Chapter 6

Application Of Fixed Point In Fractal Theory

6.1 Abstract

Iterated Function System are offered in 2-metric space, D-metric space, commuting mapping, Kannan’s mappings and under Reich condition in order to propose further investigation for image processing behavior within fractal theory via various fixed point principle.

6.2 Introduction

Iterated Function System theory defines mathematically some concepts of Chaos and irregularity. The research done mainly by Bransley led to significant new methods for image understanding [48, 50]. Other researchers allowed these ideas and focussed on special characteristics of Iterated Function System. Fractals such as measures over IFS attractors [50]. IFS description provides a potential new method for researching
the image shape and texture. It forms, through a set of simple geometric transformations, a basic set of tools for interactive image construction.

Iterated Function Systems are based on the mathematical foundations laid by Hutchinson [27]. Fractals have an elegant recursive definition: A Fractal is constructed from a collage of transformed copies of itself, it is inherently self-similar and infinitely scalable.

The transformation is preformed by a set of affine maps. An affine mapping of the plane is a combination of a rotation, scaling, a sheer and a translation in $\mathbb{R}^2$.

Any affine transformation $w : R^2 \rightarrow R^2$ of the plane has the form.

$$
\begin{pmatrix}
x' \\
y'
\end{pmatrix} = w \begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix} \begin{pmatrix}
x \\
y
\end{pmatrix} + \begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
$$

where $(x, y), (x', y') \in R^2$ are any points on a plane.

The matrix

$$
\begin{pmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{pmatrix}
$$

is a combination of a rotation, rescaling and a sheer, usually described by linear equations.

The vector

$$
\begin{pmatrix}
b_1 \\
b_2
\end{pmatrix}
$$

defines the translation part of the transformation.

**Definition 6.2.1.** Let $(F, d)$ be a complete metric space. Let $w_i : F \rightarrow F$ be a collection of mappings ($w_i : i = 1, 2, ..., N$). Then

$$
\Omega = \{ F, (W_i; i = 1, 2, ..., N) \}$$
is called the Iterated Function System.

The function \( w_i \) needs to be affine and contractive.

**Definition 6.2.2.** Let \((F,d)\) be a complete metric space. Let \( W_i : F \to F \) be a collection of mappings \((w_i; i = 1, 2, \ldots N)\). Then a mapping \( w_i \) is called contractive if there exists a positive real number \( 0 \leq s_i < 1 \) such that

\[
\forall (x, y) \in F, d(w_i(x), w_i(y)) \leq s_i d(x, y)
\]

**Definition 6.2.3.** Let \( \Omega = \{F_j(w_i); i = 1, 2, \ldots N\} \) be a Hyperbolic Iterated Function System with contraction factors \((s_i, i = 1, 2, \ldots N)\). An IFS \( \omega \) is hyperbolic when all transformations \( w_i \) are contractive, i.e. all \( 0 \leq s_i < 1 \). The contraction ratio of \( \omega \) is the minimum of all contraction ratios \( s_i \):

\[
s = \{\min(s_i, i = 1, 2, \ldots)\}
\]

**Definition 6.2.4.** If an IFS \( \Omega = \{F_i(w_i); i = 1, 2, \ldots, N\} \) is contractive, there exists a unique set

\[
A = w(A) = \bigcup_{i=1}^{N} w_i(A)
\]

called the fixed point of (the attractor) of \( A \).

The uniqueness of an attractor \( A \) for contractive iterated function systems is a result of contractive mapping fixed point theorem for the mapping \((w_i)\) acting on a space \((P(F), d(H))\) which is contractive according to the Hausdorff distance \( d(H) \).

### 6.2.1 The Hausdorff Distance

The proof of the Banach fixed point theorem uses the distance function only as a tool for discussing the closeness of two elements of \( K \). When we talk of the
convergence of sequence of set \( B_n \in K \) to some set of \( A \), intuitively we wish to show that for sufficiently large \( n \), the set \( B_n \) strongly resembles \( A \).

Thus, we wish to quantify the notion of closeness between two sets \( B_1 \) and \( B_2 \), such that we can say precisely when two sets are within some distance \( \epsilon \) of each other. One way of doing this is to consider “inflating” the set \( B_1 \) by an amount \( \epsilon \). That is, we consider the set of all points within a distance between \( B_1 \) and \( B_2 \) is less than \( \epsilon \), then \( B_2 \) should be entirely contained in the inflated version of \( B_1 \). The \( \epsilon \)-inflated set \( B_1 \) is given by,

\[
B_1(\epsilon) = \{ v \in \mathbb{R}^2 : \exists w \in B_1 \text{ such that } d(v, w) < \epsilon \},
\]

where \( d(v,w) \) is the usual Euclidean distance between \( b \) and \( w \), both points of \( \mathbb{R}^2 \). We require that \( B_2 \subset B_1(\epsilon) \). However, this is not sufficient. The set \( B_2 \) could have a very different form and be much smaller than \( B_1 \). Thus, we also consider inflating \( B_2 \),

\[
B_2(\epsilon) = \{ v \in \mathbb{R}^2 : \exists w \in B_2 \text{ such that } d(v, w) < \epsilon \},
\]

and requiring that \( B_1 \subset B_2(\epsilon) \). We denote by \( d_H(B_1, B_2) \) the Hausdorff distance between \( B_1 \) and \( B_2 \), which remains to be precisely defined. We want that

\[
d_H(B_1, B_2) < \epsilon \iff (B_1 \subset B_2(\epsilon)) and (B_2 \subset B_1(\epsilon))
\]

Thus, intuitive idea of inflating a set until it subsumes another helps to make sense of the formal definition of the Hausdorff distance.

**Definition 6.2.5.** (1) Let \( B \) be a compact (closed and bounded) subset of \( \mathbb{R}^2 \) and let \( v \in \mathbb{R}^2 \). the distance of \( v \) to \( B \), denoted by \( d(v,B) \), is

\[
d(v, B) = \min_{w \in B} d(v, w),
\]
(2) The Hausdorff distance between two compact sets $B_1$ and $B_2$ of $\mathbb{R}^2$ is

$$d_H(B_1, B_2) = \max(\max_{v \in B_1} d(v, B_2), \max_{w \in B_2} d(w, B_1))$$

**Remark 6.2.1.** The condition that $B$, $B_1$, and $B_2$ be compact ensures that the minima and maxima in definition 6.2.5 do indeed exist.

**Remark 6.2.2.** Given the following fact regarding maxima,

$$\max(a, b) < \epsilon \iff (a < \epsilon \text{ and } b < \epsilon)$$

We have that

$$d_H(B_1, B_2) < \epsilon$$

if and only if

$$\max_{v \in B_1} d(v, B_2) < \epsilon \text{ and } \max_{w \in B_2} d(w, B_1) < \epsilon$$

if and only if

$$B_1 \subset B_2(\epsilon) \text{ and } B_2 \subset B_1(\epsilon).$$

Thus, the Hausdorff distance is intimately related to the concept of inflated sets.

**Definition 6.2.6.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an affine contraction.

(1) A real number $r \in (0, 1)$ is a contraction for $T$, if for all $v, w \in \mathbb{R}^2$ we have that

$$d(T(v), T(w)) \leq rd(v, w).$$

(2) A contraction factor $r$ is an exact contraction factor if for all $v, w \in \mathbb{R}^2$ we have that

$$d(T(v), T(w)) \leq rd(v, w).$$

**Remark 6.2.3.** Only affine transformations of which linear part is some composition of a homotheth, a rotation, and a reflection with respect to a line have exact contraction factors.
Definition 6.2.7. An IFS on a plane is given by a finite collection of contractive affine transformations \( w_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) for all \( (w_i; i = 1, 2, ... N) \). An IFS with probabilities is an IFS such that each \( w_i \) is assigned a probability \( p_i \) with \( p_1 + p_2 + ... p_N = 1 \):

\[
\omega = \{ \mathbb{R}^2; (w_i; p_i), i = 1, 2, ... N \}
\]

Definition 6.2.8. Let the IFS \( \omega = \{ \mathbb{R}^2, (w_i, p_i); i = 1, 2, ... N \} \) be a non hyperbolic IFS with probabilities. Let the IFS have contraction factors \( (s_i, i = 1, 2, ... N) \). The IFS model is contractive and posses a unique common fixed point called its attractor \( A \) when the converge contraction condition is obeyed.

\[
s_1p_1.s_2p_2.........s_Np_N < 1
\]

A specific class of IFS contains the so-called condensation sets.

Definition 6.2.9. Let \((F,d)\) be a complete metric space and let \( c \in H(F) \). Define a transformation \( W_0 : H(F) \rightarrow H(F) \) by \( W_0(B) = C \) for all \( B \in H(F) \). Then \( W_0 \) is called the associated condensation set.

