = 1, 2, \ldots$, where $a_n, c_n \geq 0$ with $c_n < 1$ and $\lim_{n \to \infty} \frac{a_n + c_n}{1 - c_n} = 1$.

In section-I of this chapter, first a result is established on strong convergence [Theorem 2.3.7] of the Das and Debata type iterative sequence of the uniform $(L - \alpha)$-Lipschitzian asymptotically quasi nonexpansive mappings in a nonempty, closed, convex, bounded subset of a Banach space as main content. Next, is the result [Theorem 2.3.8] derived for the weak convergence of Das and Debata type iterative sequence of uniform $(L - \alpha)$-Lipschitzian asymptotically quasi nonexpansive mappings in a nonempty, closed, convex and bounded subset of a uniformly convex Banach space involving Opial condition. Further, another result [Theorem 2.3.9] is proved for the weak convergence of Das and Debata type iterative sequence for the mappings in a particular sense of Bose and Laskar condition in a nonempty, closed, convex and bounded subset of a uniformly convex Banach space involving Opial condition. Finally, two strong convergence results first-[Theorem 2.3.10] is obtained in the nonempty, compact, convex subset of a uniformly convex Banach space for the Das and Debata type Ishikawa iterative sequences of mappings and the second-[Theorem 2.3.11] is obtained in a nonempty, compact, convex subset of a uniformly convex Banach space for the Das and Debata type Ishikawa iterative sequences of mappings satisfying Bose and Laskar condition. Both of these results, generalizes the results of Khan and Takahashi [69].

Similarly, the main content of Section-II in this chapter is the result on strong convergence [Theorem 2.4.4] of the Ishikawa iterative sequences of rank $r$ of the Lipschitz type asymptotically quasi nonexpansive mappings in a nonempty, compact, convex
subset of a uniformly convex Banach space. A corollary is obtained from this result as an improvement of the convergence result given by Qihou [42]. However, Section II ends with a strong convergence result [Theorem 2.4.6] for the modified Ishikawa iterative sequences of rank r of the Lipschitz type asymptotically quasi nonexpansive mappings in a nonempty, compact, convex subset of a uniformly convex Banach space.

In section-III of this chapter is the result on strong convergence [Theorem 2.5.4] of the Ishikawa iterative sequences of rank r with errors of the Lipschitz type asymptotically quasi nonexpansive mappings in a nonempty, compact, convex subset of a uniformly convex Banach space. A corollary is obtained from this result as an improvement of the convergence result given by Qihou [42].

2.3 Section-I

Before we prove results in this section, let us recall preliminaries as below:

2.3.1 Preliminaries

First, we give the following Lemmas which we shall need in the sequel.

**Lemma 2.3.1.** [37] Let \( \{r_n\}, \{s_n\}, \{t_n\} \) be three nonnegative sequence satisfying the following condition.

\[
r_{n+1} \leq (1 + s_n)r_n + t_n
\]

for all \( n \in N \). If \( \sum_{n=1}^{\infty} s_n < \infty \), \( \sum_{n=1}^{\infty} t_n < \infty \). Then \( \lim_{n \to \infty} r_n \) exists.

**Lemma 2.3.2.** [32] Suppose that \( X \) is a uniformly convex Banach space and \( 0 < p \leq t_n \leq q < 1 \) for all \( n \in N \). Suppose further that \( \{x_n\} \) and \( \{y_n\} \) are sequences of \( X \) such
that \( \limsup_{n \to \infty} ||x_n|| \leq r, \limsup_{n \to \infty} ||y_n|| \leq r \) and \( \lim_{n \to \infty} ||t_n x_n + (1 - t_n) y_n|| = r \) hold for some \( r \geq 0 \). Then \( \lim_{n \to \infty} ||x_n - y_n|| = 0 \).

We recall that a Banach space \( X \) is said to satisfy Opial’s condition [85] if for any sequence \( \{x_n\} \) in \( X, x_n \to x \) implies that \( \limsup_{n \to \infty} ||x_n - x|| < \limsup_{n \to \infty} ||x_n - y|| \) for all \( y \in X \) with \( y \neq x \). Moreover, we also know that a mapping \( T : D \to X \) is called demiclosed with respect to \( y \in X \) if for each sequence \( \{x_n\} \) in \( D \) and each \( x \in X \), \( x_n \to x \) and \( Tx_n \to y \) imply that \( x \in D \) and \( Tx = y \).

**Lemma 2.3.3.** [30] Let \( X \) be a uniformly convex Banach space satisfying Opial’s condition and let \( D \) be a nonempty closed convex bounded subset of \( X \). Let \( T \) be asymptotically nonexpansive mapping of \( D \) into itself. Then \( I - t \) is demiclosed with respect to zero.

We shall now prove the following lemmas on the lines similar to [69]. It will be used to prove the main results.

**Lemma 2.3.4.** Let \( D \) be a nonempty closed convex bounded subset of normed space \( X \) and let \( T, S : D \to D \) be two uniform \( (L - \alpha) \) -Lipschitz mappings. Define a sequence \( \{x_n\} \) as in (2.3.6). Then,

\[
||x_{n+1} - T x_{n+1}|| \leq d_{n+1} + L\{2d_n + Ld_n^\alpha + L(d_n + L^{\alpha}d_n^\alpha)^\alpha\}
\]

and

\[
||x_{n+1} - S x_{n+1}|| \leq d_{n+1} + L\{d_n + Ld_n^\alpha + d_n' + L(d_n + L^{\alpha}d_n^\alpha)^\alpha\}
\]

where \( d_n = ||x_n - T^n x_n|| \) and \( d_n' = ||x_n - S^n x_n|| \).
Proof. We consider,

\[
\|x_n - x_{n+1}\| = \|x_n - (1 - \alpha_n)x_n + \alpha_n T^n y_n\| \\
\leq \|x_n - T^n y_n\| \\
\leq \|x_n - T^n x_n\| + \|T^n x_n - T^n y_n\| \\
\leq d_n + L\|x_n - y_n\|^\alpha \\
\leq d_n + L\|x_n - (1 - \beta_n)x_n + \beta_n S^n x_n\|\|^\alpha \\
\leq d_n + L\|x_n - S^n x_n\|\|^\alpha \\
\leq d_n + Ld_n^\alpha \tag{2.3.1}
\]

and

\[
\|x_{n+1} - T x_{n+1}\| \leq \|x_{n+1} - T^{n+1} x_{n+1}\| + \|T x_{n+1} - T^{n+1} x_{n+1}\| \\
\leq d_{n+1} + L\|x_{n+1} - T^n x_{n+1}\|^\alpha \\
\leq d_{n+1} + L\|(x_{n+1} - x_n) + (x_n - T^n x_n) + (T^n x_n - T^n x_{n+1})\|\|^\alpha \\
\leq d_{n+1} + L\{\|x_{n+1} - x_n\| + \|x_n - T^n x_n\| + \|T^n x_n - T^n x_{n+1}\|\}\|^\alpha \\
\leq d_{n+1} + L\{d_n + Ld_n^\alpha + d_n + L\|x_n - x_{n+1}\|\}^\alpha \leq d_{n+1} + L\{2d_n + Ld_n^\alpha + L(d_n + Ld_n^\alpha)\}^\alpha.
\]

Similarly, we can prove that

\[
\|x_{n+1} - S x_{n+1}\| \leq d_{n+1}^\prime + L\{d_n + Ld_n^\alpha + d_n^\prime + L(d_n + Ld_n^\alpha)\}^\alpha
\]

\[\square\]

Lemma 2.3.5. Let X be a uniformly Banach space and let D be a nonempty closed convex bounded subset of X. Let \(T, S : D \to D\) be the continuous mappings satisfying
condition (2.2.2). Given a sequence \( \{x_n\} \) defined by (2.3.6). Then,

\[
||x_{n+1} - Tx_n|| \leq \frac{1 - c_n}{1 - 3c_n} \left[ d'_{n+1} + \frac{a_n + c_n}{1 - c_n} \left( 1 + a_n + 2c_n \cdot 1 + a_n d_n + \frac{1 + c_n}{1 - c_n} d_n \right) \right]
\]

and

\[
||x_{n+1} - Sx_n|| \leq \frac{1 - c_n}{1 - 3c_n} \left[ d_{n+1} + \frac{a_n + c_n}{1 - c_n} \left( 1 + a_n + 2c_n \cdot 1 + a_n d'_n + \frac{1 + c_n}{1 - c_n} d'_n \right) \right]
\]

where \( d_n = ||x_n - T^n x_n|| \) and \( d'_n = ||x_n - S^n x_n|| \).