The condensation transformation is a contraction mapping \( W_0 : H(F) \rightarrow H(F) \) on the metric space \((H(F), d_H)\) where the metric space \((H(F), d_H)\) where the contractive factor is equal to 0. This transformation possesses a unique fixed point called the condensation set.

### 6.3 Iterated Function System

Definition 6.3.1. A (hyperbolic) iterated function system consists of a complete metric space \((X,d)\) together with a finite set of contraction mapping \( w_n : X \rightarrow X \)
with respective contractivity factor \( s_n \), for \( n = 1,2,\ldots N \). The abbreviation “IFS” is used for “iterated function system”. The notation for the IFS just announced is \( \{X; w_n, n = 1,2,\ldots N\} \) and its contractivity factor is \( s = \text{Max}\{s_n : n = 1,2,\ldots N\} \).

The following theorem summarizes the main facts so far about a hyperbolic IFS.

**Theorem 6.3.1.** Let \( \{X; w_n, n = 1,2,\ldots N\} \) be a hyperbolic iterated function system with contractivity factor \( s \). Then the transformation \( W : H(X) \to H(X) \) defined by

\[
W(B) = \bigcup_{n=1}^{N} w_n(B)
\]

for all \( B \in H(X) \), is a contraction mapping on the complete metric space \( (H(X), h(d)) \) with contractivity factor \( s \). That is

\[
h(W(B), W(C)) \leq s \cdot h(B, C)
\]

for all \( B, C \in H(X) \). Its unique fixed point, \( A \in H(X) \), obeys

\[
A = W(A) = \bigcup_{n=1}^{N} w_n(A),
\]

and is given by \( A = \text{Lim}_{n \to \infty} W^n(B) \) for any \( B \in H(X) \).

**Definition 6.3.2.** The fixed point \( A \in H(X) \) described in the theorem is called the attractor of the IFS.

### 6.3.1 Bransley’s Collage Theorem

The following theorem is central of the design of IFS’s whose attractors are close to given sets.
Theorem 6.3.2. Let \((X, d)\) be a complete metric space. Let \(L \in H(X)\) be given, and let \(\epsilon \geq 0\) be given. Choose an IFS (or IFS with condensation) \(\{X; (w_0), w_1, w_2, \ldots w_n\}\) with contractivity factor \(0 \leq s < 1\). So, that
\[
h(L, \bigcup_{n=1}^{\infty} w_n(L)) \leq \epsilon,
\]
where \(h(d)\) is the hausdorff metric. Then
\[
h(L, A) \leq \epsilon / (1 - s)
\]
where \(A\) is the attractor of the IFS. Equivalently,
\[
h(L, A) \leq (1 - s)^{-1} h(L, \bigcup_{n=1}^{\infty} w_n(L))
\]
for all \(L \in H(X)\).

We have seen that the contraction mapping used in IFS are typically affine maps. The iteration dynamics associated with affine maps is nor very interesting but when the action of a system of contraction mappings is considered the result is quite remarkable. Also it is evident that IFS which ultimately and in collage theorem was proved in metric space. Obviously it can be asked-is it possible to develop IFS and the Collage theorems subsequently in 2-metric space setting? Although this is purely an analytical query which is answered affirmatively.

6.4 2-Metric Analogue Of Iterated Function System

The method of IFS is very useful for the approximation of Fractal sets and images. It is founded by Hutchinson [27] and further developed by Bransley in the early 1980s.
[48, 50]. In this method, the image or target is represented by a function in one or more spatial variables. The target is represented by an element \( x \) of a given metric space \( (X, d) \). Such an element is then identified as unique fixed point or attractor of a contraction map \( T : X \rightarrow X \) i.e. \( Tx = x \). This \( x \) can be generated by iterating \( T \) [27].

The IFS is based upon the Contraction Mapping Principle [CMP] given by Banach [63]. The CMP states that a contractive transformation, defined on a complete metric space, possesses a unique fixed point or attractor. For the purpose of image compression, this idea translates into finding an optimal contractive transformation whose attractor closely approximates a given target image. This problem is widely known as the inverse problem in the fractal image coding literature. The fractal-based schemes exploit the self-similarities that are inherent in many real-world images for the purpose of encoding an image as a collection of transformations. Therefore, a digitized image can be stored as a collection of IFS transformations and is easily regenerated or decoded for use or display.

As we have said above, the problem of representing a given image by IFS is a typical inverse problem. Such a problem is, in turn, related to the problem of finding image/ function as the fixed element of a given iteration algorithm as IFS which can be reduced to the mathematical problem of finding:

1. a suitable metric space \( X \) in which to represent the image,
2. a suitable metric \( d \) on \( X \),
3. a suitable contraction map \( T : X \rightarrow X \).

The Collage Theorem [48, 50] is the tool through which IFS in the metric space \( X \) and the metric \( d \) generates image iteration. In this Theorem, contraction mapping
T satisfies CMP [63].

In a paper [66], Gähler introduced the concept of 2-metric, a real valued function of a point triple on a set X of which abstract properties were suggested by the area function. For a triple determined by a triangle in Euclidean space, associated with a given 2-metric was a natural topology.

Thus in this section\(^1\), our object is to generate Iterated function system in 2-metric space in order to formulate the Collage theorem to validate such IFS. It is important to mention that the concept of 2-metric has more geometrical relevance than the distance metric in the context of fractal. For the simple reason, one represents distance between two elements as function whereas other represents area covered by three elements. The contraction mappings used in IFS are typically affine maps. The iteration dynamics associated with affine maps is not very interesting but when the action of a system of contraction mappings is considered the result is quite remarkable.

Before proving our results, we first give some basic definitions concerning 2-metric space.

### 6.4.1 2-Metric Space

**Definition 6.4.1.** A 2-metric space is a space M with a distance function (metric) \(\rho\) in which for each triple points \(a, b, c\) there exists a real function \(\rho(a, b, c)\) such that to each pair of points \(a, b(b \neq b)\) from M, there is \(c \in M\) satisfying

\[
(1) \quad \rho(a, b, c) \neq 0
\]

(2) \(\rho(a, b, c) = 0\) when at least two of the three points are equal.

(3) \(\rho(a, b, c) = \rho(a, c, b) = \rho(b, c, a)\)

(4) \(\rho(a, b, c) \leq \rho(a, b, d) + \rho(a, d, c) + \rho(d, b, c)\).

A 2-metric space is called bounded, if there exists a constant \(K\) such that

\[\rho(a, b, c) \leq K, \text{ for all } a, b, c \in M.\]

Remark 6.4.1. If \(M\) consist of two points, \(\rho\) is trivial. We suppose \(M\) contains at least three points.

Remark 6.4.2. The most natural example of such a function is the Euclidean 2-metric on \(E^m\) for \(m > 1\).

Definition 6.4.2. A sequence \(x_n\) in 2-metric space \(M\) is called a convergent sequence if there is an \(x \in M\) such that \(\lim \rho(x_n, x, y) = 0\) for all \(y \in M\). Here \(x\) is called the limit of \(x_n\).

Definition 6.4.3. A sequence \(x_n\) in a 2-metric space \(M\) is called a Cauchy sequence, if \(\rho(x_m, x_n, y) = 0\) for all \(y \in M\).

Definition 6.4.4. A 2-metric space in which every Cauchy sequence converges is called a complete 2-metric space.

We know that Banach contraction principle is used in theory of IFS. Banach Contraction Principle infect, not only ensure unique fixed point in complete metric space but also provides a constructive method to calculate it. With intention to study IFS in 2-metric space, the validity of Banach Contraction Principle in 2-metric space is required. It is already proved as below:
6.4.2 2-I terated Function System

Let us construct the Iterated Function System on a 2-metric space.

**Theorem 6.4.1.** Let \((M, \rho)\) be a 2-metric space then \((H(M), h)\) is also a 2-metric space.

**Proof** Recall the axioms for a 2-metric space.

1. \(h(A, B, C) \neq 0\)

2. \(h(A,B,C)=0\), when at least two of the three points are equal.

3. \(h(A,B,C)=h(A,C,B)=h(B,C,A)\)

4. \(h(A, B, C) \leq h(A, B, D) + h(A, D, C) + h(D, B, C)\)

**Proof of 1:** \(h(A, B, C) \neq 0\) to each pair of points \(A, B (A \neq B)\) from \(H(M)\), there is \(C \in H(M)\).

**Proof of 2:** If \(A = B, B = C, C = A\) then \(h(A, B, C) = 0\), because every \(a \in A\) satisfies \(m(a, B, C) = 0\), conversely, if \(h(A, B, C) = 0\), then all terms of the max expression are equal to zero, and thus \(m(a, B, C) = 0\) for every \(a\). Every such point \(a\) is a limit point of \(B\) and \(C\) since any neighborhood of \(a\) must contain a point of \(b\) and \(c\) if \(m(a, B, C) = \inf_{b \in B, c \in C} h(a, b, c)\) is to equal to 0. so \(a\) is in \(B\) and in \(C\) because \(B\) and \(C\) are by definition closed. Since \(a \in A\) was arbitrary, \(A \subseteq B\) and \(A \subseteq C\). By symmetry of our definition \(B \subseteq A\) and \(C \subseteq A\). Similarly, we can prove that \(C \subseteq B\) and \(B \subseteq C\). Thus \(A = B, B = C\) and \(C = A\).