**Proof.** We have

\[
||x_n - x_{n+1}|| = ||x_n - \{(1 - \alpha_n)x_n + a_n T^n y_n\}||
\]

\[
\leq ||x_n - T^n y_n||
\]

\[
\leq ||x_n - S^n x_n|| + ||S^n x_n - T^n y_n|| \quad \text{(2.3.2)}
\]

From condition (2.2.2), we have,

\[
||S^n x_n - T^n y_n|| \leq a_n ||x_n - y_n|| + c_n (||x_n - T^n y_n|| + ||y_n - S^n x_n||)
\]

\[
\leq a_n ||x_n - y_n|| + c_n (||x_n - S^n x_n|| + ||S^n x_n - T^n y_n||)
\]

\[
+ ||y_n - x_n|| + ||x_n - S^n x_n||
\]

Then,

\[
||S^n x_n - T^n y_n|| \leq \frac{a_n + c_n}{1 - c_n} ||x_n - y_n|| + \frac{2c_n}{1 - c_n} ||x_n - S^n x_n|| \quad \text{(2.3.3)}
\]

From (2.3.2) and (2.3.3), we obtain

\[
||x_n - x_{n+1}|| \leq ||x_n - S^n x_n|| + \frac{a_n + c_n}{1 - c_n} ||x_n - y_n|| + \frac{2c_n}{1 - c_n} ||x_n - S^n x_n||
\]

\[
\leq \frac{1 + a_n + 2c_n}{1 - c_n} d'_n \quad \text{(2.3.4)}
\]
\[ ||x_{n+1} - T x_{n+1}|| \leq ||x_{n+1} - S^{n+1} x_{n+1}|| + ||T x_{n+1} - S^{n+1} x_{n+1}|| \]

\[ \leq d_{n+1} + \frac{a_n + c_n}{1 - c_n} ||x_{n+1} - S^n x_{n+1}|| + \frac{2c_n}{1 - c_n} ||x_{n+1} - T x_{n+1}|| \]

\[ \leq d'_{n+1} + \frac{a_n + c_n}{1 - c_n} (||x_{n+1} - x_n|| + ||x_n - T^n x_n||) + ||T^n x_n - S^n x_{n+1}|| \]

\[ \leq d'_{n+1} + \frac{a_n + c_n}{1 - c_n} (||x_{n+1} - x_n|| + d_n + \frac{a_n + c_n}{1 - c_n} ||x_n - x_{n+1}||) + \frac{2c_n}{1 - c_n} ||x_n - T^n x_n|| \]

\[ \leq d'_{n+1} + \frac{a_n + c_n}{1 - c_n} \left[ \frac{1 + a_n}{1 - c_n} ||x_n - x_{n+1}|| \right] + \frac{1 + c_n}{1 - c_n} d_n. \] (2.3.5)

Substituting (2.3.4) into (2.3.5), we get

\[ ||x_{n+1} - T x_{n+1}|| \leq \frac{1 - c_n}{1 - 3c_n} \left[ d'_{n+1} + \frac{a_n + c_n}{1 - c_n} \left( \frac{1 + a_n}{1 - c_n} + \frac{2c_n}{1 - c_n} d'_n + \frac{1 + c_n}{1 - c_n} d_n \right) \right]. \]

Similarly, we can prove that

\[ ||x_{n+1} - S x_{n+1}|| \leq \frac{1 - c_n}{1 - 3c_n} \left[ d_{n+1} + \frac{a_n + c_n}{1 - c_n} \left( \frac{1 + a_n}{1 - c_n} + \frac{2c_n}{1 - c_n} d'_n + \frac{1 + c_n}{1 - c_n} d'_n \right) \right] \]

\[ \leq \frac{1 - c_n}{1 - 3c_n} \left[ d_{n+1} + \frac{a_n + c_n}{1 - c_n} \left( \frac{1 + a_n}{1 - c_n} + \frac{a_n + 2c_n}{1 - c_n} d'_n + \frac{1 + c_n}{1 - c_n} d'_n \right) \right]. \] (2.3.6)

\[ \leq \frac{1 - c_n}{1 - 3c_n} \left[ d_{n+1} + \frac{a_n + c_n}{1 - c_n} \left( \frac{1 + a_n}{1 - c_n} + \frac{2c_n}{1 - c_n} d'_n + \frac{1 + c_n}{1 - c_n} d'_n \right) \right]. \] (2.3.7)

\[ \square \]

**Lemma 2.3.6.** Let \( X \) be a Banach space and \( D \) a nonempty subset of \( X \). Let \( S, T : D \to D \) be two mappings such that

\[ ||T^n x - S^n y|| \leq \alpha_n ||x - y|| + c_n (||x - T^n y|| + ||y - S^n x||) \]

for all \( x, y \in D \) and \( n \in N \), where \( a_n, c_n \geq 0 \) and satisfying the following condition

(i) \( c_n < 1 \) for all \( n \in N \)

(ii) \( \frac{a_n + 2c_n}{1 - c_n} \leq 1 \) for all \( n \in N \)
If $F(T) \cap F(S) \neq \emptyset$.

Then,

$$||T^n x - p|| \leq \frac{a_n + 2c_n}{1 - c_n} ||x - p||.$$

and

$$||S^n x - p|| \leq \frac{a_n + 2c_n}{1 - c_n} ||x - p||,$$

for all $x \in D$ and $n \in N$.

Proof. (a) Let $p \in F(T) \cap F(S)$. Then,

$$||T^n x - p|| \leq a_n ||x - p|| + c_n (||x - p|| + ||S^n x - p||)$$

$$\leq (a_n + c_n)||x - p|| + c_n ||S^n x - p||$$

and

$$||S^n x - p|| \leq (a_n + c_n)||x - p|| + c_n ||T^n x - p||.$$

Now,

$$||T^n x - p|| \leq (a_n + c_n)||x - p|| + c_n ((a_n + c_n)||x - p|| + c_n ||T^n x - p||)$$

$$\leq (1 + c_n)(a_n + c_n)||x - p|| + c_n^2 ||T^n x - p||,$$

which implies that

$$||T^n x - p|| \leq \frac{a_n + c_n}{1 - c_n} ||x - p||.$$

Similarly,

$$||S^n x - p|| \leq \frac{a_n + c_n}{1 - c_n} ||x - p||,$$

for all $x \in D$ and $n \in N$. \qed
In a paper, Tan and Xu [37] studied the asymptotically nonexpansive mapping for convergence of Ishikawa iterative sequence. Later, it was extended for quasinonexpansive iterative sequences with error term by Qihou [42]. In a different paper, Qihou [44] introduced \((L - \alpha)\) uniform Lipschitz asymptotically quasinonexpansive mapping to study the convergence of its Ishikawa iterative sequence with error term.

First, we study the problems of the approximation of common fixed points for uniform \((L - \alpha)\) Lipschitz asymptotically quasi-nonexpansive mappings and also for the continuous mappings which satisfy the condition (2.2.2). Our scheme is given by the sequence \(\{x_n\}\) in \(D\) defined as follows:

\[
\begin{align*}
    x_1 &\in D \\
    x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n \\
    y_n &= (1 - \beta_n)x_n + \beta_n S^n x_n, \quad n = 1, 2, ... \\
\end{align*}
\]

Where \(\{\alpha_n\}\) and \(\{\beta_n\}\) are sequences in \([0, 1]\). Our results improve and extend the corresponding previously known results of Khan and Takahashi [69].

### 2.3.2 Main Results

As first result of section-I\(^1\) we prove the following:

**Theorem 2.3.7.** Let \(X\) be a uniformly convex Banach space and \(D\) be a nonempty closed convex bounded subset of Banach space \(X\). Let \(T, S : D \rightarrow D\) be the asymptotically quasi-nonexpansive mappings with sequence \(\{k_n\}\) such that \(\sum_{n=1}^{\infty} (k_n - 1) < \infty\) and \(F(T) \cap F(S) \neq \emptyset\). Define a sequence \(\{x_n\}\) in \(D\) as (2.3.6) Then the following hold:

\(\text{(a)}\) \(\lim_{n \to \infty} \|x_n - p\| = \lim_{n \to \infty} \|x_n - p\| \) exists.

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\(^1\)S.C. Shrivastava, “Approximation of common fixed point of certain class of nonlinear mappings”, Accepted 2011, Journal of Applied Mathematical Sciences.
(b) \( \lim_{n \to \infty} ||x_n - Tx_n|| = 0 = ||x_n - Sx_n|| \) if \( S \) and \( T \) is a uniform \((L - \alpha)\)-Lipschitzian.