**Proof of 3:** The max operation is symmetric so, \(h\) is symmetric.

**Proof of 4:** Let \(A, B, C\) and \(D\) be the elements of \(M\). Let \(a \in A\) and \(b \in B\) be
arbitrary elements. then there must exists \( d \in D \) so that
\[
h(a, b, d) < h(A, B, D)
\]
similarly, for \( a \in A \) and \( d \in D \) there must be \( c \in C \) so that
\[
h(a, d, c) < h(A, D, C)
\]
and for \( d \in D \) and \( b \in B \) there must be \( c \in C \) so that
\[
h(d, b, c) < h(D, B, C)
\]
adding and applying triangle inequality
\[
h(a, b, c) < h(A, B, D) + h(A, D, C) + h(D, B, C)
\]
Since for \( a \in A \) and \( b \in B \) there must be \( c \in C \) so that
\[
h(a, b, c) < h(A, B, C)
\]
Thus,
\[
h(A, B, C) \leq h(A, B, D) + h(A, D, C) + h(D, B, C)
\]

**Theorem 6.4.2.** Let \( M \) be a complete bounded 2-metric space, and let \( f_n(x)(n = 1, 2, ..) \) be a family of mappings of \( M \) into itself. Suppose that there exist a sequence of non-negative integers \( \{m_n\} \) and non-negative numbers \( \alpha, \beta \) such that for all \( x, y, a \in X \) and every pair \( i, j \) with \( i \neq j \)
\[
\rho(f_i^{m_i}(x), f_j^{m_j}(y), a) \leq \alpha(\rho(x, f_i^{m_i}(x), a) + \rho(y, f_j^{m_j}(y), a) + \beta\rho(x, y, a)
\]
where \( 2\alpha + \beta < 1 \). Then the sequence of mappings \( \{f_n\} \) has a unique common fixed point.
The proof of this theorem can be found in [35][pg.No.68] from this theorem
\[ \rho(x_n, x_{n+1}, a) \leq \left( \frac{\alpha + \beta}{1 - \alpha} \right)^n \rho(x_0, x_1, a) \]

**Corollary 6.4.3.** Let \((M, \rho)\) be a 2-metric space and \(f : M \to M\) be a distance decreasing mapping. Let \(w : h(M) \to H(M)\) be defined as follows:

\[ w(B) = f(B), \quad \text{for} \quad B \in H(M) \]

Then \(w\) be a distance decreasing mapping on \((H(M), h)\).

**Proof** Let \(B, C, D \in H(M)\). Let us define

\[ \rho(x, C, D) = \inf \{ \rho(x, y, D) : y \in C \}, \quad x \in B, \]

\[ L(B, C, D) = \max \{ \rho(x, y, D) : x \in B, y \in C \} \]

If \(B \neq C \neq D\), Then,

\[ L(w(B), w(C), D) < \max \{ \min \{ \rho(x, y, a) : y \in C, x \in B \} : a \in D \} \quad (6.4.1) \]

\[ = L(B, C, D) \quad (6.4.2) \]

Similarly,

\[ L(w(C), w(B), D) < L(C, B, D) \]

Hence

\[ h(w(B), w(C), D) = \max \{ L(w(B), w(C), D), L(w(C), w(B), D) \} \]

\[ < \max \{ L(B, C, D), L(C, B, D) \} \]

\[ \leq h(B, C, D) \]

This completes the proof.
Corollary 6.4.4. Let \((M, \rho)\) be a 2-metric space and let \(\{w_n : H(M) \to H(M), n = 1, 2, ...N\}\) be a finite family of distance decreasing mappings. Let us define \(w : H(M) \to H(M)\) as follows:

\[
W(B) = \bigcup_{n=1}^{N} w_n(B) \quad \text{for} \quad B \subset H(M)
\]

Then \(W\) be a distance decreasing mapping.

**Proof** We demonstrate the claim for \(M = 2\). An inductive argument completes the proof. Let \(B, C, D \in H(M), \quad B \neq C \neq D\), we have

\[
h(w(B), w(C), D) = h((w_1(B) \subset w_2(B) \subset D), (w_1(C) \subset w_2(B) \subset D))
\]

using a property of hausdorff metric we obtain

\[
h(w(B), w(C), D) \leq \max\{h(w_1(B), w_1(C), D), h(w_2(B), w_2(C), D)\}
\]

\[
< h(B, C, D)
\]

This completes the proof.

**Definition 6.4.5.** A 2-metric space \((M, \rho)\) with a finite family \(\{f_n : M \to M, n = 1, 2, ...N\}\) of distance decreasing mapping shall be called an Iterated Function System on 2-metric space-2IFS and denoted by \(\{(M, \rho); f_n; n = 1, 2, ...N\}_2\).

Thus from all the above results and definition of 2-IFS, we are in the position to present the following theorem for 2-IFS.

**Theorem 6.4.5.** Let \(\{(M, \rho); f_n; n = 1, 2, ...N\}_2\) be a 2-IFS. Then a mapping \(W : H(M) \to H(M)\) defined as

\[
W(B) = \bigcup_{n=1}^{N} w_n(B), \quad B \in H(M);
\]
has a unique fixed point $A \in H(M)$,

such that $A = W(A) = \bigcup_{n=1}^{N} w_n(A), \quad A \in H(M)$.

**Proof** Corollary 6.4.3 implies that $W$ is a distance decreasing mapping and, by theorem 6.4.1 $(H(M), h)$ is a 2-metric space. Hence the assumptions of Theorem 6.4.5 are satisfied and the proof is completed.

6.4.3 Collage Theorem In 2-Metric Space

Now we are in a position to formulate Collage theorem in 2-Metric space as below:

**Theorem 6.4.6.** Let $M$ be a complete 2-metric space. Let $L, K \in H(M)$ be given. Choose an 2-IFS $\{(M, \rho); f_n : n = 1, 2, ..., N\}_2$ with non-negative numbers $\alpha, \beta$ such that

$$\rho(L, \bigcup_{n=1}^{N} \rho_n(L), K) \leq \frac{\alpha + \beta}{1 - \alpha} \rho(w(L), (L), K).$$

where $A$ is the attractor of the 2-IFS.

**Proof**

$$\rho(w^n(L), L, K) \leq \sum_{m=1}^{n} \rho(w^m(L), w^{\alpha(m-1)}(L), K) \leq \frac{\alpha + \beta}{1 - \alpha} \rho(w(L), (L), K)$$

from which, on taking the limit as $n \to \infty$, we obtain

$$\rho(A, L, K) \leq \frac{\alpha + \beta}{1 - \alpha} \rho(w(L), (L), K)$$

This completes the proof.
6.5 Iterated Function System In D-Metric Space

As we have raised a question with reference to IFS in the context of 2-metric space and answered it affirmatively in para 6.4.

6.5.1 D-Metric Space

Before proving our results, we first give some basic definitions and theorems concerning Iterated Function System and D-metric space.

**Definition 6.5.1.** In his paper Dhage [4] introduced a generalized metric space or D-metric space as follows. Let \( D : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \) is called a D-metric space if it satisfies the following properties:

(i) \( D(x, y, z) \geq 0 \), for each \( x, y, z \in \mathcal{X} \) with equality if and only if \( x = y = z \)

(ii) \( D(x, y, z) = D(y, x, z) = D(x, z, y) = \ldots \ldots \) (symmetry),

(iii) \( D(x, y, z) \leq D(x, y, a) + D(x, a, z) + D(a, y, z) \), for each \( x, y, z, a \in \mathcal{X} \)

**Definition 6.5.2.** A sequence \( \{x_n\} \) of points of a D-metric space \( \mathcal{X} \) converges to a point \( x \in \mathcal{X} \), if for an arbitrary \( \epsilon > 0 \), there exists positive integer \( n_0 \) such that for all \( n, m \geq n_0 \), \( D(x_m, x_n, x) < \epsilon \).

**Definition 6.5.3.** A sequence \( \{x_n\} \) of points of a D-metric space \( \mathcal{X} \) is a cauchy sequence if for an arbitrary \( \epsilon > 0 \), here exists a positive integer \( n_0 \) such that \( p, n, m \geq n_0 \),

\[
D(x_m, x_n, x_p) < \epsilon
\]

**Definition 6.5.4.** D-metric space \( \mathcal{X} \) is complete if Cauchy sequence \( \{x_n\} \) in \( \mathcal{X} \) converges in \( \mathcal{X} \).
Definition 6.5.5. A set $S \subset X$ is said to be bounded if there exist a constant $K > 0$, such that $D(x, y, z) \leq k$, for all $x, y, z \in S$ and the constant $k$ is called a D-bound of $S$.