**Proof.** Let \( p \in F(T) \cap F(S) \). Then

\[
||x_{n+1} - p|| = ||(1 - \alpha_n)x_n + \alpha_n T^n y_n - p|| \\
\leq ||x_n - p|| + ||T^n y_n - p|| \\
\leq ||x_n - p|| + k_n ||y_n - p||
\]

(2.3.9)

\[
||y_n - p|| = ||(1 - \beta_n)x_n + \beta_n S^n x_n - p|| \\
\leq (1 - \beta_n)||x_n - p|| + \beta_n ||S^n x_n - p|| \\
\leq k_n ||x_n - p||
\]

(2.3.10)

From (2.3.7) and (2.3.8), we get,

\[
||x_{n+1} - p|| \leq k_n^2 ||x_n - p||.
\]

Since,

\[
\sum_{n=1}^{\infty} k_n^2 = \sum_{n=1}^{\infty} [1 + (k_n - 1)(k_n + 1)] \\
\leq \sum_{n=1}^{\infty} [1 + (k_n - 1) \sup_{n \in N} (k_n + 1)].
\]

Now by lemma 2.3.1, we conclude that

\[
\lim_{n \to \infty} ||x_n - p|| \text{ exists.}
\]

Suppose \( \lim_{n \to \infty} ||x_n - p|| = d \), for some \( d > 0 \). Since

\[
||T_n y^n - p|| \leq k_n ||y_n - p||,
\]

it follows that

\[
\limsup_{n \to \infty} ||T^n y_n - p|| \leq \limsup_{n \to \infty} (k_n ||y_n - p||).
\]
We observe that,

\[ \lim_{n \to \infty} ||x_n - p|| \leq d, \]

and

\[ \lim_{n \to \infty} ||T^n y_n - p|| \leq d. \]

Then,

\[ \lim_{n \to \infty} ||x_{n+1} - p|| = \lim_{n \to \infty} ||\alpha_n(T^n y_n - p) + (1 - \alpha_n)(x_n - p)|| - d. \]

From Lemma 2.3.2, we obtain,

\[ \lim_{n \to \infty} ||x_n - T^n y_n|| = 0. \] (2.3.11)

Further,

\[ ||x_n - p|| \leq ||x_n - T^n y_n|| + ||T^n y_n - p||, \]

\[ \leq ||x_n - T^n y_n|| + k_n ||y_n - p||, \]

gives that

\[ d \leq \liminf_{n \to \infty} ||y_n - p|| \leq \limsup_{n \to \infty} ||y_n - p|| \leq d. \]

Hence,

\[ \lim_{n \to \infty} ||y_n - p|| = d, \]

implies that

\[ \lim_{n \to \infty} ||\beta_n(S^n x_n - p) + (1 - \beta_n)(x_n - p)|| = d. \]

Using Lemma 2.3.2, we get,

\[ \lim_{n \to \infty} ||x_n - S^n x_n|| = 0. \] (2.3.12)
Again,

\[ ||T^n x_n - x_n|| \leq ||T^n x_n - T^n y_n|| + ||T^n y_n - x_n|| \]
\[ \leq k_n ||x_n - y_n|| + ||T^n y_n - x_n|| \]
\[ \leq k_n ||x_n - (1 - \beta_n) x_n + \beta_n S^n x_n|| + ||T^n y_n - x_n|| \]
\[ \leq k_n ||x_n - S^n x_n|| + ||T^n y_n - x_n||, \]

implies together with (2.3.9) and (2.3.10) that

\[ \lim_{n \to \infty} ||x_n - T^n x_n|| = 0 = \lim_{n \to \infty} ||x_n - S^n x_n||. \]

Applying Lemma 2.3.4 it can be shown that

\[ \lim_{n \to \infty} ||x_n - T x_n|| = 0 = \lim_{n \to \infty} ||x_n - S x_n||. \]

Next result we prove as below:

**Theorem 2.3.8.** Let \( X \) be a uniformly convex Banach space satisfying Opial’s condition and let \( D, T, S \) and \( \{x_n\} \) are same as in Theorem 2.3.7. If \( F(T) \cap F(S) \neq \emptyset \) then \( \{x_n\} \) converges weakly to a common fixed point of \( T \) and \( S \).

**Proof.** We prove that \( \{x_n\} \) has a unique weak subsequential limit in \( F(T) \cap F(S) \).

To prove this, let \( u \) and \( v \) be weak limits of the subsequences \( \{x_{n_i}\} \) and \( \{x_{n_j}\} \) of \( \{x_n\} \) respectively. By Theorem 2.3.7,

\[ \lim_{n \to \infty} ||x_n - T x_n|| = 0 = \lim_{n \to \infty} ||x_n - S x_n|| \]

and \( I-T, I-S \) are demiclosed with respect to zero by Lemma 2.3.3, we obtain that \( Tu = u \) and \( Su = u \). Similarly, we can prove that \( v \in F(T) \cap F(S) \). If \( u \neq v \), then
by Opial’s condition.

\[
\lim_{n \to \infty} \|x_n - u\| = \lim_{n_i \to \infty} \|x_{n_i} - u\| < \lim_{n_i \to \infty} \|x_{n_i} - u\|
\]

\[
= \lim_{n \to \infty} \|x_n - v\| < \lim_{n_j \to \infty} \|x_{n_j} - u\|
\]

\[
< \lim_{n_j \to \infty} \|x_{n_j} - u\| = \lim_{n \to \infty} \|x_n - u\|
\]

This is contradiction and hence the proof is complete.

Another important result we prove as below:

**Theorem 2.3.9.** Let \(X\) be uniformly convex Banach space and let \(D\) be a nonempty closed convex bounded subset of \(X\) which satisfying Opial’s condition. Let \(T, S : D \to D\) be the continuous mappings satisfying condition (2.2.2). Given a sequence \(\{x_n\}\) as in (2.3.6), then \(\{x_n\}\) converges weakly to a common fixed point of \(T\) and \(S\).

**Proof.** Since \(p \in F(T) \cap F(S)\). Then,

\[
\|x_{n+1} - p\| = \|(1 - \alpha_n)x_n + \alpha_n T^n y_n - p\|,
\]

\[
\leq \|x_n - p\| + \|T^n y_n - p\|.
\]

Using Lemma 2.3.6, we obtain,

\[
\|x_{n+1} - p\| \leq \|x_n - p\| + \frac{\alpha_n + c_n}{1 - c_n} \|p - y_n\|
\]

\[
\leq \|x_n - p\| + \frac{\alpha_n + c_n}{1 - c_n} (\|p - x_n\| + \|p - S^n x_n\|).
\]

Now, again using Lemma (2.3.6),

\[
\|x_{n+1} - p\| \leq \|x_n - p\| + \frac{\alpha_n + c_n}{1 - c_n} \|x_n - p\| + \left(\frac{\alpha_n + c_n}{1 - c_n}\right)^2 \|x_n - p\|
\]

Let \(\frac{\alpha_n + c_n}{1 - c_n} = L_n\). Then,

\[
\|x_{n+1} - p\| \leq (1 + L_n + L_n^2) \|x_n - p\|.
\]
From Lemma 2.3.1, we get $\lim_{n \to \infty} ||x_n - p||$ exists. Let $\lim_{n \to \infty} ||x_n - p|| = d$ for some $d > 0$.

Since,

$$||y_n - p|| = ||(1 - \beta_n)x_n + \beta_n S_n x_n - p|| \leq ||x_n - p|| + L_n ||x_n - p||.$$ 

Now,

$$\lim \sup_{n \to \infty} ||y_n - p|| \leq \lim \sup_{n \to \infty} ||x_n - p|| \leq d,$$

and

$$||T^n y_n - p|| = L_n ||y_n - p||.$$ 

Then,

$$\lim \sup_{n \to \infty} ||T^n y_n - p|| \leq \lim \sup_{n \to \infty} ||x_n - p|| \leq d.$$ 

Now consider, we have

$$\lim_{n \to \infty} ||x_{n+1} - p|| = \lim_{n \to \infty} ||\alpha_n (T^n y_n - p) + (1 - \alpha_n) (x_n - p)||.$$ 

From Lemma 2.3.2, we obtain,

$$\lim_{n \to \infty} ||x_n - T^n y_n|| = 0.$$ 

Next,

$$||x_n - p|| \leq ||x_n - T^n y_n|| + ||T^n y_n - p|| \leq ||x_n - T^n y_n|| + L_n ||y_n - p||.$$ 

Note that,

$$||x_n - p|| \leq \lim \inf_{n \to \infty} ||y_n - p|| \leq \lim \sup_{n \to \infty} ||y_n - p|| \leq d.$$
Hence,

\[
\lim_{n \to \infty} ||y_n - p|| = d.
\]

That is,

\[
\lim_{n \to \infty} ||\beta_n (S^n x_n - p) + (1 - \beta_n)(x_n - p)|| = d.
\]

Since

\[
\limsup_{n \to \infty} ||S^n x_n - p|| \leq d,
\]

and

\[
\limsup_{n \to \infty} ||x_n - p|| \leq d.
\]

From Lemma 2.3.2, we obtain

\[
\lim_{n \to \infty} ||x_n - S^n x_n|| = 0 \tag{2.3.13}
\]

Now, again

\[
\limsup_{n \to \infty} ||S^n y_n - p|| \leq d,
\]

and

\[
\limsup_{n \to \infty} ||x_n - p|| \leq d.
\]

From Lemma 2.3.2, we get,

\[
\lim_{n \to \infty} ||x_n - S^n y_n|| = 0 \tag{2.3.14}
\]

We obtain that

\[
||x_n - T^n x_n|| \leq ||x_n - S^n x_n|| + ||S^n y_n - T^n x_n||
\]

\[
\leq ||x_n - S^n y_n|| + L_n ||x_n - y_n|| + \frac{2c_n}{1 - c_n} ||x_n - T^n x_n||
\]

\[
\leq \frac{1 - c_n}{1 - 3c_n} \{L_n ||x_n - S^n x_n|| + ||x_n - S^n y_n||\}.
\]
implies together with (2.3.11) and (2.3.12) that
\[ \lim_{n \to \infty} ||x_n - S^n x_n|| = o = \lim_{n \to \infty} ||x_n - T^n x_n||. \]

Lemma 2.3.5 reveals that
\[ \lim_{n \to \infty} ||x_n - T x_n|| = o = \lim_{n \to \infty} ||x_n - S x_n||. \]

The rest of the proof follows the lines similar to Theorem 2.3.8 and is therefore omitted. This completes the proof of the theorem.

Following we obtain as consequence:

**Theorem 2.3.10.** Let \( D \) be a nonempty compact convex subset of a uniformly convex Banach space \( X \) and \( T, S \) and \( \{x_n\} \) as in Theorem 2.3.7. If \( F(T) \cap F(S) \neq \emptyset \), then \( \{x_n\} \) converges strongly to a common fixed point of \( T \) and \( S \).

**Theorem 2.3.11.** Let \( D \) be a nonempty compact convex subset of a uniformly convex Banach space \( X \) and let \( T, S : D \to D \) be the continuous mappings satisfying condition (2.2.1). Given a sequence \( \{x_n\} \) as in (2.3.6), the \( \{x_n\} \) converges strongly to a common fixed point of \( T \) and \( S \).

**Remark 2.3.1.** Theorem 2.3.10 and Theorem 2.3.11 generalize the results of Khan and Takahashi [69].

### 2.4 Section-II

In this Section, we study the convergence of the sequence of Ishikawa iteration of rank- \( r \) to the common fixed points of a finite family of asymptotically quasi- nonexpansive
mappings in uniformly convex Banach spaces. Our results extend and improve some
known recent results. Let $C$ be a subset of normed space $X$ and $T : C \to C$ be a
mapping. Then $T$ is said to be an asymptotically quasi-nonexpansive mapping, if
$F(T) \neq \phi$ and there is a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
\[
||T^nx - p|| \leq k_n||x - p|| \text{ for all } x \in C \text{ and } p \in F(T),
\]
($F(T)$ denotes the set of fixed points of $T$). $T$ is an asymptotically nonexpansive
mapping [25], if there is a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that
\[
||T^nx - T^ny|| \leq k_n||x - y|| \text{ for all } x, y \in C.
\]
If for each $n \in N$, there are constants $L > 0$ and $\alpha > 0$ such that
\[
||T^nx - T^ny|| \leq L||x - y||^\alpha \text{ for all } x, y \in C.
\]
Then $T$ is called uniformly $(L - \alpha)$-Lipschitz. Every asymptotically nonexpansive
mapping is uniformly $(L - \alpha)$-Lipschitz mapping.

In [72], Ishikawa introduced a new iteration process as follows:
\[
\begin{align*}
x_{n+1} &= (1 - a_n)x_n + a_nTy_n, \\
y_n &= (1 - b_n)x_n + b_nTx_n, n = 1, 2, \ldots,
\end{align*}
\]
where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$ satisfying certain restrictions.

In 1973, Petryshyn and Williamson [84] gave necessary and sufficient conditions for
the convergence of Mann iterative sequence (cf.[81]) to the fixed points of quasi nonex-
pansive mappings. In 1997 Ghosh and Debnath [51] extended the results of Petryshyn
and Williamson [84] and gave necessary and sufficient conditions for Ishikawa itera-
tive sequence to converge to the fixed point for quasi-nonexpansive mappings.

Qihou [44] extended the results of [51, 84] and gave the necessary and sufficient
conditions for Ishikawa iterative sequence to converge to the fixed point of asymptotically quasi-nonexpansive mappings.

In a paper [11] Ishikawa iteration process of rank-r was introduced which is similar to the following:

\[
\begin{align*}
    x_1 & \in C \\
    x_{n+1} & = (1 - a_{n,i})x_n + a_{n,i}T_i y_{n,i}; \\
    y_{n,i} & = (1 - a_{n,i+1})x_n + a_{n,i+1}T_i y_{n,i+1}; i = 1, 2, 3, ..., r - 1 \\
    y_{n,r} & = x_n
\end{align*}
\]

(2.4.2)

The modified Ishikawa iteration process of rank r is the following:

\[
\begin{align*}
    x_1 & \in C \\
    x_{n+1} & = (1 - a_{n,i})x_n + a_{n,i}T^n_i y_{n,i}; \\
    y_{n,i} & = (1 - a_{n,i+1})x_n + a_{n,i+1}T^n_i y_{n,i+1}; i = 1, 2, 3, ..., r - 1 \\
    y_{n,r} & = x_n
\end{align*}
\]

(2.4.3)

It is very useful in computing to common fixed points of nonlinear mappings.

We therefore study the convergence of Ishikawa iteration of rank 3 for three uniformly \((L - \alpha)\)-Lipschitz type asymptotically quasi-nonexpansive mappings on a compact convex subset of a uniform convex Banach space in this section. Our scheme is given as follows:

Let C be a nonempty compact convex subset of a uniformly convex Banach space \(X\) and for \(i = 1, 2, 3\), let \(T_i : C \to C\) be uniformly \((L_i - \alpha_i)\)-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence \(\{k_i^n\}\) such that \(\sum_{n=1}^{\infty} (k_i^n - 1) < \infty\).

Define a sequence \(\{x_n\}\) in C as follows:

\[
\begin{align*}
    x_1 & \in C \\
    x_{n+1} & = (1 - a_n)x_n + a_nT^n_1 y_n, \\
    y_n & = (1 - b_n)x_n + b_nT^n_2 z_n, \\
    z_n & = (1 - c_n)x_n + c_nT^n_3 x_n, \text{ for all } n \in N,
\end{align*}
\]

(2.4.4)
Where \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are sequences in \((0,1)\). Our results generalize and improve the results of [42, 44, 37].

### 2.4.1 Preliminaries

Let us recall following lemmas which will be used to prove the main theorems, of this section.

**Lemma 2.4.1.** [42] Let \( \{\alpha_n\}_{n=1}^\infty, \{\beta_n\}_{n=1}^\infty, \{\gamma_n\}_{n=1}^\infty \) be three sequences of nonnegative numbers satisfying
\[
\alpha_{n+1} \leq (1 + \beta_n)\alpha_n + \gamma_n \quad \forall n \in \mathbb{N}
\]
and
\[
\sum_{n=1}^\infty \beta_n < +\infty, \sum_{n=1}^\infty \gamma_n < +\infty.
\]
Then \( \lim_{n \to \infty} \alpha_n \) exists.

**Lemma 2.4.2.** [32] Let \( X \) be a uniformly convex Banach space, \( 0 < \alpha \leq t_n \leq \beta < 1, x_n, y_n \in X, \limsup_{n \to \infty} \|x_n\| \leq a, \limsup_{n \to \infty} \|y_n\| \leq a, \lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = a, a \geq 0 \). Then \( \lim_{n \to \infty} \|x_n - y_n\| = 0 \).

**Lemma 2.4.3.** Let \( C \) be a nonempty convex subset of a uniformly convex Banach space \( X \) and for \( i = 1, 2, 3 \), let \( T_i : C \to C \) be uniformly \( (L_i - \alpha_i) \)-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence \( \{k_n^i\} \) such that \( \sum_{n=1}^\infty (k_n^i - 1) < \infty \). Define a sequence \( \{x_n\} \) in \( C \) as follows:
\[
\begin{cases}
  x_1 \in C \\
  x_{n+1} = (1 - a_n)x_n + a_n T_1^n y_n, \\
  y_n = (1 - b_n)x_n + b_n T_2^n z_n, \\
  z_n = (1 - c_n)x_n + c_n T_3^n x_n, \text{ for all } n \in \mathbb{N},
\end{cases}
\]  
(2.4.5)

where \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \) are sequences in \((0,1)\). If \( F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset \), then, \( \lim_{n \to \infty} \|x_n - p\| \) exists for all \( p \in F(T_1) \cap F(T_2) \cap F(T_3) \).
Proof. Let \( p \in \bigcap_{i=1}^{3} F(T_i) \). Then,

\[
||z_n - p|| = ||(1 - c_n)x_n + c_nT_3^n x_n - p|| \\
\leq (1 - c_n)||x_n - p|| + c_n||T_3^n x_n - p|| \\
\leq k_n^3||x_n - p||. \quad (2.4.6)
\]

and

\[
||y_n - p|| = ||(1 - b_n)x_n + b_nT_2^n z_n - p|| \\
\leq (1 - b_n)||x_n - p|| + b_n||T_2^n z_n - p|| \\
\leq k_n^{(2)}k_n^{(3)}||x_n - p||. \quad (2.4.7)
\]