Definition 6.5.6. Let $X$ denotes a complete D-metric space and $T$ be a mapping from $X$ into itself. Then $T$ is called a contraction mapping if there is a constant $0 \leq \alpha < 1$ such that

$$D(T(x)+T(y)+T(z)) \leq \alpha D(x, y, z)D(x, T(x), z)D(y, T(y), z)D(x, T(y), z)D(y, T(x), z)$$

for each $x, y, z \in X$, The constant $\alpha$ is called contractivity factor for $T$.

Definition 6.5.7. Let $X$ be a complete D-metric space and $H(X)$ denotes the space whose points are the compact subsets of $X$, other than the empty set. Let $a, b, c \in X$ and $A, B, C \in H(x)$ . Then

(i) Distance from the point $a$ to the set $B$ is defined as

$$D(a, B, C) = \max\{D(a, b, c) : b \in B\}$$

(ii) Distance from the set $A$ to set $B$ is defined as

$$D(A, B, C) = \max\{D(a, B, C) : a \in A\}$$

(iii) Hausdorff distance between set $A$ to set $B$ is defined as

$$h_D(A, B, C) = D(A, B, C) \lor D(A, C, B) \lor D(B, C, A)$$

Then the function $h_d(D)$ is the D-metric defined on the space.

Note: Through out this paper the notation $u \lor v$ means the maximum and $u \land v$ denotes the minimum of pair of real numbers $u$ and $v$. 
We know that Banach contraction principle is used in theory of IFS. Banach Contraction Principle in fact, not only ensure unique fixed point in complete metric space but also provides a constructive method to calculate it. With intention to study IFS in D-metric space, the validity of Banach Contraction Principle in D-metric space is required. It is already proved as below:

**Theorem 6.5.1.** Let $T : X \to X$ be a contraction mapping with contractivity factor $\alpha$, on a complete D-metric space $X$. Then $T$ possesses exactly one fixed point $x^* \in X$, and moreover for any point $x \in X$, the sequence $\{T^n(x) : n = 0, 1, 2, 3, \ldots\}$ converges to $x^*$. That is $\lim_{n \to \infty} T^n(x) = x^*$, for each $x \in X$.

### 6.5.2 D-Iterated Function System

Next as its immediate consequence of 2-metric we now propose to establish the IFS in D metric space\(^2\) and answering it accordingly as below:

To establish IFS in D-Metric space we need to prove following lemmas.

**Lemma 6.5.2.** Let $T : X \to X$ be a contraction mapping on the D-metric space $X$ with contractivity factor ‘$\alpha’$, then $T : H(X) \to H(X)$ defined by

$$T(B) = \{T(x) : x \in B\}, \text{ for every } B \in H(X)$$

is a contraction mapping on $(H(X), h_D)$ with contractivity factor ‘$\alpha’$.

**Proof.** Let $A, B, C \in H(X)$, let us define

$$L(A, B, C) = \max \{ \sup_{a \in A, b \in B} D(a, b, C), \sup_{b \in B, c \in C} D(b, c, A), \sup_{c \in C, a \in A} D(c, a, B) \}$$

---

Where $D(a, b, c) = \inf \{\rho(a, b, c) | c \in C\}$

Define $D(A, B, C) = \inf \{\rho(a, b, c) | a \in A, b \in B, c \in C\}$

Then,

$$L(T(A), T(B), T(C)) \leq \alpha L(A, B, C)$$

Similarly,

$$L(T(A), T(C), T(B)) \leq \alpha L(A, C, B) \text{ and }$$

$$L(T(B), T(C), T(A)) \leq \alpha L(B, C, A)$$

Hence,

$$h_D(T(A), T(B), T(C)) < \max \{L(T(A), T(B), T(C)), L(T(A), T(C), T(B)), L(T(B), T(C), T(A)) \}$$

$$< \max \{\alpha L(A, B, C), \alpha L(A, C, B), \alpha L(B, C, A)\}$$

$$\leq \alpha h_D(A, B, C)$$

\[ \square \]

**Lemma 6.5.3.** Let $X$ be a D-metric space. Let $\{T_n : n = 1, 2, 3...N\}$ be a mappings on $(H(X), h_D)$. Let the contractivity factor for $T_n$ be denoted by $\alpha_n$ for each $n$. Define $W : H(X) \to H(X)$ by

$$W(B) = T_1(B) \cup T_2(B) \cup T_3(B) \ldots \cup T_N(B)$$

$$= \bigcup_{n=1}^{N} T_n(B), \text{ for each } B \in H(X)$$

Then $W$ is a contraction mapping with contractivity factor $\alpha = \max \{\alpha_n : n = 1, 2, \ldots N\}$

**Proof.** Let $A, B, C \in H(X), \quad A \neq B \neq C,$
we have
\[ h_D(T(A), T(B), T(C)) = h_D(T_1(A) \cup T_2(A) \cup \ldots \cup T_n(A), T_1(B) \cup T_2(B) \cup \ldots \cup T_n(B), T_1(C) \cup T_2(C) \cup \ldots \cup T_n(C)) \]

Using the property of Hausdorff metric space we obtain,
\[ h_D(T(A), T(B), T(C)) \leq \max h_D(T_1(A), T_1(B), T_1(C)), (T_2(A), T_2(B), T_2(C)), \ldots \]
\[ \ldots (T_n(A), T_n(B), T_n(C)) \]
\[ h_D(T(A), T(B), T(C)) \leq \max h_D(T_1(A), T_1(B), T_1(C)), (T_2(A), T_2(B), T_2(C)), \ldots \]
\[ \leq \alpha^n h_D(A, B, C) \]

This completes the proof.

**Definition 6.5.8.** A D-iterated function system consists of a D-complete metric space together with a finite set of contraction mapping \( T_n : X \rightarrow X \) with contractivity factor \( \alpha_n \) for \( n=1,2,3,N \).

The notation for the IFS just introduced is \( \{X : T_n : n = 1,2,\ldots,N\} \) and its contractivity factor is \( \alpha = \max \{\alpha_n : n = 1,2,\ldots,N\} \).

Thus from all the above results and definition of D-IFS, we are in the position to present the following theorem for D-IFS.

**Theorem 6.5.4.** Let \( \{X : T_n : n = 1,2,\ldots,N\} \) be an IFS with contractivity factor \( \alpha \). Then the transformation \( W \), defined by \( W(A) = \bigcup_{n=1}^{N} T_n(A) \) for \( A \in H(X) \); is a
contraction mapping on the complete $D$-metric space $(H(X), h_D)$ with contractivity factor $\alpha$. That is

$$h_D(W(A), W(B), W(C)) \leq \alpha h_D(A, B, C)$$

By Banach contraction principle it has the unique fixed point $\hat{A} \in H(X)$ called the attractor of the IFS given by

$$\hat{A} = \lim_{n \to \infty} W^n(A)$$

for any $A \in H(X)$, where $W^n(A)$ denotes the $n$-th iteration of $W$.

Following Lemma is required in mathematical formulation of Collage theorem.

**Lemma 6.5.5.** Let $T : X \to X$ be a contraction mapping on a complete $D$-metric space $X$ with contractivity factor $\alpha$ and let $x^* \in X$ be the fixed point of $T$. Then

$$D(x, x^*, a) \leq (1 - \alpha)^{-1} D(x, T(x), a)$$

for all $x \in X$.

### 6.5.3 Collage Theorem In D Metric Space

Now we are in a position to formulate Collage theorem in D-Metric space as below:

**Theorem 6.5.6.** Let $X$ be a complete $D$-metric space. Let $L \in H(X)$ be given. Choose an $D$-IFS $\{X : T_n : n = 1, 2, \ldots, N\}$ with contractivity factor $\alpha$ such that

$$h(L, \bigcup_{n=1}^{N} T_n(L), K) < \epsilon$$

for some $\epsilon > 0$ then,

$$h(L, A, K) < \epsilon/(1 - \alpha),$$

Where $A$ is the attractor of the $D$-IFS.
Proof.

\[ h(W^0(L), L, K) \leq \sum_{m=1}^{n} h(W^0(W^0(L), W^0(L), W^0(K))) \]

\[ \leq \sum_{m=1}^{n} \alpha^{m-1} h(W(L), L, K) \]

\[ \leq \frac{1 - \alpha^n}{1 - \alpha} h(W(L), L, K) \]

From which on taking the limit as \( n \to \infty \), we obtain

\[ h(A, L, K) \leq \frac{1}{1 - \alpha} h(W(L), L, K) \]

\[ \blacksquare \]

6.6 An Iterated Function System For Commuting Mapping

With an open question - is it possible to compress one image containing the other and commuting with each other? we establish the College Theorem and iterated function system (key to image compression) for two such mappings under the contraction condition in the sense of Jungck [19].

In this section, we shall try to explore the possibility of improvement in IFS by replacing contraction mapping by a more useful mapping known as Commuting mapping.

\[ \text{S.C. Shrivastava, Padmavati, "An Iterated Function System For Commuting Mapping" Pure} \]
\[ \text{and Appl. Math., vol. 70, No. 7 (2011).} \]
6.6.1 Commuting Mapping

In 1976, Jungck [19] introduced a mapping, which was defined as follows:

Let g be a continuous mapping of a complete metric space \((X, d)\) into itself. Then \(f\) has a fixed point in \(X\) iff there exist \(\alpha \in (0, 1)\) and a mapping \(g : X \to X\) which commutes with \(f\) and satisfies

\[ g(X) \subset f(X) \]

and

\[ d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \forall x, y \in X \]

On the basis of definition of (hyperbolic) iterated function system given by Barnsley [50], we now introduce Iterated function system with Commuting mapping as below:

A Iterated function system with Commuting mappings consists of a complete metric space \((X, d)\) together with mapping \(f : X \to X\) and \(g : X \to X\) with contractivity factor \(\alpha\).

**Proposition 6.6.1.** Let \(f\) be a continuous mapping of a complete metric space \((X, d)\) into itself. Then \(f\) has a fixed point in \(X\) iff there exists \(\alpha \in (0, 1)\) and a mapping \(g : X \to X\) which commutes with \(f\) and satisfies

\[ g(X) \subset f(X) \]

and

\[ d(g(x), g(y)) \leq \alpha d(f(x), f(y)) \forall x, y \in X \quad (\ast) \]

Indeed, \(f\) and \(g\) have a unique common fixed point if the above condition holds.

**Proof.** To see that the stated condition is necessary, suppose that \(f(a) = a\) for some \(a \in X\). Define \(g : X \to X\) by \(g(x) = a\) for all \(x \in X\). Then \(g(f(x)) = a\) and
\( f(g(x)) = f(a) = a(x \in X) \), so \( g(f(x)) = f(g(x)) \forall x \in X \) and \( g \) commutes with \( f \). Moreover, \( g(x) = a = f(a) \forall x \in X \) so that \( g(X) \subset f(X) \). Finally, for any \( \alpha \in (0,1) \) we have for all \( x, y \in X \):

\[
d(g(x), g(y)) = d(a,a) = 0 \leq \alpha d(f(x), f(y))
\]

Thus the above condition holds.

on the other hand, suppose there is a mapping \( g \) of \( X \) into itself which commutes with \( f \) and for which eq.\[*\] holds. We show that this condition is sufficient to ensure that \( f \) and \( g \) have a unique common fixed point.

To this end, let \( x_0 \in X \) and let \( x_1 \) be such that \( f(x_1) = g(x_0) \). In general, choose \( x_n \) so that (1) \( f(x_n) = g(x_{n-1}) \). We can do this since \( g(X) \subset f(X) \). The relation \([*]\) and (1) imply that \( d(f(x_{n+1}), f(x_n)) \leq \alpha d(f(x_n), f(x_{n-1})) \) for all \( n \). The lemma yields \( t \in X \) such that (2) \( f(x_n) \to t \). But then (1) implies that (3) \( g(x_n) \to t \).

Now since \( f \) is continuous, \([*]\) implies that both \( f \) and \( g \) are continuous. Hence (2) and (3) demand that \( g(f(x_n)) \to g(t) \) and \( f(g(x_n)) \to f(t) \). But \( f \) and \( g \) commute so that \( g(f(x_n)) = f(g(x_n)) \) for all \( n \). Thus \( f(t) = g(t) \), and consequently \( f(f(t)) = f(g(t)) = g(g(t)) \) by commutativity. We can therefore infer

\[
d(g(t), g(g(t))) \leq \alpha d(f(t), g(t)) = \alpha d(g(t), g(g(t))).
\]

Hence \( d(g(t), g(g(t)))(1 - \alpha) \leq 0 \). Since \( \alpha \in (0,1), g(t) = g(g(t)) \). We now have \( g(t) = g(g(t)) = f(g(t)) \); i.e. \( g(t) \) is a common fixed point of \( f \) and \( g \). To see that \( f \) and \( g \) can have only one common fixed point, suppose that \( x = f(x) = g(x) \) and \( y = g(y) = f(y) \). Then \([*]\) implies \( d(x,y)(1 - \alpha) \leq 0 \). Since \( \alpha < 1, x = y \).

\( \square \)
6.6.2 IFS for commuting Mapping

To establish IFS for commuting mapping we need to prove following lemmas.

**Lemma 6.6.2.** Let $f$ be a mapping, of a set $X$ into itself. Then $f$ has a fixed point iff there is a constant map $g : X \rightarrow X$ which commutes with $f$ (i.e., $g(f(x)) = f(g(x))) \forall x \in X$

Then $f : H(X) \rightarrow H(X)$ and $g : H(X) \rightarrow H(X)$ defined by $f(B) = \{f(x) : x \in B\}$ and $g(B) = \{g(x) : x \in B\} \forall B \in H(X)$ is a commuting mapping on $(H(X), h(d))$ with contractivity factor $\alpha$

**Proof.** Since $f$ and $g$ are continuous mappings such that $g : H(X) \rightarrow H(X)$ and $g(X) \subset f(X)$

Let $B, C \in H(X)$, then

\[ h(g(B), g(C)) = d(g(B), g(C)) \vee d(g(C), g(B)) \]
\[ \leq \alpha d((f(B), f(C)) \vee d(f(C), f(B))) \]
\[ = \alpha d((f(B), f(C)) \vee d(f(C), f(B))) \]
\[ \leq \alpha h(f(B), f(C)) \]

This completes the proof.

**Lemma 6.6.3.** Let $(X,d)$ be a metric space. Let $g_n : n = 1, 2, 3, ...N$ be a continuous Reich mappings on $(H(X), h)$. Let the contractivity factor for $g_n$ be denoted by $\alpha_n$ for each $n$. Define $g' : H(X) \rightarrow H(X)$ by $g'(B) = g_1(B) \cup g_2(B) \cup ... \cup g_n(B) = \bigcup_{n=1}^{N} g_n(B)$ for each $B \in H(X)$. Then $g'$ is a commuting mapping with contractivity factor $\alpha = \max\{\alpha_n : n = 1, 2, ...N\}$
Proof. We shall prove the theorem using mathematical induction method using the properties of metric \( h \). For \( N = 1 \), the statement is obviously true. Now for \( N = 2 \), we see that

\[
\begin{align*}
\ h(g'(B), g'(C)) & = h(g_1(B) \cup g_2(B), g_1(C) \cup g_2(C)) \\
& \leq h(g_1(B), g_1(C)) \vee h(g_2(B), g_2(C)) \\
& \leq [\alpha_1 h(f_1(B), f_1(C)) \vee \alpha_2 h(f_2(B), f_2(C))] \\
& \leq (\alpha_1 \vee \alpha_2)[h(f_1(B) \vee f_2(B)), h(f_1(C) \vee f_2(C))] \\
& = (\alpha_1 \vee \alpha_2)[h(f_1(B) \cup f_2(B)), h(f_1(C) \cup f_2(C))] \\
& = \alpha[h(f'(B) \cup f'(C))]
\end{align*}
\]

Therefore,

\[
\ h(g'(B), g'(C)) \leq \alpha[h(f'(B) \cup f'(C))]
\]

By the condition of mathematical induction Lemma 6.6.3 is proved.

\[\square\]

Thus, from all the above results and the definition of Iterated function system for Commuting mapping, we are in the position to present the following theorem for Iterated function system for Commuting mapping.

**Theorem 6.6.4.** Let \( \{X; (g_0), g_1, g_2, ..., g_N\} \), where \( g_0 \) is the condensation mapping be a Iterated function system for commuting mapping with contractivity factor \( \alpha \). Then the transformation \( g' : H(X) \to H(X) \) defined by \( g'(B) = \bigcup_{n=1}^{\infty} g_n(B) \) for all \( B \in H(X) \) is a continuous Commuting mapping on the complete metric space \( (H(X), h(d)) \) with contractivity factor ‘\( \alpha ' \). Its unique fixed point, which is also called
an attractor \( A \in H(X) \), obeys

\[
A = g'(A) = \bigcup_{n=1}^{N} A,
\]

and is given by \( A = \lim_{n \to \infty} g'^{\text{on}}(B) \) for any \( B \in H(X) \).

### 6.6.3 Collage Theorem For Commuting Mapping

Based on above mathematical formulation of **Proposition 6.6.1**, we can prove the following **collage theorem** for IFS due to Commuting mapping.