From (2.4.6) and (2.4.7), we have,

\[
||x_{n+1} - p|| \leq (1 - a_n)||x_n - p|| + a_n||T_1^n y_n - p|| \\
\leq k_n^{(1)}k_n^{(2)}k_n^{(3)}||x_n - p||. \quad (2.4.8)
\]

Observe that,

\[
\sum_{n=1}^{\infty} (k_n^{(1)}k_n^{(2)}k_n^{(3)} - 1) = \sum_{n=1}^{\infty} [(k_n^{(1)}k_n^{(2)}k_n^{(3)} - 1) + k_n^{(1)}(k_n^{(2)} - 1) + k_n^{(1)} - 1] \\
\leq K_1 \sum_{n=1}^{\infty} (k_n^{(3)} - 1) + K_2 \sum_{n=1}^{\infty} (k_n^{(2)} - 1) \\
+ \sum_{n=1}^{\infty} (k_n^{(1)} - 1) < \infty,
\]

for some constants \( K_1, K_2 > 0 \). Using Lemma 2.4.1, we obtain that \( \lim_{n \to \infty} ||x_n - p|| \) exists. \( \Box \)
2.4.2 Main Results

Now, we prove main result of section-II as below:

**Theorem 2.4.4.** Let $C$ be a nonempty compact convex subset of a uniformly convex Banach space $X$ and for $i = 1, 2, 3$, let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty$. Define a sequence $\{x_n\}$ in $C$ as follows:

\[
\begin{align*}
    x_1 \in C \\
x_{n+1} &= (1 - a_n)x_n + a_nT_1^n y_n, \\
y_n &= (1 - b_n)x_n + b_nT_2^n z_n, \\
z_n &= (1 - c_n)x_n + c_nT_3^n x_n, \text{ for all } n \in \mathbb{N},
\end{align*}
\]

(2.4.9)

where $\{a_n\}, \{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ such that $0 < a < a_n \leq \bar{a} < 1$, $0 < b_n \leq \bar{b} < 1$ and $0 < c \leq c_n \leq \bar{c} < 1$.

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$ then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2$ and $T_3$.

**Proof.** By lemma 2.4.3, we have $\lim_{n \to \infty} ||x_n - p||$ exists for all $p \in F(T_1) \cap F(T_2) \cap F(T_3)$.

Set $\lim_{n \to \infty} ||x_n - p|| = d$ for some $d > 0$. Then, from (2.4.2) and (2.4.3) we have

\[
\limsup_{n \to \infty} ||z_n - p|| \leq \limsup_{n \to \infty} ||x_n - p|| = d,
\]

and

\[
\limsup_{n \to \infty} ||y_n - p|| \leq \limsup_{n \to \infty} ||x_n - p|| = d,
\]

---

respectively.

Note that,

$$\limsup_{n \to \infty} ||T_n y_n - p|| \leq \limsup_{n \to \infty} (k_n^{(1)}||y_n - p||) \leq d,$$

and

$$\lim_{n \to \infty} ||x_{n+1} - p|| = \lim_{n \to \infty} ||(1 - a_n)(x_n - p) + a_n(T_n y_n - p)|| = d.$$

Thus, from Lemma 2.4.2, we get

$$\lim_{n \to \infty} ||x_n - T_n y_n|| = 0. \quad (2.4.10)$$

Next,

$$||x_n - p|| \leq ||x_n - T_n y_n|| + ||T_n y_n - p||$$

$$\leq ||x_n - T_n y_n|| + k_n^{(1)}||y_n - p||,$$

which gives that,

$$d \leq \liminf_{n \to \infty} ||y_n - p|| \leq \limsup_{n \to \infty} ||y_n - p|| \leq d,$$

and hence,

$$\lim_{n \to \infty} ||y_n - p|| = d.$$

Note that,

$$\limsup_{n \to \infty} ||T_n z_n - p|| \leq \limsup_{n \to \infty} (k_n^2||z_n - p||) \leq d,$$

and

$$d = \lim_{n \to \infty} ||y_n - p|| = \lim_{n \to \infty} ||(1 - b_n)(x_n - p) + b_n(T_n z_n - p)||.$$

Thus, from Lemma 2.4.2 , we get,

$$\lim_{n \to \infty} ||x_n - T_n z_n|| = 0 \quad (2.4.11)$$
Note that,

\[ ||x_n - p|| \leq ||x_n - T^n_2 z_n|| + ||T^n_2 z_n - p|| \]
\[ \leq ||x_n - T^n_2 z_n|| + k^{(2)}_n ||z_n - p||, \]

which gives that,

\[ d \leq \liminf_{n \to \infty} ||z_n - p|| \leq \limsup_{n \to \infty} ||z_n - p|| \leq d, \]

and hence,

\[ \lim_{n \to \infty} ||z_n - p|| = d. \]

Since,

\[ \limsup_{n \to \infty} ||T^n_3 x_n - p|| \leq \limsup_{n \to \infty} (k^{(3)}_n ||x_n - p||) \leq d, \]

and

\[ d = \lim_{n \to \infty} ||z_n - p|| = \lim_{n \to \infty} ||(1 - c_n)(x_n - p) + c_n(T^n_3 x_n - p)||. \]

Thus, from Lemma 2.4.2, we get,

\[ \lim_{n \to \infty} ||x_n - T^n_3 x_n|| = 0. \tag{2.4.12} \]

Since, \( C \) is compact, \( \{x_n\}_{n=1}^\infty \) has a convergent subsequence \( \{x_{n_k}\}_{n=1}^\infty \). Let

\[ \lim_{n \to \infty} x_{n_k} = p. \tag{2.4.13} \]

Thus from (2.4.9),(2.4.10) and (2.4.11), we have

\[ ||x_{n_k+1} - x_{n_k}|| \leq a_{n_k} ||T_1^{n_k} y_{n_k} - x_{n_k}|| \to 0, \tag{2.4.14} \]

and

\[ ||y_n - x_n|| \leq b_n ||T_2^n z_n - x_n|| \to 0. \tag{2.4.15} \]
again from (2.4.10) and (2.4.13), we have,

$$\lim_{k \to \infty} T_{1k} y_{nk} = p. \quad (2.4.16)$$

Since,

$$\lim_{k \to \infty} x_{nk+1} = p,$$

we have,

$$\lim_{k \to \infty} T_{1k+1} y_{nk+1} = p. \quad (2.4.17)$$

From (2.4.14), (2.4.15), (2.4.16) and (2.4.17) therefore we have,

$$0 \leq ||p - T_1 p|| = ||p - T_{1k+1} y_{nk+1} + T_{1k} y_{nk+1} - T_{1k} y_{nk+1} - T_{1k+1} x_{nk+1} + T_{1k+1} x_{nk+1} - T_{1k+1} x_{nk} + T_{1k+1} x_{nk} - T_{1k} y_{nk} + T_{1k+1} y_{nk} - T_1 p||$$

$$\leq ||p - T_{1k+1} y_{nk+1}|| + ||T_{1k+1} y_{nk+1} - T_{1k+1} x_{nk+1}|| + ||T_{1k+1} x_{nk+1} - T_{1k} x_{nk} + ||T_{1k} x_{nk} - T_{1k+1} y_{nk}||$$

$$+ ||T_{1k+1} y_{nk} - T_1 p||$$

$$\leq ||p - T_{1k+1} y_{nk+1}|| + L_1 |y_{nk+1} - x_{nk+1}|^{\alpha_1}$$

$$+ L_1 |x_{nk+1} - x_{nk}|^{\alpha_1} + L_1 |x_{nk} - y_{nk}|^{\alpha_1}$$

$$+ L_1 |T_{nk} y_{nk} - p|^{\alpha_1} \to 0 \text{ as } n \to \infty.$$

Next,

$$||z_n - x_n|| \leq c_n ||T_{nk} x_n - x_n|| \to 0. \quad (2.4.18)$$

From (2.4.11) and (2.4.13), we have

$$\lim_{k \to \infty} T_{2k} z_{nk} = p. \quad (2.4.19)$$
Since,

$$\lim_{k \to \infty} x_{n_k+1} = p,$$

then we have,

$$\lim_{k \to \infty} T_2^{n_k+1} z_{n_k+1} = p. \quad (2.4.20)$$

From (2.4.14), (2.4.19) and (2.4.20), we have,

$$0 \leq ||p - T_2 p|| = ||p - T_2^{n_k+1} z_{n_k+1} + T_2^{n_k+2} z_{n_k+1} - T_2^{n_k+1} x_{n_k+1} + T_2^{n_k+1} x_{n_k+1} - T_2^{n_k+1} x_{n_k} + T_2^{n_k+1} x_{n_k} - T_2^{n_k+1} z_{n_k} + T_2^{n_k+1} z_{n_k} - T_2 p||$$

$$\leq ||p - T_2^{n_k+1} z_{n_k+1}|| + ||T_2^{n_k+1} z_{n_k+1} - T_2^{n_k+1} x_{n_k+1}|| + ||T_2^{n_k+1} x_{n_k+1} - T_2^{n_k+1} x_{n_k}|| + ||T_2^{n_k+1} x_{n_k} - T_2^{n_k+1} z_{n_k}|| + ||T_2^{n_k+1} z_{n_k} - T_2 p||$$

$$\leq ||p - T_2^{n_k+1} z_{n_k+1}|| + L_2 ||z_{n_k} + 1 - x_{n_k+1}||^{\alpha_2} + L_2 ||x_{n_k} - z_{n_k}||^{\alpha_2} + L_2 ||T_2^{n_k} z_{n_k} - p||^{\alpha_2} \to 0 \text{ as } n \to \infty.$$