**Theorem 6.6.5.** Let \((X, d)\) be a complete metric space. Let \( L \in H(X) \) be given and \( \epsilon \geq 0 \) be given. Choose an IFS for commuting mapping \( \{x; (g_0, g_1, \ldots, g_N)\} \), where \( g_0 \) is the condensation mapping with contractivity factor \( '\alpha' \), so that

\[
h(L, \bigcup_{n=0}^{N} g_n(L)) \leq \epsilon,
\]

Then

\[
h(L, A) \leq \epsilon \frac{1}{1 - \alpha},
\]

where \( A \) is the attractor of the IFS for commuting mapping.

**Proof.**

\[
h(L, A) \leq \sum_{m=1}^{n} h(g'^{\text{on}}(L), g'^{\text{on}}(A))
\]

\[
\leq \sum_{m=1}^{n} \alpha(h(f'^{\text{on}}(L), f'^{\text{on}}(A))
\]

\[
\leq \sum_{m=1}^{n} \frac{\alpha^m}{1 - \alpha} (h(f'(L), f'(A))
\]

on taking the limit \( n \to \infty \), we obtain

\[
h(L, A) \leq \frac{1}{1 - \alpha} (h(f'(L), f'(A)))
\]
This completes the proof.

6.7 Iterated Function System Due To Kannan

In this section\textsuperscript{4}, we shall explore the possibility of improvement in IFS by replacing contraction condition by the condition known as Kannan condition. In 1968, Kannan [61, 62] introduced the condition with two mappings as follows:

6.7.1 Kannan Mapping

Let \( T_1 \) and \( T_2 \) are two mappings of a complete metric space \( X \) into itself and if

\[
d(T_1x, T_2y) \leq a[d(x, T_1x) + d(y, T_2y)]
\]

(6.7.1)

for all \( x, y \) in \( X \) and \( 0 \leq a < \frac{1}{2} \) then \( T_1 \) and \( T_2 \) have a common fixed point.

The two mappings \( T_1 \) and \( T_2 \) are called mappings with Kannan’s condition. Let us name the value \( a \) as contractivity factor of Kannan mappings \( T_1 \) and \( T_2 \).

On the basis of definition of (hyperbolic) iterated function system given by Barnsley [50], we now introduce Iterated function system for Kannan’s two mappings as below:

A Iterated function system due to Kannan [62] consists of a complete metric space \((X, d)\) together with two self mappings \( T_1 \) and \( T_2 \) with contractivity factor \( a \).

Proposition 6.7.1. \( T_1 \) and \( T_2 \) are two mappings with contractivity factor \( a \), on a metric space \((X, d)\) and \( x, y \in X, x \neq y \) and \( 0 \leq a < 1 \). Then \( T_1 \) and \( T_2 \) satisfies the following conditions:

Proof. Since $T_1$ and $T_2$ are Kannan contraction mappings. We have

\[ d(T^m_1(x), T^m_2(x)) \leq a(d(T^{m-1}_1(x), T^m_1(x))) + a(d(T^{m-1}_2(x), T^m_2(x))) \]

\[ \leq a\left(\frac{a^{m-1}}{1-a}d(x, T_1(x)) + \frac{a^{m-1}}{1-a}d(x, T_2(x))\right) \]

\[ = \frac{a^m}{1-a}(d(x, T_1(x)) + d(x, T_2(x))) \]

Taking limit as $m \to \infty$, we have

\[ \lim_{m \to \infty} d(T^m_1(x), T^m_2(x)) \leq \lim_{m \to \infty} \frac{a^m}{1-a}(d(x, T_1(x)) + d(x, T_2(x))) \]

Therefore, $\lim_{m \to \infty} d(T^m_1(x), T^m_2(x)) = 0$, since $a < 1/2$.

Proposition 6.7.2. Let $T_1$ and $T_2$ are two self mappings defined on $X$ such that either $T_1$ or $T_2$ is continuous and

\[ d(T_1x, T_2y) \leq a[d(x, T_1x) + d(y, T_2y)] \]

for all $x, y \in X, x \neq y$

If for some $x_0 \in X$, the sequence $x_n$ defined by

\[ x_n = \begin{cases} 
T_1(x_{n-1}), & \text{if } n \text{ is odd} \\
T_2(x_{n-1}), & \text{if } n \text{ is even}
\end{cases} \]

has a limit point $z$ then at least one of $T_1$ and $T_2$ possesses a fixed point in $X$. Further, if both $T_1$ and $T_2$ then $z$ is a unique fixed point of each one of them and in this case $x_n \to z$. 

\[ \square \]
Proof. Let \( x_n \neq x_{n+1} \) for all \( n \). For odd \( n \) we have

\[
d(x_n, x_{n+1}) = d(T_1(x_{n-1}), T_2(x_n)) < a[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]
\]

\[
< d(x_n, x_{n+1}) \quad \text{if } 0 < a \leq 1/2
\]
similarly,

\[
d(x_n, x_{n+1}) < d(x_{n-1}, x_n)
\]

for even \( n \).

Thus, \( \{d(x_n, x_{n+1})\} \) is a monotonically decreasing, bounded sequence and so there exists a real number \( r \) such that

\[
d(x_n, x_{n+1}) \to r
\]

Let \( \{x_{nk}\} \) be a subsequence of \( \{x_n\} \) converging to \( z \). Choose a subsequence \( \{x_{nk_i}\} \) of \( \{x_{nk}\} \) such that \( n_{ki}'s \) are either all even or all odd. Let all \( n_{ki}'s \) be even. Since \( x_{nk_i} \to z \), the continuity of \( T_1, T_2 \) implies that

\[
T_1(x_{nk_i}) = (x_{nk_i+1}) \to f_1(z)
\]

and

\[
T_2(x_{nk_i+1}) = (x_{nk_i+2}) \to f_2(f_1(z))
\]
If \( z \neq T_1(z) \), then
\[
d(z, T_1(z)) = \lim_{i \to \infty} d(x_{n_ki}, x_{n_ki+1})
\]
\[
= r
\]
\[
= \lim_{i \to \infty} d(x_{n_ki+1}, x_{n_ki+2})
\]
\[
= d(T_1(z), T_2(T_1(z)))
\]
\[
< a[d(z, T_1(z)) + d(T_1(z), T_2(T_1(z)))]
\]
thus,
\[
d(z, T_1(z)) < d(z, T_1(z)),
\]
a contradiction.

Similarly, by choosing \( n'_{k_i} \) all odd we see that \( T_2(z) = z \). Consequently \( z \) is a fixed point of either \( T_1\) or \( T_2 \).

**Proposition 6.7.3.** Let \( T_1 \) and \( T_2 \) are two selfmappings and if
\[
d(T_1x, T_2y) \leq a[d(x, T_1x) + d(y, T_2y)]
\]
with contractivity factor \('a'\) on a metric space \((X, d)\). If \( T_1 \) and \( T_2 \) has a common fixed point, then it is unique.

**Proof.** On the contrary, let \( x^* \) and \( y^* \) be two common fixed points of \( T_1 \) and \( T_2 \). Then
\[
x^* = T_1(x^*), y^* = T_2(y^*) , \quad \text{and}
\]
\[
d(x^*, y^*) = d(T_1(x^*), T_2(y^*))
\]
\[
\leq a[d(x^*, T_1(x^*)) + d(y^*, T_2(y^*))]
\]
\[
\leq a[d(x^*, x^*) + d(y^*, y^*)]
\]
\[
= 0
\]
Therefore, \( x^* = y^* \).

### 6.7.2 IFS Due To Kannan

**Lemma 6.7.4.** Let \( T_1 : X \to X \) and \( T_2 : X \to X \) are two continuous Kannan mapping on the metric space \((X, d)\) with contractivity factor \('a'\). Then \( T_1 : H(x) \to H(x) \) and \( T_2 : H(x) \to H(x) \) defined by \( T_1(B) = \{ T_1(x) : x \in B \} \) for every \( B \in H(X) \) and \( T_2(C) = \{ T_2(x) : x \in C \} \) for every \( C \in H(X) \) are Kannan mappings on \((H(X), h(d))\) with contractivity factor \('a'\).

**Proof.** Since \( T_1 : H(X) \to H(X) \) and \( T_2 : H(X) \to H(X) \) are continuous Kannan mappings. Let \( B, C \in H(X) \), then

\[
h(T_1(B), T_2(C)) = d(T_1(B), T_2(C)) \vee d(T_2(C), T_1(B))
\]

\[
\leq a[d(B, T_1(B)) + d(C, T_2(C))]
\]

\[
\vee a[d(C, T_2(C)) + d(B, T_1(B))]
\]

\[
= a[d(B, T_1(B)) + d(C, T_2(C))]
\]

\[
= a[h(B, T_1(B)) + h(C, T_2(C))]
\]

This completes the proof. \( \square \)

**Lemma 6.7.5.** Let \((X, d)\) be a complete metric space. Let \( T_{1n} \) and \( T_{2n} \) are two continuous Kannan mappings on \((H(X), h)\). Let the contractivity factor for \( T_{1n} \) and \( T_{2n} \) be denoted by \( a_n \) for each \( n \). Define \( T_1' : H(X) \to H(X) \) and \( T_2' : H(X) \to H(X) \) by \( T_1'(B) = T_1(B) \cup T_2(B) \cup \ldots \cup T_n(B) = \bigcup_{j=1}^{n} T_n(B) \) for each \( B \in H(X) \). Then \( T_1' \) and \( T_2' \) are two Kannan mappings with contractivity factor \( a = \max\{a_n : n = 1, 2, \ldots, N\} \).
Proof. We shall prove the theorem using mathematical induction method using the properties of metric $h$. For $N = 1$, the statement is obviously true. Now for $N = 2$, we see that

$$h(T'_1(B), T'(C)) = h(T_{11}(B) \cup T_{12}(B), T_{21}(C) \cup T_{22}(C))$$

$$\leq h(T_{11}(B), T_{21}(C)) \lor h(T_{12}(B), T_{22}(C))$$

$$\leq [a_1(h(B, T_{11}(B)) + h(C, T_{21}(C)))]$$

$$\lor [a_2(h(B, T_{12}(B)) + h(C, T_{22}(C)))]$$

$$\leq a_1h(B, T_{11}(B)) \lor a_2h(B, T_{12}(B))$$

$$+ a_1h(C, T_{21}(C)) \lor a_2h(C, T_{22}(C))$$

$$= [(a_1 \lor a_2)\{h(B, T_{11}(B)) \lor h(B, T_{12}(B))\}]$$

$$+ [(a_1 \lor a_2)\{h(C, T_{21}(C)) \lor h(C, T_{22}(C))\}]$$

Therefore,

$$h(T'_1(B), T'_2(C)) \leq a[h(B, T'_1(B)) + h(C, T'_2(C))$$

By the condition of mathematical induction Lemma 6.7.5 is proved.

Using results given above and the formulation of Iterated function system for Kannan’s two self mappings, now we are in the position to present the following theorem for Iterated function system for Kannan’s two self mappings.

**Theorem 6.7.6.** Let $IFS = (X :, T'_1, T'_2)$ where $T'_1 : H(X) \rightarrow H(X) and T'_2 : H(X) \rightarrow H(X)$ are two self mappings with contractivity factor $'a'$. Then the transformation $T'_1(B) = \bigcup_{n=1}^{N} T_{1n}(B) and T'_2(B) = \bigcup_{n=1}^{N} T_{2n}(B)$ for all $B \in H(X)$ are continuous mappings on the complete metric space $H(X), h(d)$ with contractivity factor $'a'$. Its
unique common fixed point, which is also called an attractor \( A \in H(X) \), obeys

\[
\lim_{n \to \infty} T_1^{on}(B) = \lim_{n \to \infty} T_2^{on}(B) = A
\]

### 6.7.3 Collage Theorem Due To Kannan

Finally we give collage theorem as below:

**Theorem 6.7.7.** Let \((X,d)\) be a complete metric space. Let \( L \in H(X) \) be given and \( \epsilon \geq 0 \) be given. Choose an IFS \( \{X : T_1, T_2\} \). Let a subset \( L \) of \( X \) be such that

\[
h(L, A) < \epsilon
\]

for some \( \epsilon > 0 \). Then

\[
h(L, A) < \epsilon / (1 - a)
\]

where \( A \) is the Attractor of the IFS.

**Proof.**

\[
h(L, A) \leq \sum_{m=1}^{n} h(T_1^{(on)}(L), T_2^{(on)}(A))
\]

\[
\leq \sum_{m=1}^{n} a^m (h(L, T_1^{(on)}(L)) + h(A, T_2^{(on)}(L)))
\]

\[
\leq \sum_{m=1}^{n} \frac{a^m}{1 - a} (h(L, T_1^{(on)}(L)) + h(A, T_2^{(on)}(L)))
\]

on taking the limit \( n \to \infty \), we obtain

\[
h(L, A) \leq \frac{1}{1 - a} (h(L, T_1^{(on)}(L)) + h(A, T_2^{(on)}(L)))
\]

This completes the proof. \( \square \)
6.8 Iterated Function System Due To Reich

As we have raised a question with reference to IFS in the context of 2-Metric space and answered it affirmatively in para 6.4. Next, as its immediate consequence we now question the IFS in D-metric space and answer it accordingly as below:

6.8.1 Reich Mapping

In this section\(^5\), we shall try to explore the possibility of improvement in IFS by replacing contraction condition by a more useful condition known as Reich condition. In 1971, Reich [68] introduced a mapping, which was defined as follows:

Let \( T : X \to X \) be such that

\[
    d(T(x), T(y)) \leq ad(x, T(x)) + b(y, T(y)) + cd(x, y)
\]

(6.8.1)

for \( x, y \in X \) and \( a, b, c \) are nonnegative and \( a + b + c < 1 \). Then \( T \) has a unique fixed point.

Note that \( a = b = 0 \) yields Banach's fixed point theorem [63], while \( a = b, c = 0 \) yields Kannan's fixed point theorem [62]. Of course, we may assume always that \( a = b \), but this is not essential. Hence, Reich's theorem is better than Banach's and Kannan's theorem. On the basis of definition of (hyperbolic) iterated function system given by Barnsley [50], we now introduce Iterated function system due to Reich as below:

A Iterated function system due to Reich consists of a complete metric space \((X, d)\) together with a finite set of Reich mappings \( T_n : X \to X \).

First of all, we state and prove the two propositions which will establish a relation between $T^n : n = 1, 2, ..., N$ and $\alpha$ and uniqueness of fixed point of $T$ if it exists, respectively.

**Proposition 6.8.1.** Let $T : X \to X$ be a Reich mapping, with contractivity factor '\(\alpha\)', on a metric space $(X, d)$ and $x \in X$, Then $T$ satisfied the following condition:

$$d(T^{n+1}(x), T^n(x)) \leq \alpha^n d(x, T(x))$$

Moreover, $\lim_{n \to \infty} d(T^{n+1}(x), T^n(x)) = 0$

**Proof.** Take any point $x \in X$ and consider the sequence $\{T^n(x)\}$. Putting $x = T^n(x)$, $y = T^{n-1}(x)$ in 6.8.1 we obtain for $n \geq 1$,

$$d(T^{n+1}(x), T^n(x)) \leq ad(T^n(x), T^{n+1}(x)) + bd(T^{n-1}(x), T^n(x)) + cd(T^n(x), T^{n-1}(x)).$$

Hence,

$$d(T^{n+1}(x), T^n(x)) \leq \alpha d(T^n(x), T^{n-1}(x)),$$

where $\alpha = (b + c)/(1 - a)$. Note that $\alpha < 1$. It follows that

$$d(T^{n+1}, T^n(x)) \leq \alpha^n d(x, T(x))$$

Taking limit as $n \to \infty$, we have

$$\lim_{n \to \infty} d(T^{n+1}(x), T^n(x)) \leq \lim_{n \to \infty} \alpha^n d(x, T(x))$$

Therefore, $\lim_{n \to \infty} d(T^{n+1}(x), T^n(x)) = 0$, since $\alpha < 1$. \qed
Proposition 6.8.2. Let $T : X \to X$ be a Reich mapping, with contractivity factor $\alpha$ on a metric space $(X, d)$. If $T$ has a fixed point, then it is unique.

Proof. On the contrary, let $x^*$ and $y^*$ be two fixed points of $T$. Then $x^* = T(x^*), y^* = T(y^*)$, and

$$d(x^*, y^*) = d(T(x^*), T(y^*))$$

$$\leq ad(x^*, T(x^*)) + b(y^*, T(y^*)) + cd(x^*, y^*)$$

$$= ad(x^*, T(x^*)) + b(y^*, T(y^*)) + cd(x^*, y^*)$$

$$= cd(x^*, y^*)$$

where $d(x, y)$ is nonzero, we would have $1 \leq c$, a contradiction.

Therefore $x^* = y^*$

Next, we prove the following proposition, which shows the principle underlying the Collage theorem for Reich mapping.