Now from (2.4.12) and (2.4.13), we have

$$\lim_{k \to \infty} T_3^{n_k} x_{n_k} = p. \quad (2.4.21)$$

Since $$\lim_{k \to \infty} x_{n_k+1} = p,$$ it follows from (2.4.12) that

$$\lim_{k \to \infty} T_2^{n_k+1} x_{n_k+1} = p. \quad (2.4.22)$$
From (2.4.14) and (2.4.22), we obtain,
\[
0 \leq ||p - T_{3}p|| = ||p - T_{3}^{n_{k}+1}x_{n_{k}+1} + T_{3}^{n_{k}+2}x_{n_{k}+1} - T_{3}^{n_{k}+1}x_{n_{k}} + T_{3}^{n_{k}+1}x_{n_{k}} - T_{3}p|| \\
\leq ||p - T_{3}^{n_{k}+1}x_{n_{k}+1}|| + L_{3}||x_{n_{k}+1} - x_{n_{k}}||^{\alpha_{3}} + L_{3}||T_{3}^{n_{k}}x_{n_{k}} - p||^{\alpha_{3}} \to 0 \text{ as } n \to \infty.
\]
Thus, p is a common fixed point of \(T_{1}, T_{2}\) and \(T_{3}\). Since the subsequence \(\{x_{n_{k}}\}_{k=1}^{\infty}\) of \(\{x_{n}\}_{n=1}^{\infty}\) converges to p and \(\lim_{n \to \infty} ||x_{n} - p||\) exists, we conclude that \(\lim_{n \to \infty} x_{n} = p\).

**Corollary 2.4.5.** Let \(C\) be a nonempty compact convex subset of a uniformly convex Banach space and for \(i = 1, 2\), let \(T_{i} : C \to C\) be uniformly \((L_{i} - \alpha_{i})\)-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence \(\{k_{n}^{(i)}\}\) such that \(\sum_{n=1}^{\infty} (k_{n}^{(i)} - 1) < \infty\). Define a sequence \(\{x_{n}\}\) in \(C\) as follows:
\[
\begin{align*}
x_{1} & \in C \\
x_{n+1} & = (1 - a_{n})x_{n} + a_{n}T_{1}^{n}y_{n}, \\
y_{n} & = (1 - b_{n})x_{n} + b_{n}T_{2}^{n}x_{n}, \text{ for all } n \in \mathbb{N},
\end{align*}
\]
where \(\{a_{n}\}\) and \(\{b_{n}\}\) are sequences in \([0, 1]\) such that \(0 < a < a_{n} \leq \overline{a} < 1\) and \(0 < b < b_{n} \leq \overline{b} < 1\).

If \(F(T_{1}) \cap F(T_{2}) \neq \emptyset\), then the sequence \(\{x_{n}\}\) converges strongly to a common fixed point of \(T_{1}\) and \(T_{2}\).

**Remark 2.4.1.** Corollary 2.4.5 is an improvement of the results of Qihou [44] when \(c_{n}\) and \(c'_{n} = 0\).

In the same manner we have the following theorem.
Theorem 2.4.6. Let $C$ be a nonempty compact convex subset of a uniformly convex Banach space and for $i = 1, 2, \ldots, r$; let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty$. Let $\{x_n\}$ be an iterative sequence of the modified Ishikawa iteration process of rank $r$ defined in $C$ by (2.4.3), where $\{a_{n,i}\}$ $(i=1, 2, \ldots, r)$ be sequences of real numbers in $[0, 1]$ such that $0 < a_i \leq a_{n,i} \leq \alpha_i < 1$ for all $i \in 1, 2, \ldots, r$ and $n \in \mathbb{N}$. If $F(T_1) \cap F(T_2) \cap \ldots \cap F(T_r) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_r$.

2.5 Section-III

In this section, we study the convergence of Ishikawa iteration with an error of rank-3 for three uniformly $(L - \alpha)$ lipschitz type asymptotically quasi nonexpansive mappings on a compact convex subset of uniformly convex Banach space.

Our scheme is given as follows:

Let $C$ be a nonempty compact convex subset of a uniformly convex Banach space $X$ and for $i = 1, 2, 3$ let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$ Lipschitz and asymptotically quasinonexpansive mappings with sequence $\{k_n^{(i)}\}$ such that $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty$.

Define a sequence $\{x_n\}$ in $C$ as follows.

\[
\begin{cases}
  x_1 \in C \\
  x_{n+1} = a_n x_n + b_n T_1^n y_n + c_n u_n, \\
  y_n = a'_n x_n + b'_n T_2^n z_n + c'_n v_n, \\
  z_n = a''_n x_n + b''_n T_3^n x_n + c''_n w_n, \forall n \in \mathbb{N}
\end{cases}
\]

where $\{u_n\}, \{v_n\}, \{w_n\}$ are bounded sequences in $C$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{a''_n\}, \{b''_n\}, \{c''_n\}$ are sequences in $(0,1)$ satisfying, $a_n + b_n + c_n = a'_n + b'_n + c'_n =$
\[ a''_n + b''_n + c''_n = 1, \text{ for all } n \in N \text{ and } \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c'_n < \infty \text{ and } \sum_{n=1}^{\infty} c''_n < \infty. \]

Our results extend and improve the results of [37], [42], [33] and [44].

### 2.5.1 Preliminaries

The following lemmas will be used to prove the main theorems.

**Lemma 2.5.1.** [42] Let \( \{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}, \{\gamma_n\}_{n=1}^{\infty} \) be three sequences of nonnegative numbers satisfying \( \alpha_{n+1} \leq (1 + \beta_n)\alpha_n + \gamma_n \forall n \in N \) and \( \sum_{n=1}^{\infty} \beta_n < +\infty, \sum_{n=1}^{\infty} \gamma_n < +\infty \). Then \( \lim_{n \to \infty} \alpha_n \) exists.

**Lemma 2.5.2.** [33] Let \( X \) be a uniformly convex Banach space, \( 0 < \alpha \leq t_n \leq \beta < 1, x_n, y_n \in X, \limsup_{n \to \infty} ||x_n|| \leq a, \limsup_{n \to \infty} ||y_n|| \leq a, \lim_{n \to \infty} ||t_n x_n + (1 - t_n) y_n|| = a, a \geq 0 \). Then \( \lim_{n \to \infty} ||x_n - y_n|| = 0 \).

**Lemma 2.5.3.** Let \( C \) be a nonempty convex subset of a uniformly convex Banach space \( X \) and for \( i = 1, 2, 3 \), let \( T_i : C \to C \) be uniformly \((L_i - \alpha_i)\)-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence \( \{k_i^j\} \) such that \( \sum_{n=1}^{\infty} (k_i^j - 1) < \infty \). Define a sequence \( \{x_n\} \) in \( C \) as follows:

\[
\begin{align*}
x_1 &\in C, \\
x_{n+1} &= a_n x_n + b_n T_1^n y_n + c_n w_n, \\
y_n &= a'_n x_n + b'_n T_2^n z_n + c'_n v_n, \\
z_n &= a''_n x_n + b''_n T_3^n x_n + c''_n w_n, &\forall n \in N
\end{align*}
\]

where \( \{a_n\}, \{b_n\}, \{c_n\}, \{a'_n\}, \{b'_n\}, \{c'_n\}, \{a''_n\}, \{b''_n\}, \{c''_n\} \) are sequences in \((0,1)\) satisfying, \( a_n + b_n + c_n = a'_n + b'_n + c'_n = a''_n + b''_n + c''_n = 1, \) for all \( n \in N \). If \( F(T_1) \cap F(T_2) \cap F(T_3) \neq 0 \) then \( \lim_{n \to \infty} ||x_n - p|| \) exists for all \( p \in F(T_1) \cap F(T_2) \cap F(T_3) \).
Proof. Let \( p \in \bigcap_{i=1}^{3} F(T_i) \). Since \( C \) is compact, there exists a constant \( M \geq 0 \) such that \( ||u_n - p|| \leq M, ||v_n - p|| \leq M, ||w_n - p|| \leq M \), for all \( n \in \mathbb{N} \).