Proposition 6.8.3. Let $T : X \to X$ be a Reich mapping, with contractivity factor $\alpha$ on a metric space $(X, d)$. and let $x^* \in X$ be the fixed point of $T$. Then

$$d(x, x^*) \leq \frac{1}{1 - \alpha} d(x, T(x)), \quad \forall x \in X$$

Proof. For $x \in X$, we have $\lim_{n \to \infty} T^n(x) = x^*$. Taking the point $a \in X$ as fixed, we
know that the distance function \(d(a, b)\) is continuous at the point \(b \in X\), we conclude

\[
d(x, x^*) = d(x, \lim_{n \to \infty} T^n(x))
\]

\[
= \lim_{n \to \infty} d(x, T^n(x))
\]

\[
< \lim_{n \to \infty} \sum_{m=1}^{n} d(T^{m-1}(x), T^m(x))
\]

\[
\leq \lim_{n \to \infty} d(x, T(x)) \times (1 + \alpha + \ldots + \alpha^{n-1})
\]

\[
\leq (1 - \alpha)^{-1}d(x, T(x))
\]

This completes the proof. \(\square\)

Using Propositions 6.8.1 and 6.8.2 we now prove the following theorem which is an generalization of Contraction mapping theorem for Reich mapping.

**Theorem 6.8.4.** Let \(T : X \to X\) be a Reich mapping, with contractivity factor \(\alpha\), on a complete metric space \((X, d)\). Then \(T\) possesses exactly one fixed point \(x \in X\), the sequence \(\{T^n(x) : n = 0, 1, 2, \ldots\}\) converges to \(x^*\). That is \(\lim_{n \to \infty} T^n(x) = x^*\), for each \(x \in X\).

**Proof.** Let \(x \in X\). Since \(T\) is a Reich mapping with contractivity factor \(\alpha\), we have

\[
d(T^{n+1}(x), T^n(x)) \leq \alpha^n d(x, T(x)) \quad \forall m = 0, 1, 2, \ldots
\]

Then for any \(x \in X\), we get

\[
d(T^n(x), T^m(x)) \leq \alpha^{m\wedge n} d(x, T^{|n-m|(x)}) \tag{6.8.2}
\]

where \(m, n = 0, 1, 2, \ldots\). In particular, let us take \(k = |n - m|\), for \(k = 0, 1, 2, \ldots\) we
have
\[
d(x, T^k(x)) \leq d(x, T(x)) + d(T(x), T^2(x)) \\
+ \ldots + d(T^{k-1}(x), T^k(x))
\]
\[
\leq (1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1})d(x, T(x))
\]
\[
\leq \left(\frac{1 - \alpha^k}{1 - \alpha}\right)d(x, T(x)),
\]
On substituting in Eq.6.8.2, we obtain
\[
d(T^n(x), T^m(x)) \leq \frac{\alpha^{m\wedge n}(1 - \alpha^k)}{1 - \alpha}(x, T(x))
\]
it immediately follows that \(\{T^n(x)\}_{n=0}^{\infty}\) is a Cauchy sequence. Since \(X\) is a complete metric space, this Cauchy sequence has a limit \(x^* \in X\) and we have
\[
\lim_{n \to \infty} T^n(x) = x^*
\]
(6.8.3)
Now to prove that \(x^*\) is a fixed point of \(T\) we see that
\[
d(x^*, T(x^*)) \leq d(x^*, T^n(x)) + d(T^n(x), T(x^*))
\]
\[
\leq d(x^*, T^n(x)) + ad(T^{n-1}(x), T^n(x)) + bd(x^* + T(x^*))
\]
\[
+ cd(T^{n-1}(x), x^*)
\]
Taking limit as \(n \to \infty\), on considering Equation 6.8.3 and proposition (6.8.1), we get
\[
d(x^*, T(x^*)) \leq (1 + b)d(x^*, T(x^*))\].
Hence \(x^* = T(x^*)\). By proposition 6.8.2, \(x^*\) is unique. This completes the proof.

6.8.2 IFS Due To Reich

**Lemma 6.8.5.** Let \(T : X \to X\) be a continuous Reich mapping on the metric space \((X, d)\) with contractivity factor \(\alpha\). Then \(T : H(X) \to H(X)\) defined by \(T(B) = \)
\{T(x) : x \in B\} for every B \in H(X) is a Reich mapping on (H(X), h(d)) with contractivity factor \'\alpha\'.

**Proof.** Since T is a continous mapping, therefore by lemma 2 of [50], T maps H(X) into itself.

Let B, C \in H(X). Then

\[
h(T(B), T(C)) = d(T(B), T(C)) \vee d(T(C), T(B))
\leq [ad(B, T(B)) + bd(C, T(C)) + cd(B, C)]
\vee [ad(C, T(C)) + bd(B, T(B)) + cd(C, B)]
= ad(B, T(B)) + bd(C, T(C)) + cd(B, C)
\leq ah(B, T(B)) + bh(C, T(C)) + ch(B, C)
\]

Therefore,

\[
h(T(B), T(C)) \leq ah(B, T(B)) + bh(C, T(C)) + ch(B, C)
\]

This completes the proof. \qed

**Lemma 6.8.6.** Let (X,d) be a metric space. Let \(T_n : n = 1, 2, 3, ... N\) be a continuous Reich mappings on (H(X), h). Let the contractivity factor for \(T_n\) be denoted by \(\alpha_n\) for each \(n\). Define \(T' : H(X) \to H(X)\) by \(T'(B) = T_1(B) \cup T_2(B) \cup ... \cup T_n(B) = \bigcup_{n=1}^{N} T_n(B)\) for each \(B \in H(X)\). Then \(T'\) is a Reich mapping with contractivity factor \(\alpha = \max\{\alpha_n : n = 1, 2, ... N\}\).

**Proof.** We shall prove the theorem using mathematical induction method using the properties of metric h. For \(N = 1\), the statement is obviously true. Now for \(N = 2\),
we see that

\[
\begin{align*}
    h(T'(B), T'(C)) &= h(T_1(B) \cup T_2(B), T_1(C) \cup T_2(C)) \\
    &\leq h(T_1(B), T_1(C)) \lor h(T_2(B), T_2(C)) \\
    &\leq [a_1h(B, T_1(B)) + b_1h(C, T_1(C)) + c_1h(B, C)] \\
    &\lor [a_2h(B, T_2(B)) + b_2h(C, T_2(C)) + c_2h(B, C)] \\
    &\leq a_1h(B, T_1(B)) \lor a_2h(B, T_2(B)) + b_1h(C, T_1(C)) \lor b_2h(C, T_2(C)) \\
    &+ C_1h(B, C) \lor C_2h(B, C) \\
    &= [(a_1 \lor a_2)\{h(B, T_1(B)) \lor h(B, T_2(B))\}] \\
    &+ [(b_1 \lor b_2)\{h(C, T_1(C)) \lor h(C, T_2(C))\}] + [(c_1 \lor c_2)\{h(B, C) \lor h(B, C)\}]
\end{align*}
\]

Therefore,

\[
h(T'(B), T'(C)) \leq ah(B, T'(B)) + bh(C, T'(C)) + ch(B, C)
\]

By the condition of mathematical induction Lemma 6.8.6 is proved.

Thus, from all the above results and the definition of Iterated function system due to Reich, we are in the position to present the following theorem for Iterated function system due to Reich.

\textbf{Theorem 6.8.7.} Let \(\{X; (T_0), T_1, T_2, ..., T_N\}\), where \(T_0\) is the condensation mapping be a Iterated function system due to Reich with contractivity factor \(\alpha\). Then the transformation \(T' : H(X) \rightarrow H(X)\) defined by \(T_n(B) = \bigcup_{n=1}^{N} T_n(B)\) for all \(B \in H(X)\) is a continuous Reich mapping on the complete metric space \((H(X), h(d))\) with contractivity factor \('\alpha'\). Its unique fixed point, which is also called an attractor \(A \in H(X)\), obeys

\[
A = T'(A) = \bigcup_{n=1}^{N} A,
\]
and is given by $A = \lim_{n \to \infty} T^n(B)$ for any $B \in H(X)$.

### 6.8.3 Collage Theorem Due To Reich

Based on above mathematical formulation of Proposition 6.8.3, we can prove the following *collage theorem* for IFS due to Reich.

**Theorem 6.8.8.** Let $(X, d)$ be a complete metric space. Let $L \in H(X)$ be given and $\epsilon \geq 0$ be given. Choose an IFS due to Reich $\{x; (T_0, T_1, \ldots, T_N)\}$, where $T_0$ is the condensation mapping with contractivity factor $\alpha$, so that

$$h(L, \bigcup_{n=0, n=1}^N T_n(L)) \leq \epsilon,$$

Then

$$h(L, A) \leq \epsilon \frac{1}{1 - \alpha},$$

where $A$ is the attractor of the IFS due to Reich. Equivalently,

$$h(L, A) \leq \frac{1}{1 - \alpha} h(L, \bigcup_{n=0, n=1}^N T_n(L)),$$

for all $L \in H(X)$.

### 6.8.4 Conclusion

Finally we conclude that the IFS proposed by Bransley [50] with the help of Banach Contraction Principle can also be extended to design IFS using 2-metric space, D-metric space, Commuting mappings, Kannan condition and Reich condition in order to invite images to be compressed within Fractal Theory.