Then we have,

\[
||z_n - p|| = ||a''_n x_n + b''_n T_3^n x_n + c''_n w_n - p|| \quad (2.5.1)
\]

\[
\leq a''_n ||x_n - p|| + b''_n ||T_3^n x_n - p|| + c''_n ||w_n - p||
\]

\[
\leq a''_n ||x_n - p|| + b''_n k_n^{(3)} ||x_n - p|| + c''_n M
\]

\[
\leq k_n^{(3)} ||x_n - p|| + c''_n M
\]

and

\[
||y_n - p|| \leq a'_n ||x_n - p|| + b'_n ||T_2^n z_n - p|| + c'_n ||v_n - p|| \quad (2.5.2)
\]

\[
\leq a'_n ||x_n - p|| + b'_n k_n^{(2)} ||z_n - p|| + c'_n M
\]

\[
\leq a'_n ||x_n - p|| + b'_n k_n^{(2)} (k_n^{(3)} ||x_n - p|| + c''_n M) + c'_n M
\]

\[
\leq k_n^{(2)} k_n^{(3)} ||x_n - p|| + c''_n \bar{M} + c'_n M
\]

for some \( \bar{M} \geq 0 \) since \( k_n^2 \to 1 \)

From (2.5.1) and (2.5.2), we have

\[
||x_{n+1} - p|| \leq a_n ||x_n - p|| + b_n ||T_1^n y_n - p|| + c_n ||u_n - p|| \quad (2.5.3)
\]

\[
\leq a_n ||x_n - p|| + b_n k_n^{(1)} ||y_n - p|| + c_n M
\]

\[
\leq a_n ||x_n - p|| + b_n k_n^{(1)} (k_n^{(2)} k_n^{(3)} ||x_n - p|| + c''_n \bar{M} + c'_n M) + c_n M
\]

\[
\leq k_n^{(1)} k_n^{(2)} k_n^{(3)} ||x_n - p|| + c_n M + c'_n M' + c''_n M''
\]

for some constants \( M, M', M'' \geq 0 \).
Observe that
\[
\sum_{n=1}^{\infty} (k_n^{(1)}k_n^{(2)}k_n^{(3)} - 1) = \sum_{n=1}^{\infty} [(k_n^{(1)}k_n^{(2)}k_n^{(3)} - 1) + k_n^{(1)}(k_n^{(2)} - 1) + k_n^{(1)} - 1]
\leq K_1 \sum_{n=1}^{\infty} (k_n^{(3)} - 1) + K_2 \sum_{n=1}^{\infty} (k_n^{(2)} - 1)
+ \sum_{n=1}^{\infty} (k_n^{(1)} - 1) < \infty
\]
for some constants $K_1, K_2 > 0$. Using Lemma 2.5.1, we obtain that $\lim_{n \to \infty} ||x_n - p||$ exists.

\[\square\]

2.5.2 Main Results

Now, we prove main result of section-III\(^3\) as below:

**Theorem 2.5.4.** Let $C$ be a nonempty compact convex subset of a uniformly convex Banach space $X$, and for $i=1,2,3$ let $T_i : C \to C$ be uniformly $(L_i - \alpha_i)$ Lipschitz and asymptotically quasi-nonexpansive mappings with sequence $\{k_n^{(i)} \}$ such that $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty$. Define a sequence $\{x_n \}$ in $C$ as follows:

\[
\begin{cases}
x_1 \in C \\
x_{n+1} = a_n x_n + b_n T_1^n y_n + c_n u_n, \\
y_n = a_n' x_n + b_n' T_2^n z_n + c_n' v_n, \\
z_n = a_n'' x_n + b_n'' T_3^n x_n + c_n'' w_n, \text{ for all } n \in N,
\end{cases}
\]

where $\{u_n\}, \{v_n\}, \{w_n\}$ in $C$ and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}, \{c_n'\}, \{a_n''\}, \{b_n''\}, \{c_n''\}$ are sequences in $[0,1]$ satisfying the following, $a_n + b_n + c_n = a_n' + b_n' + c_n' = a_n'' + b_n'' + c_n'' = 1$, for all $n \in N$ and $\sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} c_n' < \infty, \sum_{n=1}^{\infty} c_n'' < \infty.$

If $F(T_1) \cap F(T_2) \cap F(T_3) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2$ and $T_3$.

Proof. By lemma 2.5.3, we have $\lim_{n \to \infty} \|x_n - p\|$ exists for all $p \in F(T_1) \cap F(T_2) \cap F(T_3)$. Set $\lim_{n \to \infty} \|x_n - p\| = d$ for some $d > 0$. Then, from (2.5.1) and (2.5.2) we have

$$\limsup_{n \to \infty} \|z_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| = d$$

and

$$\limsup_{n \to \infty} \|y_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| = d$$

Respectively note that,

$$\limsup_{n \to \infty} \|T_n^1 y_n - p\| \leq \limsup_{n \to \infty} (k^{(1)}_n \|y_n - p\|) \leq d$$

and

$$\lim_{n \to \infty} \|x_{n+1} - p\| = \lim_{n \to \infty} \|a_n x_n + b_n T_n^1 y_n + c_n u_n - p\|$$

$$= \lim_{n \to \infty} \|a_n [x_n - p + \frac{c_n}{2a_n} (u_n - p)] + b_n [T_n^1 y_n - p + \frac{c_n}{2b_n} (u_n - p)]\|$$

$$= \lim_{n \to \infty} \|x_n - p\|$$

Thus from Lemma 2.5.2, we have,

$$\lim_{n \to \infty} \|x_n - T_n^1 y_n + (\frac{c_n}{2a_n} - \frac{c_n}{2b_n}) (u_n - p)\| = 0$$

Note that,

$$\lim_{n \to \infty} \|\frac{c_n}{2a_n} - \frac{c_n}{2b_n} (u_n - p)\| = 0$$

therefore we have

$$\lim_{n \to \infty} \|x_n - T_n^1 y_n\| = 0 \quad (2.5.4)$$
Next,

\[ |x_n - p| \leq |x_n - T^n_1 y_n| + |T^n_1 y_n - p| \]
\[ \leq |x_n - T^n_1 y_n| + k^{(1)}_n ||y_n - p|| \]

which gives that

\[ d \leq \lim\inf_{n \to \infty} ||y_n - p|| \leq \lim\sup_{n \to \infty} ||y_n - p|| \leq d \]

and hence,

\[ \lim_{n \to \infty} ||y_n - p|| = d \]

Note that,

\[ \lim\sup_{n \to \infty} ||T^n_2 z_n - p|| \leq \lim\sup_{n \to \infty} (k^{(2)}_n ||z_n - p||) \leq d \]

\[ \lim_{n \to \infty} ||y_n - p|| = \lim_{n \to \infty} ||x_n - p|| \]

Thus from Lemma 2.5.2, we have

\[ \lim_{n \to \infty} ||x_n - T^n_2 z_n + (\frac{c'_n}{2a'_n} - \frac{c'_n}{2b'_n})(v_n - p)|| = 0 \]

Note that

\[ \lim_{n \to \infty} ||(\frac{c'_n}{2a'_n} - \frac{c'_n}{2b'_n})(v_n - p)|| = 0 \]

therefore we have,

\[ \lim_{n \to \infty} ||x_n - T^n_2 z_n|| = 0 \quad (2.5.5) \]
Next,

\[ ||x_n - p|| \leq ||x_n - T^m_2 z_n|| + ||T^m_2 z_n - p|| \]
\[ \leq ||x_n - T^m_2 z_n|| + k^{(2)}_n||z_n - p|| \]

which gives that

\[ d \leq \liminf_{n \to \infty} ||z_n - p|| \leq \limsup_{n \to \infty} ||z_n - p|| \leq d, \]

and hence

\[ \lim_{n \to \infty} ||z_n - p|| = d \]

Since,

\[ \limsup_{n \to \infty} ||T^m_3 x_n - p|| \leq \limsup_{n \to \infty} (k^{(3)}_n||x_n - p||) \leq d, \]

and

\[ \lim_{n \to \infty} ||z_n - p|| = \lim_{n \to \infty} ||x_n - p|| \]

Thus from Lemma 2.5.2, we have,

\[ \lim_{n \to \infty} ||x_n - T^m_3 x_n + \left( \frac{c''_n}{2a''_n} - \frac{c''_n}{2b''_n} \right)(w_n - p)|| = 0 \]

Since,

\[ \lim_{n \to \infty} ||\left( \frac{c''_n}{2a''_n} - \frac{c''_n}{2b''_n} \right)(w_n - p)|| = 0 \]

therefore we have,

\[ \lim_{n \to \infty} ||x_n - T^m_3 x_n|| = 0 \quad (2.5.6) \]
Next,

\[ \|x_n - p\| \leq \|x_n - T^nx_n\| + \|T^n y_n - p\| \]

\[ \leq \|x_n - T^nx_n\| + k(3) \|x_n - p\| \]

which gives that

\[ d \leq \liminf_{n \to \infty} \|x_n - p\| \leq \limsup_{n \to \infty} \|x_n - p\| \leq d \]

and hence,

\[ \lim_{n \to \infty} \|x_n - p\| = d \]

Since \( C \) is compact, \( \{x_n\}_{n=1}^\infty \) has a convergent subsequence \( \{x_{n_k}\}_{k=1}^\infty \). Let

\[ \lim_{k \to \infty} x_{n_k} = p. \tag{2.5.7} \]

Thus from (2.5.5) and \( \lim_{n \to \infty} c_n = 0 \), we have

\[ \|y_{n_k+1} - x_{n_k}\| \leq b_{n_k} \|T_{n_k} y_{n_k} - x_{n_k}\| + c_{n_k} \|u_{n_k} - x_{n_k}\| \] \tag{2.5.8}

Note that, \( \lim_{n \to \infty} b'_n = 0, \lim_{n \to \infty} c'_n = 0 \) therefore we have,

\[ \|y_n - x_n\| \leq b'_n \|T_{n} z_n - x_n\| + c'n \|v_n - x_n\| \to 0 \]

Thus from (2.5.5) and (2.5.8),

\[ \lim_{k \to \infty} T_{n_k} y_{n_k} = p. \tag{2.5.9} \]

Thus

\[ \lim_{k \to \infty} x_{n_k+1} = p. \]

Similarly,

\[ \lim_{k \to \infty} x_{n_k+2} = p. \]
and

\[
\lim_{k \to \infty} T_1^{n_k+1} y_{n_k+1} = p. \tag{2.5.10}
\]

From (2.5.5), (2.5.7-2.5.10) we have

\[
0 \leq ||p - T_1 p|| = ||p - T_1^{n_k+1} y_{n_k+1} + T_1^{n_k+1} y_{n_k+1} - T_1^{n_k+1} x_{n_k+1} + T_1^{n_k+1} x_{n_k+1} - T_1^{n_k+1} x_{n_k} + T_1^{n_k+1} x_{n_k} - T_1^{n_k+1} y_{n_k} + T_1^{n_k+1} y_{n_k} - T_1 p||
\]
\[
\leq ||p - T_1^{n_k+1} y_{n_k+1}|| + ||T_1^{n_k+1} y_{n_k+1} - T_1^{n_k+1} x_{n_k+1}||
+ ||T_1^{n_k+1} x_{n_k+1} - T_1^{n_k+1} x_{n_k}|| + ||T_1^{n_k+1} x_{n_k} - T_1^{n_k+1} y_{n_k}||
+ ||T_1^{n_k+1} y_{n_k} - T_1 p||
\]
\[
\leq ||p - T_1^{n_k+1} y_{n_k+1}|| + L_1 ||y_{n_k+1} - x_{n_k+1}||^{\alpha_1}
+ L_1 ||T_1^{n_k} y_{n_k} - p||^{\alpha_1}
\to 0 \text{ as } n \to \infty.
\]

Then, from (2.5.6) and \(\lim_{n \to \infty} c'_n = 0\), we have

\[
||y_{n_k} - x_{n_k}|| \leq b'_{n_k} ||T_2^{n_k} z_{n_k} - x_{n_k}|| + c'_n ||v_{n_k} - x_{n_k}|| \to 0 \tag{2.5.11}
\]

Note that, \(\lim_{n \to \infty} b''_n = 0\), \(\lim_{n \to \infty} c''_n = 0\), we have

\[
||z_n - x_n|| \leq b''_n ||T_3^{n_k} x_n - x_n|| + c''_n ||w_n - x_n|| \to 0 \tag{2.5.12}
\]

Thus from (2.5.5) and (2.5.7)

\[
\lim_{n \to \infty} T_2^{n_k} z_{n_k} = p. \tag{2.5.13}
\]

Thus \(\lim_{k \to \infty} x_{n_k+1} = p\). Similarly, \(\lim_{k \to \infty} x_{n_k+2} = p\).

\[
\lim_{k \to \infty} T_2^{n_k+1} z_{n_k+1} = p. \tag{2.5.14}
\]
From (2.5.6), (2.5.8), (2.5.11)-(2.5.14), we have
\[
0 \leq ||p - T_2 p|| = ||p - T_2^{n_k+1} z_{n_k+1} + T_2^{n_k+2} z_{n_k+1} - T_2^{n_k+1} x_{n_k+1} \\
+ T_2^{n_k+1} x_{n_k+1} - T_2^{n_k+1} x_{n_k} + T_2^{n_k+1} x_{n_k} - T_2^{n_k+1} z_n \\
+ T_2^{n_k+1} z_n - T_2 p||
\]
\[
\leq ||p - T_2^{n_k+1} z_{n_k+1}|| + ||T_2^{n_k+1} z_{n_k+1} - T_2^{n_k+1} x_{n_k+1}|| \\
+ ||T_2^{n_k+1} x_{n_k+1} - T_2^{n_k+1} x_{n_k}|| + ||T_2^{n_k+1} x_{n_k} - T_2^{n_k+1} z_n|| \\
+ ||T_2^{n_k+1} z_n - T_2 p||
\]
\[
\leq ||p - T_2^{n_k+1} z_{n_k+1}|| + L_2||z_{n_k} + 1 - x_{n_k+1}||^{\alpha_2} \\
+ L_2||x_{n_k+1} - x_{n_k}||^{\alpha_2} + L_2||x_{n_k} - z_n||^{\alpha_2} \\
+ L_2||z_{n_k} - p||^{\alpha_2} \to 0 \text{ as } n \to \infty.
\]

Then from (2.5.7) and \( \lim_{n \to \infty} c'_n = 0 \), we have
\[
||z_{n_k+1} - x_{n_k}|| \leq b'_n ||T_{n_k}^3 x_{n_k} - x_{n_k}|| + c''_n ||w_{n_k} - x_{n_k}|| \to 0 \quad (2.5.15)
\]

Thus from (2.5.7) and (2.5.8)
\[
\lim_{k \to \infty} T_{n_k}^3 x_{n_k} = p \quad (2.5.16)
\]

Thus \( \lim_{k \to \infty} x_{n_k+1} = p \). Similarly \( \lim_{k \to \infty} x_{n_k+2} = p \), and
\[
\lim_{k \to \infty} T_{n_k+1}^3 x_{n_k+1} = p \quad (2.5.17)
\]
From (2.5.7),(2.5.8),(2.5.15-2.5.17), we have

\[
0 \leq ||p - T_3 p|| = ||p - T_3^{m_k+1} x_{n_k+1} + T_3^{m_k+2} x_{n_k+1} - T_3^{m_k+1} x_{n_k} + T_3^{m_k+1} x_{n_k} - T_3 p|| \\
\leq ||p - T_3^{m_k+1} x_{n_k+1}|| + L_3 ||x_{n_k+1} - x_{n_k}||^{\alpha_3} + L_3 ||T_3^{m_k} x_{n_k} - p||^{\alpha_3} \rightarrow 0 \text{ as } n \rightarrow \infty.
\]

Thus, \(p\) is a common fixed point of \(T_1, T_2\) and \(T_3\). Since the subsequence \(\{x_{n_k}\}_{k=1}^{\infty}\) of \(\{x_n\}_{n=1}^{\infty}\) converges to \(p\) and \(\lim_{n \to \infty} ||x_n - p||\) exists, we conclude that \(\lim_{n \to \infty} x_n = p\).

\[\square\]

**Corollary 2.5.5.** Let \(C\) be a nonempty compact convex subset of a uniformly convex Banach space and for \(i = 1, 2\), let \(T_i : C \to C\) be uniformly \((L_i - \alpha_i)\)-Lipschitz and asymptotically quasi-nonexpansive mappings with sequence \(\{k_n^{(i)}\}\) such that \(\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty\). Define a sequence \(\{x_n\}\) in \(C\) as follows:

\[
\begin{align*}
x_1 & \in C \\
x_{n+1} & = a_n x_n + b_n T_1^n y_n + c_n u_n, \\
y_n & = a_n' x_n + b_n' T_2^n y_n + c_n' v_n, \quad \forall n \in \mathbb{N}
\end{align*}
\]

where \(\{u_n, \{v_n\}\}\) in \(C\) and \(\{a_n\}, \{b_n\}, \{c_n\}, \{a_n'\}, \{b_n'\}, \{c_n'\}\) are sequences in \([0, 1]\) satisfying, \(a_n + b_n + c_n = a_n' + b_n' + c_n' = 1\), for all \(n \in \mathbb{N}\) and \(\sum_{n=1}^{\infty} c_n < \infty\), \(\sum_{n=1}^{\infty} c_n' < \infty\). If \(F(T_1) \cap F(T_2) \neq \phi\), then \(\{x_n\}\) converges to a common fixed point of \(T_1\) and \(T_2\).

**Remark 2.5.1.** Corollary 2.5.5 is an improvement of the results of Qihou [44].
2.6 Conclusion

Finally we therefore conclude by saying that Das and Debata type sequences of asymptotically quasinonexpansive mappings are convergent in a domain of Banach space. Similarly, Ishikawa type sequence having different rank of $(L_i - \alpha_i)$-Lipschitzian asymptotically quasinonexpansive mappings are also convergent in domain of Banach space, under certain conditions.